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Nonlinear Ion Acoustic Waves
with Landau Damping

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Abstract

The Korteweg-deVries equation for the ion acoustic wave, modified to include the effects of ion and electron linear Landau damping, is rigorously derived from the Vlasov-Poisson equations by a multi-scale asymptotic expansion method. Careful examination of the various orderings also shows that non-linear resonance effects (trapping, reflection) can be significant.

§1. Introduction

It is well-established[1] that in a plasma of warm electrons and cold ions, the propagation of an ion acoustic wave of infinitesimal amplitude is asymptotically governed by the Korteweg-deVries (K-dV) equation

$$\frac{\partial\phi}{\partial s} + \phi\frac{\partial\phi}{\partial\zeta} + \frac{1}{2}\frac{\partial^3\phi}{\partial\zeta^3} = 0. \tag{1.1}$$

Here ζ and s are coordinates stretched by a small parameter ε and introduced by the Gardner-Morikawa transformation

$$\zeta = \varepsilon^{1/2}(x - \lambda t) \tag{1.2.a}$$

$$s = \varepsilon^{3/2}\, x, \tag{1.2.b}$$

with λ the wave speed. The small disturbance of the electric potential is proportional to $\varepsilon\phi$. Equation (1.1) is dimensionless, with quantities normalized in units of the electron Debye length $k_{D_e}^{-1}$, the inverse of the ion plasma frequency ω_{pi}^{-1}, and the electron thermal potential T_e/e, T_e in energy units.

The crucial point here is that the slowness in change of field quantities depends through eqs. (1.2) on ε, the smallness of the disturbance. This dependence is, of course, closely related to the invariance of the K-dV equation under the scaling $\zeta \to \varepsilon^{-1/2}\zeta$, $s \to \varepsilon^{-3/2}s$, $\phi \to \varepsilon^{-1}\phi$, in other words, to the similarity of the K-dV soliton when the amplitude is decreased by the factor ε while the width is stretched by the factor $\varepsilon^{-1/2}$ and the Mach number decreased by the factor ε. Namely, in the asymptotic sense, the K-dV soliton is the invariant field of the original ion acoustic wave field. Moreover, as implied by eq.(1.2), the asymptotic expansion used to derive eq.(1.1) may not be uniformly convergent; but rather, convergence is focused in the region in (x,t) space where $x-\lambda t \sim O(\varepsilon^{-1/2})$, $x \sim O(\varepsilon^{-3/2})$. Thus the K-dV soliton is a far field of the ion acoustic wave field, and consequently we may call it the "invariant far field."

Landau damping will lead to a modification of the K-dV equation, which has been studied by Ott and Sudan[2)]. However, their theory considered the effects of electron Landau damping only, being based on an expansion in powers of the mass ratio, related to the smallness of electron inertia. Hence it cannot be applied to treat ion Landau damping. Furthermore, the treatment of resonance particles is inadequate, and the ordering scheme inconsistent. Another modification was attemted by Sanuki and Todoroki[3)], who considered both ion as well as electron Landau damping. Their treatment of resonance particles is fairly improved, showing some physical aspects of the ion Landau damping of solitary waves. However, their theory is not based on explicit asymptotic expansions, even being confused with incorrect ordering. Consequently, correspondences to the fluid model description are lost, and it is difficult to view their theory in the scope of the far-field approximation.

The purpose of the present paper is to consider the effect of Landau damping on the far-field approximation (1.1) and thereby show explicit relations between the slowness of collective changes and of Landau damping and the weakness of the disturbance. As a basic idea of this approach, we first note that for sufficiently small amplitude and long wavelength, a wave will damp after a long time. That is, Landau damping is a far-field approximation of the Vlasov equation. For a plane wave launched from some point towards the positive x-direction, the far-field (Landau) approximation may be given by

$$\frac{\partial\phi}{\partial x} + i\Omega\phi = 0. \tag{1.3}$$

Here ϕ is the slowly varying amplitude of the electric potential written as

$$\Phi_k(x,t) = \phi(x)\exp[ik(x - \lambda t)], \tag{1.4}$$

and Ω is a complex quantity, its real and imaginary parts representing dispersion and damping, respectively. From the well-known linear dispersion relation of ion acoustic waves, for small k and in the reference frame moving at the ion sound speed,

$$\Omega_r \cong -\frac{1}{2} k^3. \tag{1.5a}$$

The expression for the small damping decrement is

$$\Omega_i \cong \frac{1}{2}\pi \frac{k^2}{|k|} \left[\left(\frac{\partial f_i^{(0)}}{\partial v}\right)_\lambda + \frac{m_i}{m_e}\left(\frac{\partial f_e^{(0)}}{\partial v}\right)_\lambda \right] \tag{1.5b}$$

in which $f_{i,e}^{(0)}$ are the unperturbed ion and electron distribution functions, and the factor $k/|k|$ insures that the wave is going in the k-direction.

We now wish to consider that the range of the linear far field thus obtained is the same as that of the nonlinear far field governed by the K-dV equation. From eqs. (1.2) and (1.4), it is readily seen that these two far fields will overlap if $k \sim O(\varepsilon^{1/2})$ and if the slow spatial change of Φ_k is given by $\phi(s)$. Then eq. (1.3) specifies Ω as $O(\varepsilon^{3/2})$, which ordering is consistent with that of k since for the real part $\Omega_r \sim k^3$. Then also, the small damping rate Ω_i must be of the order $\varepsilon^{3/2}$, so from eq. (1.5.b) it follows that

$$\max\left\{ \left(\frac{\partial f_i^{(0)}}{\partial v}\right)_\lambda , \frac{m_i}{m_e}\left(\frac{\partial f_e^{(0)}}{\partial v}\right)_\lambda \right. = O(\varepsilon). \tag{1.5.c}$$

for the case of overlapping far fields.

Finally, multiplying eq. (1.3) by $\exp[ik(x - \lambda t)]$, then superposing over various k, and using the expression

$$P(1/\zeta) = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{k}{|k|} e^{ik\zeta} \, dk \qquad (1.6)$$

we obtain the linear far field equation for ion Landau damping,

$$\frac{\partial \phi}{\partial s} + \frac{1}{2} \frac{\partial^3 \phi}{\partial \zeta^3} - \frac{|\Omega_i/k|}{\pi \varepsilon} P \int_{-\infty}^{\infty} \frac{1}{\zeta - \zeta'} \frac{\partial \phi}{\partial \zeta'} d\zeta' \,, \qquad (1.7)$$

which is valid in the region specified by eqs.(1.2).

However, this linear far field is remarkably different from the nonlinear one given by eq.(1.1). In particular, ε is not an arbitrary parameter measuring the smallness of the disturbance, but must be related through eq.(1.6.c) to the gradient of the unperturbed distribution functions at the ion acoustic wave speed λ. For initial Maxwellians, this means that $\varepsilon \sim (m_e/m_i)^{1/2}$, and moreover that ε depends on the small ratio of ion temperature T_i to electron temperature T_e as

$$\varepsilon \sim (T_e/T_i)^{3/2} \exp(-T_e/2T_i). \qquad (1.8)$$

Hence T_i/T_e is small and clearly an unexpandable function of ε: $T_i/T_e \approx -(\log\varepsilon)^{-1}$ for $\varepsilon \ll 1$. Consequently, for the Vlasov equation, the asymptotic limit in ε becomes rather singular. Physically speaking, for the linear far field, the slowness in behaviour of the wave is measured by the small ion temperature and must be such that, as the ions tend to become cold, the wavelength becomes infinitely long and at the same time the focusing region of convergence is shifted to infinity. Another difference is that, for the linear far field, the ordering of ϕ is arbitrary, although, of course, it must be small enough so that $\phi << \varepsilon$. Also, the similarity law for the K-dV equation does not hold for eq.(1.7), and hence there is no invariant linear far field.

We are now ready to incorporate the linear Landau damping and the collective nonlinear effects given by eq.(1.1). As a guiding principle, the modified nonlinear far field must tend to the linear one when the potential disturbance is sufficiently weak. Then, one can easily deduce

$$\frac{\partial\phi}{\partial s} + \frac{1}{2}\frac{\partial^3\phi}{\partial\zeta^3} + \phi\frac{\partial\phi}{\partial\zeta} + \frac{1}{\sqrt{8\pi}} P\int_{-\infty}^{\infty} \frac{1}{\zeta - \zeta'}\frac{\partial\phi}{\partial\zeta'}d\zeta' = 0 \qquad (1.9)$$

where ϕ is of order $\varepsilon = \sqrt{\frac{\pi}{8}}\,\left|\frac{\Omega_i}{k}\right|$. This equation is the nonlinear far field of the Vlasov equation for low temperature ions and warm electrons. A rigorous derivation of eq.(1.9), which assumed isothermal behaviour for the electrons as described by a Boltzmann distribution, has been reported briefly[4)] by one of the present authors (T.T.).

In the follow section, the details of derivation are given for the system employing the Vlasov equation for electrons as well as ions. The essential point of the method is the use of a generalization of the Gardner-Morikawa transformation which makes the method sufficiently general to be applicable to other nonlinear systems with weak Landau-like damping. Then, a K-dV equation modified to include the linear Landau damping is readily derived if nonlinear resonance effects are temporarily neglected. Section 3 next discusses these latter effects in terms of electron trapping and ion reflection, and in particular shows the connection with the study of the ion-acoustic precursor by Kato, Tajiri, and Taniuti.[5] Although it appears difficult to solve including both these nonlinear resonance effects and the linear damping, the two are found to be comparable in magnitude in the ordering which we consider. In the last section it is noted that some of the mathematical difficulties of our derivation can be avoided by decomposing the initial distribution function as in quasi-linear theory.

§2. Theoretical Formulation

We shall consider that the distribution functions f_j of both the electrons and the ions ($j = e, i$) are governed by the Vlasov equation. For a longitudinal, electrostatic wave propagating along the x-axis, this and the Poisson equation become

$$\frac{\partial f_j}{\partial t} + v\frac{\partial f_j}{\partial x} - \theta_j\left(\frac{m_i}{m_j}\right)\frac{\partial f_j}{\partial v}\,\frac{\partial \phi}{\partial x} = 0 \tag{2.1}$$

$$-\frac{\partial^2 \phi}{\partial x^2} = \sum_j \theta_j \int_{-\infty}^{\infty} f_j \, dv \,. \tag{2.2}$$

Here $\theta_j = \frac{e_j}{e} = \pm 1$ for ions and electrons, respectively. Equations (2.1) and (2.2) are dimensionless with length in units of $k_{D_e}^{-1}$, time in ω_{pi}^{-1}, potential in T_e/e, and the distribution functions in n_0/c_s where $n_0 = \lim_{t\to-\infty} \int f_j dv$; and $c_s = \omega_{pi}/k_{D_e}$ is the ion acoustic speed.

For an ion acoustic wave launched from some point into the positive x-direction, the electric potential may represent a solitary wave of small but finite amplitude with speed λ. Boundary conditions are then to be given somewhere as a function of time, while initially it will be specified that there is no disturbance:

$$f_j(x,t,v) = f_j^{(0)}(v), \quad \phi(x,t) = 0 \quad \text{for } t < 0 \tag{2.3}$$

In what follows, the $f_j^{(0)}$ will be considered to be Maxwellian-like in that as $|v| \to \infty$, they tend sufficiently rapidly to zero.

Since the wave amplitude is infinitesimally small, for $t > 0$ there will be slight deviations of order ε from the uniform initial values and we assume

$$\phi = \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \dots \tag{2.4}$$

Likewise we are led to assume

$$f_j = f_j^{(0)} + \varepsilon f_j^{(1)} + \varepsilon^2 f_j^{(2)} + \dots \tag{2.5}$$

However this expansion for f is valid only in the non-resonance region where $|v - \lambda| >> 0(\varepsilon)$. A slightly different ordering holds for the resonance region where the particle velocity closely approaches the wave speed.

The appropriate coordinate scaling was heuristically deduced in Section 1. However, the Gardner-Morikawa transformation (1.2) will not directly apply to the Vlasov equation because of singularities in the distribution functions due to the existence of resonant particles. In order properly to treat the singularities, the Laplace transform is required. Hence we first modify the Gardner-Morikawa transformation (1.2) so as to be applicable to the Vlasov equation.

Let a function $U(x,t)$ be expressed by the Fourier-Laplace transform as

$$U(x,t) = i\iint_W (w-\lambda k)^{-1}\hat{U}(k,\omega)e^{i(kx-\omega t)}\, dk\, d\omega/(2\pi)^2 \qquad (2.6)$$

where the contour W passes just above the real axis of the complex ω-plane, and $\hat{U}$ is analytic everywhere. Then U becomes a function of $x - \lambda t$ only, and eq.(2.5) represents a wave propagating in the positive x-direction with speed λ. Consequently, the Gardner-Morikawa transformation (1.2) is seen to be equivalent to the representation

$$U(\zeta,s) = i\iint_W (\omega-\lambda k)^{-1}\, \hat{U}(k,\omega,s)e^{i\varepsilon^{1/2}(kx-\omega t)}\, dk\, d\omega/(2\pi)^2. \qquad (2.6')$$

In this form, the generalization of the Gardner-Morikawa transformation is readily achieved by postulating eq. (2.6') for any $\hat{U}(k,\omega,s)$ which is defined in the upper half of the ω-plane. Thus, introducing the coordinates $\xi = \varepsilon^{1/2}x$ and $\sigma = \varepsilon^{1/2}t$, we assume the following multi-scale Fourier-Laplace transform for $(f_j - f_j^{(0)})$ and ϕ:

$$\left\{\begin{matrix} f_j^{(n)}(v,\xi,\sigma,s) \\ \\ \phi^{(n)}(\xi,\sigma,s) \end{matrix}\right\} = i(2\pi)^{-2}\int_{-\infty}^{\infty} dk \int_W d\omega \; (\omega-\lambda k)^{-1} \exp i(k\xi-\omega\sigma) \left\{\begin{matrix} \hat{f}_j^{(n)}(v,k,\omega,s) \\ \\ \hat{\phi}^{(n)}(k,\omega,s) \end{matrix}\right\} \tag{2.7}$$

Here W is the usual Laplace transform contour chosen to lie above any poles of the integral. The $\hat{\phi}^{(n)}$ will be assumed analytic for all ω.

We shall first consider the non-resonance region. Under the scaling described above, the Vlasov equation in order $\varepsilon^{3/2}$ yields

$$\frac{\partial f_j^{(1)}}{\partial\sigma} + v\frac{\partial f_j^{(1)}}{\partial\xi} - \theta_j\left(\frac{m_i}{m_j}\right)g_j(v)\frac{\partial\phi^{(1)}}{\partial\xi} = 0 \tag{2.8}$$

where, for convenience, $g_j(v) = \partial f_j^{(0)}/\partial v$. Transforming eq. (2.8) gives, in view of the conditions (2.3)

$$\hat{f}_j^{(1)} = -\theta_j\left(\frac{m_i}{m_j}\right)\frac{kg_j(v)}{\omega - kv}\,\hat{\phi}^{(1)}. \tag{2.9}$$

The lowest order of the Poisson equation yields

$$\sum_j \theta_j \int_{n.r.} f_j^{(1)}\, dv = 0 \tag{2.10}$$

which is seen to imply quasineutrality; i.e., the electron and ion densities do not differ until $0(\varepsilon^2)$. Resonance effects also, such as Landau damping, will be shown to be at least second order. Using eq.(2.9), the density perturbation can be calculated, and then eq.(2.10) immediately gives the linear dispersion relation determining the wave speed λ,

$$K(\lambda) = 0 \tag{2.11}$$

where $K(z)$ is defined as

$$K(z) = \sum_j \left(\frac{m_i}{m_j}\right) \int_{n.r.} (v-z)^{-1} g_j(v)\,dv \tag{2.12}$$

Next, expanding the Vlasov equation to $0(\varepsilon^{5/2})$ yields

$$\frac{\partial f_j^{(2)}}{\partial \sigma} + v\frac{\partial f_j^{(2)}}{\partial \xi} + v\frac{\partial f_j^{(1)}}{\partial s} - \theta_j\frac{m_i}{m_j}\left(g_j(v)\frac{\partial \phi^{(2)}}{\partial \xi} + g_j(v)\frac{\partial \phi^{(1)}}{\partial s} + \frac{\partial \phi^{(1)}}{\partial \xi}\frac{\partial f_j^{(1)}}{\partial v}\right) = 0 \tag{2.13}$$

Transforming this and eliminating $\hat{f}_j^{(1)}$ by eq.(2.9), we obtain

$$\hat{f}_j^{(2)} = \theta_j\left(\frac{m_i}{m_j}\right)(\omega-kv)^{-1}\Big[-k\, g_j(v)\hat{\phi}^{(2)} + i\omega(\omega-kv)^{-1} g_j(v)\frac{\partial \hat{\phi}^{(1)}}{\partial s}$$

$$+ i(\omega-\lambda k)\tilde{B}_j\Big] \tag{2.14}$$

where $\tilde{B}_j$ is the transform of the nonlinear term,

$$\tilde{B}_j = -i \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\sigma \quad \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_j^{(1)}}{\partial v} \quad \exp[-i(k\xi - \omega\sigma)] \tag{2.15}$$

Substituting into eq.(2.15) the integral expressions (2.7) for $f_j^{(1)}$ and $\phi^{(1)}$, we see that four of the integrations can be performed immediately; then, with the expression (2.9), the equation for $\tilde{B}_j$ becomes

$$\tilde{B}_j(\omega,k) = \theta_j(m_i/m_j) \int \int_{W'} k'(k-k')(\omega' - \lambda k')^{-1} [(\omega-\omega') - \lambda(k-k')]^{-1}$$

$$\hat{\phi}^{(1)}(\omega',k')\hat{\phi}^{(1)}(\omega-\omega',k-k')\frac{\partial}{\partial v}[\frac{g_j(v)}{\omega' - vk'}] \; d\omega' dk' (2\pi)^{-2} \tag{2.16}$$

with $\text{Im}(\omega - \omega') > 0$. The second-order density perturbation is

$$\int_{n.r.} f_j^{(2)} dv = \theta_j (\frac{m_i}{m_j}) [\phi^{(2)}(\zeta,s) \cdot \int_{n.r.} (v-\lambda)^{-1} g_j(v) dv - A_j - \theta_j (\frac{m_i}{m_j}) B_j]. \tag{2.17}$$

Here

$$A_j = (2\pi)^{-2} \int \int_W dk \; d\omega \;\; (\omega - \lambda k)^{-1} \frac{\partial \hat{\phi}^{(1)}}{\partial s} \exp i(k\xi - \omega\sigma)$$

$$\times \int_{n.r.} dv \;\; \omega(\omega - kv)^{-2} g_j(v), \tag{2.18}$$

and

$$B_j = (2\pi)^{-4} \int\!\!\int_W dk\, d\omega \int\!\!\int_{W'} dk' d\omega'\; k'(k-k')\cdot(\omega'-\lambda k')^{-1}[(\omega-\omega')-\lambda(k-k')]^{-1}$$

$$\exp i(k\xi-\omega\sigma)\; \hat{\phi}^{(1)}(\omega',k')\hat{\phi}^{(1)}(\omega-\omega',k-k')L_j(\omega,k;\omega',k') \quad (2.19)$$

with

$$L_j = -k \int_{n.r.} (\omega' - vk')^{-1}(\omega - vk)^{-2} g_j(v)\, dv \qquad (2.20)$$

The integrals denoted by A_j and B_j, respectively, may be evaluated as follows. Closing the W-contour below, and then differentiating eq.(2.18) once with respect to $\zeta = \xi - \lambda\sigma$, we find

$$\frac{\partial A_j}{\partial \zeta} = A_{0j}(\lambda)\, \frac{\partial \phi^{(1)}}{\partial s} \qquad (2.21)$$

where, using eq. (2.12), one recognizes

$$\sum_j A_{0j} = \lambda \sum_j \left(\frac{m_i}{m_j}\right) \int_{n.r.} (v-\lambda)^{-1}\, \frac{\partial g_j(v)}{\partial v}\, dv = \lambda \left(\frac{\partial K(z)}{\partial z}\right)_{z=\lambda} \qquad (2.22)$$

Now consider the ω' - integration in the expression (2.19) for B_j. Because the pole at $\omega' = \omega - \lambda(k-k')$ lies on the W contour which is above the W' contour, by closing W' in the lower half plane, only the residue at $\omega' = \lambda k'$ contributes. The ω-integration is also performed by closing W below, to obtain

$$B_j = -B_{0j}(\lambda) \int\int k^{-1}(k-k')\hat{\phi}^{(1)}(k-k')\hat{\phi}^{(1)}(k')\exp ik\xi \; dkdk'(2\pi)^{-2} \tag{2.23}$$

with

$$B_{0j}(\lambda) = \int_{n.r.} (v-\lambda)^{-3} g_j(v)\, dv \tag{2.24}$$

Differentiating eq.(2.26) and recognizing the result as a convolution, we readily find

$$\frac{\partial B_j}{\partial \zeta} = - B_{0j}(\lambda)\ \phi^{(1)}\frac{\partial \phi^{(1)}}{\partial \zeta} . \tag{2.25}$$

Note that because we are considering the Vlasov equation in the non-resonance region where $v \neq \lambda$, analytic continuation of integrals such as L_j of eq. (2.20) is unnecessary.

The Poisson equation in $0(\varepsilon^2)$ is

$$\frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} = - \sum_j \theta_j \left(\int_{n.r.} f_j^{(2)} dv + < \int_{res.} f_j dv >^{(2)} \right) \tag{2.26}$$

where the term in brackets indicates the contribution from the resonance region to the second-order density perturbation. This equation is also differentiated once, and the results (2.21) and (2.25) substituted into eq. (2.17) give the non-resonant part. By the dispersion relation (2.11), the coefficient of the term involving $\phi^{(2)}$ vanishes. Thus we obtain a modified K-dV equation of the form

$$\frac{\partial^3\phi^{(1)}}{\partial\zeta^3} + \alpha \frac{\partial\phi^{(1)}}{\partial s} + \beta\, \phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\zeta} + \frac{\partial}{\partial\zeta}\sum_j \theta_j < \int_{res.} f_j dv >^{(2)} = 0 \quad (2.27)$$

The respective coefficients are

$$\alpha = -\lambda(\partial K/\partial\lambda) \quad (2.28.a)$$

$$\beta = \sum_j \theta_j (m_i/m_j)^2 \int_{n.r.} (v-\lambda)^{-3} g_j(v)dv \quad (2.28.b)$$

with K given by eq.(2.12), and the resonance term to be determined next.

In order to evaluate the various integrals over velocity, we must specify the velocity range of the resonant particles. For trapped particles this can be achieved through the energy balance $\frac{m_j}{m_i}(v-\lambda)^2 \sim \phi \sim \varepsilon$. That is, the trapped particles depend on the magnitude of the initial disturbance, and consequently their velocity range may be made arbitrarily small. However, resonant particles also include those which resonate with the wave to cause Landau damping. The velocity range of the latter is independent of the strength of the initial disturbance and depends only on the behaviour of the unperturbed distribution function in the neighborhood of λ. For these particles, the boundaries seem quite diffusive. Following the study of the amplitude oscillation of the electron plasma wave by Imamura, Sugihara, and Taniuti[6),] we shall say that the resonance region is given by $|v-\lambda|<\delta$ where $(\frac{m_i}{m_j})g_j(\lambda\pm\delta)=0(\varepsilon)$, in accordance with eq.(1.5.c) since the parameter ε is no longer arbitrary but must be related to the smallness of the damping rate, and $1 >> \delta/\lambda > \varepsilon^{1/2}$. For example, with initial Maxwellians, an argon or cesium plasma, and λ the ion acoustic speed $\sqrt{T_e/m_i}$, then $g_i(\lambda) \sim \frac{m_i}{m_e} g_e(\lambda)$ means $T_e/T_i \sim 20$, so $\varepsilon \sim 5 \times 10^{-3}$, and hence $\delta \sim 10^{-2}\lambda$ is satisfactory. Note that velocity integrals over the non-resonant region, as in eq.(2.12), etc., differ from the principal value integral by at most $0(\delta/\lambda)$.

The ordering for the distribution function in the resonance region is different from that of eq.(2.5). From the energy balance equation, it is natural to introduce the velocity u by $v-\lambda=\varepsilon^{1/2}(\frac{m_i}{m_j})^{1/2}u$ where $u=0(1)$. Along the particle path (i.e., in the Lagrangian representation), $f_j(v,x,t)=f_j(v,0,0) =f_j^{(0)}(v)$. Then we expand as

$$f_j^{(0)}(v) \cong f_j^{(0)}(\lambda) + \varepsilon^{3/2} (\frac{m_j}{m_i})^{1/2} [u \frac{m_i}{m_j} \varepsilon^{-1} (\frac{\partial f_j^{(0)}}{\partial u})_\lambda] + \cdots$$

and see that the bracketed quantity is order unity by previous conditions. Thus in the resonance region, the proper ordering is

$$f_j = f_j^{(0)} + \varepsilon^{3/2} (\frac{m_j}{m_i})^{1/2} f_j^{(1)} + \cdots \tag{2.29}$$

Another crucial point, as mentioned by Taniuti[7)] in his sequel to ref. 6, is that $f_j^{(1)} = f_j^{(1)}(u)$ such that $(\frac{m_i}{m_j})^{1/2}(\frac{\partial f_j^{(1)}}{\partial v})_\lambda = 0(\varepsilon^{-1/2})$, a "rapid" stretching in velocity space. Incidentally, we remark that $(f_i^{(0)}/f_e^{(0)})_\lambda \doteq 0(\frac{T_i}{T_e})$; however this need not be included explicitly in the expansion (2.29) since only $\frac{m_i}{m_j}(\frac{\partial f_j^{(0)}}{\partial v})_\lambda$ will enter the following equations and this quantity is $0(\varepsilon)$ for both electrons and ions.

Then, to order $\varepsilon^{5/2}$, the Vlasov equation is

$$\varepsilon^{-1/2}(\frac{m_j}{m_i})^{1/2}[\frac{\partial f_j^{(1)}}{\partial \sigma} + v\frac{\partial f_j^{(1)}}{\partial \xi}] - \theta_j[\frac{m_i}{m_j}\frac{\partial f_j^{(0)}}{\partial v}\varepsilon^{-1}]\frac{\partial \phi^{(1)}}{\partial \xi}$$

$$- \theta_j[\varepsilon^{1/2}(\frac{m_i}{m_j})^{1/2}\frac{\partial f_j^{(1)}}{\partial v}]\frac{\partial \phi^{(1)}}{\partial \xi} = 0 \qquad (2.30)$$

By our conditions, all of these terms are easily seen to be of the same order. The second and third terms of eq.(2.30) will be identified, respectively, as Landau damping and trapping. In this section, we wish to focus attention on the effect of the linear damping, hence we shall neglect the last term temporarily. Then eq.(2.30) becomes

$$\varepsilon^{-1/2}(\frac{\partial f_j^{(1)}}{\partial \sigma} + v\frac{\partial f_j^{(1)}}{\partial \xi}) - \theta_j\frac{m_i}{m_j}\varepsilon^{-1}g_j(v)\frac{\partial \phi^{(1)}}{\partial \xi} = 0 \qquad (2.31)$$

which is easily solved by Fourier-Laplace analysis. Transforming

$$\hat{f}_j^{(1)} = -\varepsilon^{1/2}\theta_j(\frac{m_i}{m_j})\varepsilon^{-1}g_j(v)\frac{k}{\omega - kv}\hat{\phi}^{(1)}, \qquad (2.32)$$

hence

$$<\int_{res.} f_j dv>^{(2)} = \varepsilon^{-1/2} \int_{res.} f_j^{(1)} dv = i(2\pi)^{-2}\varepsilon^{-1}\ \theta_j (m_i/m_j)$$

$$\times \int\int_W dk\ d\omega (\omega-\lambda k)^{-1} e^{i(k\xi-\omega\sigma)} \hat{\phi}^{(1)} \int_{res.} (v-\omega/k)^{-1} g_j(v) dv \quad (2.33)$$

The velocity integral in eq.(2.33) can be managed by rewriting

$\int_{res.} dv = \int_{-\infty}^{\infty} dv - \int_{n.r.} dv$, approximating $\int_{n.r.} dv \cong P\int_{-\infty}^{\infty} dv$, and analytically continuing the $\int_{-\infty}^{\infty} dv$ integral over the entire ω-plane by use of the well-known Landau contour. Then the ω-integration can be performed by closing W in the lower half plane, encircling the pole at $\omega=\lambda k$, to give

$$\lim_{\omega\to\lambda k} \int_{-\infty}^{\infty} dv (v-\frac{\omega}{k})^{-1}\ g_j(v) = P\int (v-\lambda)^{-1}\ g_j(v) dv + i\pi\ g_j(\lambda)\cdot\frac{k}{|k|}. \quad (2.34)$$

Finally differentiating eq.(2.33) with respect to ζ and using the expression (1.6), we obtain the last term of the modified K-dV equation (2.27) to be

$$\frac{\partial}{\partial\zeta} \sum_j \theta_j < \int_{res.} f_j dv>^{(2)} = -\gamma\cdot P\int_{-\infty}^{\infty} (\zeta-\zeta')^{-1}\ \frac{\partial\phi^{(1)}}{\partial\zeta'}\ d\zeta' \quad (2.35)$$

with

$$\gamma = \sum_j \left(\frac{m_i}{m_j}\right) \varepsilon^{-1} g_j(\lambda). \qquad (2.28.c)$$

Taking the unperturbed electron and ion distribution functions to be Maxwellians,

$$f_j^{(0)}(v) = \frac{1}{c_j\sqrt{\pi}} \exp(-v^2/c_j^2) \qquad (2.36)$$

where $c_j = [2T(ev)/m_j]^{1/2} c_s^{-1}$, then the coefficients (2.28.a ~ b) can be expressed in terms of the plasma dispersion function Z as

$$\alpha = -\lambda \sum_j \left(\frac{m_i}{m_j}\right) c_j^{-3} Z''\left(\frac{\lambda}{c_j}\right) \qquad (2.28.a')$$

$$\beta = \frac{1}{2} \sum \left(\frac{m_i}{m_j}\right)^2 c_j^{-4} \quad Z'''\left(\frac{\lambda}{c_j}\right). \qquad (2.28.b')$$

The dispersion relation becomes

$$\sum_j \left(\frac{m_i}{m_j}\right) c_j^{-2} Z'\left(\frac{\lambda}{c_j}\right) = 0 \qquad (2.11')$$

where the prime indicates differentiation with respect to the argument.

For the ion wave in a two-component plasma with $T_e \simeq T_i$, our ordering scheme clearly shows that the effects of non-linearity, dispersion, and damping are not matched. Rather, the ion Landau damping becomes large, $g_i(\lambda) = 0(1)$, and the ion wave is unobservable. For T_e/T_i sufficiently large, we may take $c_i << \lambda << c_e$, expecting λ to be close to the ion acoustic speed c_s. By use of appropriate limiting expressions for the Z-function, we find from eq.(2.11')

$$\lambda = 1 + \frac{3}{2}\left(\frac{T_i}{T_e}\right) \tag{2.37}$$

The coefficients of the modified K-dV equation become

$$\alpha = 2\lambda^{-2} - 3\lambda^{-4}\left(\frac{T_i}{T_e}\right) \tag{2.38.a}$$

$$\beta = 3\lambda^{-4} - 1 + 30\lambda^{-6}\left(\frac{T_i}{T_e}\right) \tag{2.38.b}$$

$$\gamma = -\frac{\lambda}{\sqrt{2\pi}}\left[\left(\frac{m_e}{m_i}\right)^{1/2} + \left(\frac{T_e}{T_i}\right)^{3/2}\exp\left(-\frac{\lambda^2}{2}\frac{T_e}{T_i}\right)\right]\varepsilon^{-1} \tag{2.38.c}$$

Then, for $\lambda \cong 1$ and choosing ε as in section 1, we easily obtain from the expressions (2.38.a ~ c) the equation, (1.9), derived by Taniuti[4]. And, for cold ions and negligible electron inertia, i.e., $T_i=0$ and $(m_e/m_i)^{1/2}=0$, it becomes the unmodified K-dV equation, (1.1), of Washimi and Taniuti[1].

Also, in the limit $T_i=0$, the modified K-dV equation derived here becomes that of Ott and Sudan[2]. However, that derived by Sanuki and Todoroki[3] differs in two respects: (i) their treatment allows the wave speed λ to depend on the slow coordinate s, which leads to extra terms involving the factor $\frac{\partial\lambda}{\partial s}$ in their modified K-dV equation. However, when our theory is modified to let $\lambda = \lambda(s)$, slightly different terms result; and, anyway, the amplitude decay due to Landau damping, proportional to $\frac{\partial\lambda}{\partial s}$, can be just as readily obtained by the method presented in Ott and Sudan's paper (ref.2, Sec.V). Also, (ii), their K-dV equation contains another term, apparently of incorrect order and whose coefficient is the quantity $K(\lambda)$ defined by eq.(2.12), which in our theory determines the wave speed according to $K(\lambda) = 0$.

Recently Kono and Sanuki[8)] have shown that a K-dV equation for the ion acoustic wave, which includes the linear Landau damping, can be derived from one of the fundamental equations in the theory of weak turbulence. This is done by expanding the plasma dielectric function. We note that their basic equation is quite similar to our eq.(2.14). Also, their approximate evaluation of a velocity integral could be done rigorously if the modified Gardner-Morikawa transformation (2.7) were used. The ordering of physical quantities is not clearly stated in their paper, and trapped particle effects are not taken into account.

In the next section, we turn to a consideration of these nonlinear resonance effects which have been hitherto neglected.

§3. Effects of Nonlinear Resonant Particles

The derivation of the modified K-dV equation (2.27) purposely neglected the third term of the equation (2.30) for $f_j^{(1)}$ in the resonance region in order to examine the effect of the Landau damping more clearly. That this hitherto neglected term represents the effects of nonlinear resonant (e.g., trapped) particles is easily understood by inspecting eq.(2.30) without the linear damping term:

$$u \frac{\partial f_j^{(1)}}{\partial \zeta} - \theta_j \frac{\partial f_j^{(1)}}{\partial u} \frac{\partial \phi^{(1)}}{\partial \zeta} = 0 \tag{3.1}$$

The general solution of eq.(3.1) is of the Bernstein-Green-Kruskal type,

$$f_j^{(1)} = f_j^{(1)} \left(\frac{1}{2} u^2 + \theta_j \phi\right) \tag{3.2}$$

where, as previously defined, $u = \sqrt{\frac{m_j}{m_i}}(v-\lambda)\varepsilon^{-1/2}$. Therefore, the equation (2.30) is readily solved when either the nonlinear resonance effects or the Landau damping is neglected. It appears difficult, however, to solve analytically the full equation containing both terms.

The important point is that the Landau damping and trapped particle effects enter equation (2.30) in the same order of magnitude. Now, this equation was derived under the two assumptions $\frac{m_i}{m_j}\left(\frac{\partial f_j^{(0)}}{\partial v}\right)_\lambda = 0(\varepsilon)$ which brought in the damping, and $\left(\frac{m_i}{m_j}\right)^{1/2}\left(\frac{\partial f_j^{(1)}}{\partial v}\right)_\lambda = 0(\varepsilon^{-1/2})$ which brought in the nonlinear trapping term. Independent of this second condition, we may estimate the magnitude of the effect of trapping as

$$\int_{\text{res.}} f_j dv \sim \left(\frac{\partial f_j^{(0)}}{\partial v}\right)_\lambda \int_{v_-}^{v_+} \varepsilon^{1/2}\sqrt{\frac{m_i}{m_j}}(u^2+2\theta_j\phi^{(1)})^{1/2} dv \sim \varepsilon^2\left[\varepsilon^{-1}\frac{m_i}{m_j}\left(\frac{\partial f_j^{(0)}}{\partial v}\right)_\lambda\right]$$

$$\times \quad Q(\phi^{(1)}, \phi_M^{(1)}) \tag{3.3}$$

Here $\phi_M^{(1)}$ refers to the maximum of the electric potential $\phi^{(1)}$, and we have followed the analysis of Kato, Tajiri, and Taniuti[5], estimating

$$v_\pm = \lambda + \varepsilon^{1/2}\sqrt{2\theta_j(\frac{m_i}{m_j})(\phi_M^{(1)} - \phi^{(1)}}\ , \text{ with}$$

$$Q(\ \phi^{(1)},\ \phi_M^{(1)}) \leq 0\ (\phi_M^{(1)}) = 0(1). \tag{3.4}$$

Dispersive effects, which in the modified K-dV equation (2.27) are balanced with the effects of nonlinearity and Landau damping, are proportional to $\partial^2\phi/\partial x^2$, and this in turn is related to the effects of trapped particles, eq.(3.3), through the Poisson equation. Thus, taking $\frac{m_i}{m_j}(\frac{\partial f_j^{(0)}}{\partial v}\ )_\lambda = 0(\varepsilon)$ means that for both ions and electrons these trapped-particle resonance effects are comparable in magnitude.

In the treatment by Ott and Sudan[2], a small amount of noise (e.g. Coulomb collisions) which would scatter trapped particles in presumed to be present, and these resonant electron effects are summarily dismissed. It is of interest to examine the validity of this presumption. The effective electron-electron collision frequency is $\nu_{eff} \cong \nu_c(e\phi/Te)^{-1}$, where $\nu_c \approx \omega_{pe}(n_0L_D^3)^{-1}$ is the usual collision frequency calculated for a 90° deflection but the resonant electrons need only deviate from the straight line orbits of linear theory by the small angle $e\phi/T_e = 0(\varepsilon)$ to de-trap. Furthermore, taking the trapping time of an electron by a solitary pulse to be approximately

ω_B^{-1} with $\omega_B = k\sqrt{e\phi/me}$ and k^{-1} the soliton width, then by the ordering for ϕ we find $\omega_B \sim \omega_{pe}\varepsilon$. Consequently, if $(n_0 L_D^3)^{-1} > \varepsilon^2$, electron trapping will be destroyed. For a laboratory plasma, wall effects may also be important. That is, let L be the relevant dimension of the containing device. Then if $\omega_B^{-1} c_e > L$, or equivalently $L < \varepsilon^{-1} L_D$, by their velocity component transverse to the motion of trapping, and as long as any applied magnetic field is weak, the electrons will collide with the side walls and quickly thermalize.

For ions, collisional effects are less significant because of their larger mass. Also, in contrast to electrons, ions are not trapped but rather reflected by a solitary pulse and propagate as a precursor. How this precursor modifies the ion acoustic K-dV equation has been investigated by Kato, Tajiri, and Taniuti[5], who in their study neglected resonant Landau damping and assumed a Maxwell-Boltzmann distribution function for the electrons. They found that under the condition $(\frac{\partial f_i^{(0)}}{\partial v})_\lambda = 0(\varepsilon)$, which has been assumed throughout the present paper also, the resonant ions reflected by the soliton potential hump contribute to the K-dV equation a correction term $(\varepsilon^{-1} \cdot \partial f_i^{(0)}(\lambda)/\partial v) Q(\phi, \phi_M) \approx \varepsilon^{-1} (\frac{\partial f_i^{(0)}}{\partial v})_\lambda [\phi_M - \frac{1}{2}\phi \ln|\phi|]$ for quite small ϕ. Consequently, under the assumed smallness of $(\partial f_i^{(0)}/\partial v)_\lambda$, the effect of the reflected ions is seen to be of the same order as that of Landau damping, consistent with the order estimate (3.3) and our previous result. We note, however, that the approximately BGK solution which they obtain holds only for a quasi-stationary region extending from the precursor front to the main part of the wave but is invalid for the far upstream region where reflected ions are virtually nonexistent. The estimate (3.3), also, is valid only in this precurson region.

The competition between nonlinear resonance effects and Landau damping may be further considered since the K-dV equation also admits a periodic solution. In this case ions can really be trapped and their effect perhaps estimated by comparing the ion bounce frequency with the linear Landau damping. We recall[9] that the periodic (cnoidal) solution to the K-dV equation is characterized by a modulus $0 \leq m \leq 1$ and for $m \simeq 1$ can be approximately regarded as a sequence of solitons, each of width proportional to $\phi^{-1/2}$ but separated by the logarithmically large distance $d \sim \phi^{-1/2} |\ln(1-m^2)|$. Then $\omega_B \sim d^{-1}\phi^{1/2}$ whereas as in the discussion preceding eq.(1.5.c) $\Omega_i \sim O(\varepsilon^{3/2})$. Neverthless, the effect of the "trapped" particles on the K-dV equation may not become negligible, because in the limit m=1 (i.e., a soliton), by the result of Kato et al.[5] for the precursor, reflected ions are still significant. Thus it appears that the usual criterion of comparing ω_B and Ω_i must be applied with care in the case of long wavelength.

§4. Outline of a Regorous Perturbation Theory

It was briefly noted in Section 1 that assuming $\frac{m_i}{m_j}\left(\frac{\partial f_j^{(0)}}{\partial v}\right)_\lambda = O(\varepsilon)$ leads to asymptotically singular dependences on ε.

Actually, in the theory of Section 2, the small quantities $(me/mi)^{1/2}$ and $(Te/Ti)^{3/2}\exp(-Te/2Ti)$ were merely assigned to order ε when they appeared (cf.eq.(2.37)). These were numerical orderings, with no consideration of any functional dependences on ε which would have to be included along with (2.4), (2.5), and (2.29) in any rigorous perturbation method applied to the Vlasov-Poisson system (2.1) and (2.2).

It may be mentioned that a rigorous expansion theory which remedies both the difficulty of singular ε-dependences and that of specifying δ, the resonance velocity interval, is possible if we decompose the initial ion distribution function as

$$f_i^{(0)} = F^{(0)} + \varepsilon\tilde{F}^{(0)} \tag{4.1}$$

in which both $F^{(0)}$ and $\tilde{F}^{(0)}$ are normalized to n_0 over v: $(-\infty,\infty)$, and $F^{(0)}$ differs appreciably from $f_i^{(0)}$ only in the neighborhood of $v=\lambda$ where there is a plateau such that the derivatives of $F^{(0)}$ vanish. Thus $F^{(0)}$ will be responsible for the effects of the non-resonant ions while $\tilde{F}^{(0)}$ represents the resonant ones. (An unperturbed distribution function of this type has been used by Frieman and Rutherford[10)] in their quasi-linear theory of a weakly unstable plasma.)

For the electron distribution function, a separation like eq.(4.1) is inappropriate because for Te/Ti reasonably large, although $\frac{m_i}{m_e}\left(\frac{\partial f_e^{(0)}}{\partial v}\right)_\lambda$ is small, $f_e^{(0)}(\lambda)$ is not. Hence, in this section we shall focus attention on ion effects and regard the electrons as a dynamic background described by a Boltzmann distribution normalized to $n_0(1+\varepsilon)$. This choice, of course, will mean that electron Landau damping of the ion wave is suppressed.

For the ion distribution function,

$$f_i = f_i^{(0)}(v) + \sum_{n=1} \varepsilon^n f^{(n)}(v,\xi,\sigma,s) + \varepsilon \sum_{n=1} \varepsilon^{n/2} f_r^{(n)}(u,\xi,\sigma,s) \tag{4.2}$$

with $u=(v-\lambda)\varepsilon^{-1/2}$. The components $f_r^{(n)}$ denote the resonant part of the distribution function, where $\partial f_r^{(n)}/\partial u = 0(1)$.

The procedure of solution resembles that presented in Section 2 and we shall merely give an outline. Order $\varepsilon^{3/2}$ of the Vlasov equation is

$$\frac{\partial f^{(1)}}{\partial \sigma} + v\,\frac{\partial f^{(1)}}{\partial \xi} - G(v)\,\frac{\partial \phi^{(1)}}{\partial \xi} = 0 \tag{4.3}$$

where $G(v) = (\partial F^{(0)}/\partial v)$. This leads to the dispersion relation for λ:

$$\int_{-\infty}^{\infty} (v-\lambda)^{-1}\, G(v)\, dv - 1 = 0 \tag{4.4}$$

By the properties of $F^{(0)}$, the integral is well defined. Order $\varepsilon^{5/2}$ of the Vlasov equation is

$$\frac{\partial f^{(2)}}{\partial \sigma} + v\frac{\partial f^{(2)}}{\partial \xi} + v\frac{\partial f^{(1)}}{\partial s} - G(v)\frac{\partial \phi^{(2)}}{\partial \xi} + G(v)\frac{\partial \phi^{(1)}}{\partial s} + \frac{\partial \phi^{(1)}}{\partial \xi}\frac{\partial f^{(1)}}{\partial v} +$$

$$\varepsilon^{-1/2}\left(\frac{\partial f_r^{(1)}}{\partial \sigma} + v\frac{\partial f_r^{(1)}}{\partial \xi}\right) - \tilde{G}\,\frac{\partial \phi^{(1)}}{\partial \xi} - \frac{\partial f_r^{(1)}}{\partial u}\frac{\partial \phi^{(1)}}{\partial \xi} = 0 \qquad (4.5)$$

Grouping $f_i = f_{n.r.} + f_{r.}$ where $f_{n.r.} = F^{(0)} + \Sigma\varepsilon^n f^{(n)}$ and $f_r = \varepsilon\{\tilde{F}^{(0)} + \Sigma\varepsilon^{n/2} f_r^{(n)}\}$ and requiring both f_{nr} and f_r to satisfy the Vlasov equation, we can divide eq.(4.5) into two equations which have been encountered before, namely eqs.(2.13) and (2.30). The equations, and thus their solutions, are identical except for the replacement of g(v) by G(v) in eq. (2.13) and by $\tilde{G}(v)$ in eq.(2.30). As before, we shall drop the term representing nonlinear resonance effects and solve for the modification due to Landau damping only. A modified K-dV equation is easily derived:

$$\frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} + \alpha\frac{\partial \phi^{(1)}}{\partial s} + \alpha^*\frac{\partial \phi^{(1)}}{\partial \zeta} + \beta\phi^{(1)}\frac{\partial \phi^{(1)}}{\partial \zeta} - \gamma\cdot P\int_{-\infty}^{\infty} (\zeta-\zeta')^{-1}$$

$$\times\ \frac{\partial \phi^{(1)}}{\partial \zeta'}\, d\zeta' = 0 \qquad (4.6)$$

with

$$\alpha = -\lambda\left[\frac{\partial}{\partial z}\int_{-\infty}^{\infty} (v - z)^{-1}\, G(v)dv\right]_{z=\lambda} \qquad (4.7.a)$$

$$\alpha^* = - P \int_{-\infty}^{\infty} (v - \lambda)^{-1} \tilde{G}(v) \, dv \qquad (4.7.b)$$

$$\beta = \int_{-\infty}^{\infty} (v - \lambda)^{-3} G(v) \, dv - 1 \qquad (4.7.c)$$

$$\gamma = \tilde{G}(\lambda) \qquad (4.7.d)$$

It is important to note that $G(v)$ in the integral of eq. (4.4) is functionally independent of ε. Also, any difficulty with specifying the region of resonant particles is neaty avoided.

Moreover, since $\tilde{F}^{(0)}$ is appreciable only in the immediate vicinity of $v=\lambda$, $\alpha^*\cong 0$. By the same reasoning,

$$P \int (v - \lambda)^{-1} G(v) \cong P \int (v - \lambda)^{-1} g_j(v) dv \text{ , etc.,}$$

so the dispersion relation and modified K-dV equation derived above are nearly identical to those of Section 2.

In conclusion, however, we would emphasize that the modified K-dV equation derived here represents a partial understanding, and that eq. (4.6) should be modified further to incorporate nonlinear resonance effects in order correctly to describe the behaviour of the ion acoustic wave.

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