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Nonlinear Wave Modulation in a Magnetized

Collisionless Plasma

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Abstract

Nonlinear modulation of quasi-monochromatic electromagnetic waves propagating parallel to an external magnetic field is investigated with particular attention to contributions of resonant particles at the group velocity. The contributions of these resonant particles give rise to a nonlocal nonlinear term and modify the local nonlinear term of the nonlinear Schrödinger equation. An explicit expression of a coefficient of the nonlocal nonlinear term is given for a plasma characterized by isotropic Maxwellian velocity distribution functions. In a vanishing temperature limit this coefficient becomes zero and the nonlinear Schrödinger equation agrees with that obtained by using a fluid model.

§1. Introduction

It has been well established that in dispersive media a modulated nonlinear wave is asymptotically governed by a nonlinear Schrödinger equation.[1),2)] As far as electrostatic waves in collisionless plasmas are concerned, examining the resonance effects of particles moving at the group velocity of the wave, Ichikawa and Taniuti[3)] have shown that their contributions modify not only a coupling coefficient of a local-nonlinear term but also bring about a nonlocal-nonlinear term. In the present paper, applying their extended reductive perturbation method, we investigate nonlinear wave modulation of electromagnetic waves propagating in a magnetized collisionless plasma. In particular, we focus our attention to electron cyclotron waves (so called the whistler modes), since there have been increasing interests to understand observed pulsations of the amplitude spectrum of whistlers.[4)~6)] To confine ourselves within the wave-wave coupling, we disregard the wave-particle interaction relevant to the cyclotron damping and the trapped particle effects at the resonant velocity $v_R = (\omega - \omega_c)/k$, where ω_c is the cyclotron frequency and (ω, k) are the frequency and the wave number of the wave, but take into account the wave-wave-particle interaction relevant to the resonant contributions of particles at the group velocity of the wave.

In §2, we present a general treatment for a quasi-monochromatic electromagnetic wave propagating parallel to an external magnetic field. After lengthy calculations, we obtain a modified nonlinear Schrödinger equation with the nonlocal-nonlinear term associated with the resonant particles at the group velocity. In §3, coefficients of the nonlinear Schrödinger equation are calculated for a Maxwellian velocity distribution function of plasma particles. In the limit of vanishing temperature, the result is shown to be in agreement with that obtained for a cold plasma.[7] The expression of the nonlocal nonlinear coupling coefficient is in agreement with the one obtained by Suzuki and Ichikawa, who have derived a three dimensional nonlinear Schrödinger equation by applying a heuristic Fourier expansion method to separate slow processes from a nonlinear system. Thus, the present rigorous approach, though restricted to the one dimensional case, justifies the result of their heuristic derivation. In the last section, we present conclusing discussions.

§2. Perturbation Method for Cyclotron Waves

We assume that a quasi-monochromatic electromagnetic wave (called the carrier wave) is propagating parallel to an external constant magnetic field B_0, directed to the z-axis in a collisionless two-component plasma. Then, the Vlasov-Maxwell equations are reduced to

$$(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z})F_\alpha - \Omega_\alpha \frac{\partial}{\partial \theta} F_\alpha = \frac{e_\alpha}{m_\alpha} (\frac{\partial}{\partial z} \phi + \frac{1}{c} \frac{\partial}{\partial t} A_z) \frac{\partial}{\partial v_z} F_\alpha$$

$$+ \frac{e_\alpha}{2m_\alpha c} \sum_{\pm} \exp(\mp i\theta) [(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}) A_\pm \cdot (\frac{\partial}{\partial v_\perp} \mp \frac{i}{v_\perp} \frac{\partial}{\partial \theta})$$

$$- \frac{\partial}{\partial z} A_{\pm} \cdot v_{\perp} \frac{\partial}{\partial v_z}] F_{\alpha} , \tag{1}$$

$$\left(\frac{\partial^2}{\partial z^2} - c^{-2}\frac{\partial^2}{\partial t^2}\right) A_{\pm} = - \frac{4\pi}{c} \sum_{\alpha} e_{\alpha} \int dv_z \int dv_{\perp} v_{\perp}^2 \int d\theta \exp(\pm i\theta) F_{\alpha} , \tag{2}$$

$$\left(\frac{\partial^2}{\partial z^2} - c^{-2} \frac{\partial^2}{\partial t^2}\right) \phi = -4\pi \sum_{\alpha} e_{\alpha} \int dv_z \int dv_{\perp} v_{\perp} \int d\theta F_{\alpha} , \tag{3}$$

where the suffix α denotes ions and electrons, the variable θ is an azimuthal angle in velocity space and $\Omega_{\alpha} = (e_{\alpha}/m_{\alpha}c)B_0$ is the cyclotron frequency of the α-species in the system. Furthermore, F_{α} is a distribution function of the each species, ϕ is a scalar potential, A_z and $A_{\pm} = A_x \pm iA_y$ are a longitudinal and transverse components of a vector potential. They are subject to the supplementary condition

$$c^{-1} \frac{\partial}{\partial t} \phi + \frac{\partial}{\partial z} A_z = 0 . \tag{4}$$

Applying the extended reductive perturbation method to the set of equations (1) ~ (4), we introduce the stretched variables as

$$\eta = \varepsilon z , \quad \zeta = \varepsilon^2 z , \qquad \sigma = \varepsilon t , \tag{5}$$

and expand the distribution function and the electromagnetic potentials into the following series,

$$F_{\alpha} = F_{\alpha}^{(0)}(v_{\perp}, v_z) + \sum_{n=1}^{\infty} \sum_{\ell=-\infty}^{\infty} \varepsilon^n f_{\alpha,\ell}^{(n)}(\underset{\sim}{v}, \eta, \sigma, \zeta) \exp[i\ell(kz-\omega t)], \tag{6}$$

$$X = \sum_{n=1}^{\infty} \sum_{\ell=-\infty}^{+\infty} \varepsilon^n X_\ell^{(n)}(\eta,\sigma,\zeta)\exp[i\ell(kz-\omega t)], \tag{7}$$

where X represents $A_\pm$, A_z and ϕ. The unperturbed plasma is characterized by the particle distribution functions $F_\alpha^{(0)}(v_\perp,v_z)$, axially symmetric one in velocity space. The reality condition requires

$$\begin{aligned} f_{\alpha,-\ell}^{(n)} &= f_{\alpha,\ell}^{(n)*}, \quad \phi_{-\ell}^{(n)} = \phi_\ell^{(n)*}, \\ A_{z,-\ell}^{(n)} &= A_{z,\ell}^{(n)*}, \quad A_{\pm,-\ell}^{(n)} = A_{\mp,\ell}^{(n)*} \end{aligned} \tag{8}$$

where the asterisk denotes the complex conjugate of the attached quantity. After expanding and collecting terms of the n-th powers of ε, we obtain an infinite series of equations for the n-th order component.

For the first order in ε, with the help of (4), eq.(1) determines $f_{\alpha,\ell}^{(1)}$ in terms of $A_{\pm,\ell}^{(1)}$ and $\phi_\ell^{(1)}$. Integrating with respect to θ, we obtain

$$f_{\alpha,\ell}^{(1)} = f_0\,\phi_\ell^{(1)} + \frac{e_\alpha}{2m_\alpha c}\left(e^{i\theta} f_-(1) A_{-,\ell}^{(1)} + e^{-i\theta} f_+(1) A_{+,\ell}^{(1)}\right), \tag{9}$$

where

$$f_0 = -\left(\frac{e_\alpha}{m_\alpha}\right)\frac{k}{\omega-kv_z}\{1-(\omega/kc)^2\}\frac{\partial}{\partial v_z}F_\alpha^{(0)}, \tag{10}$$

$$f_\pm(n) = n d_\pm^{-1}(n)\mathcal{D}F_\alpha^{(0)}, \tag{11}$$

with the abbreviations of

$$d_{\pm}(n) = n(\omega - kv_z) \mp \Omega_\alpha , \tag{12}$$

$$\mathscr{D} = (\omega - kv_z) \frac{\partial}{\partial v_\perp} + kv_\perp \frac{\partial}{\partial v_z} . \tag{13}$$

We note that a homogeneous solution is dropped in (9) since in the initial state only the carrier wave exists. Substitution of (9) into the first order terms of (2) and (3) yields

$$\ell^2 k^2 \varepsilon_{\pm}(\ell\omega, \ell k) A^{(1)}_{\pm,\ell} = 0, \tag{14}$$

and

$$\ell^2 k^2 [1 - (\omega/kc)^2] \varepsilon_{\parallel}(\ell\omega, \ell k) \phi^{(1)}_{\ell} = 0, \tag{15}$$

where the dielectric functions $\varepsilon_{\pm}$ and $\varepsilon_{\parallel}$ are defined as

$$\varepsilon_{\pm}(\omega,k) = 1 - \left(\frac{\omega}{kc}\right)^2 - \left(\frac{2\pi}{kc}\right)^2 \sum_\alpha \frac{e^2_\alpha}{m_\alpha} \int dv_z \int dv_\perp v_\perp^2 f_{\pm}(1), \tag{16}$$

$$\varepsilon_{\parallel}(\omega,k) = 1 + \frac{8\pi^2}{k} \sum_\alpha \frac{e^2_\alpha}{m_\alpha} \int dv_z \int dv_\perp v_\perp \frac{1}{\omega - kv_z} \frac{\partial}{\partial v_z} F^{(0)}_\alpha . \tag{17}$$

In the present analysis, the carrier wave is specified by $A^{(1)}_{+,1}$ or $A^{(1)}_{-,1}$; thr former is associated with a left circularly polarized wave, i.e., an ion cyclotron wave and the latter with a right circularly polarized wave, i.e., an electron cyclotron wave (so called whistler mode). When either polarized wave $A^{(1)}_{+,1}$ or $A^{(1)}_{-,1}$ is considered to be the carrier wave, $\varepsilon_+(\omega, k)$ or $\varepsilon_-(\omega, k)$ does vanish and $\varepsilon_-(\omega, k)$ or $\varepsilon_+(\omega, k)$ does not vanish, respectively. At the same time $\varepsilon_\pm(\ell\omega, \ell k)$ with $\ell \neq \pm 1$ and $\varepsilon_\parallel(\ell\omega, \ell k)$ never vanish irrespective to the polarization of the carrier wave. Hence, $A^{(1)}_{-,1}$ (or $A^{(1)}_{+,1}$), $A^{(1)}_{\pm,\ell}$ ($\ell \neq \pm 1$) and $\phi^{(1)}_\ell$ are determined to be zero from eqs. (14) and (15). As for the $\ell = 0$ component is concerned, we set $A^{(1)}_{\pm,0}$ and $\phi^{(1)}_0$ are equal to zero. This is consistent with the present problem to examine the nonlinear wave modulation of the carrier waves $A^{(1)}_{+,1}$ (or $A^{(1)}_{-,1}$).

To the second order in ε, firstly we consider the components of $\ell = 0$. Eq. (1) gives

$$\Omega_\alpha \frac{\partial}{\partial\theta} f^{(2)}_{\alpha,0} = 0 \, , \tag{18}$$

while eq.(2) is automatically satisfied for $f^{(2)}_{\alpha,0}$ specified by (18). Eq. (3) requires that

$$\sum_\alpha e_\alpha \int dv_z \int dv_\perp v_\perp f^{(2)}_{\alpha,0} = 0 \, , \tag{19}$$

yet the zeroth harmonic component $f^{(2)}_{\alpha,0}$, called a slow mode, can not be determined explicitly at this stage.

Turning to the components of $\ell = 1$, solving eq. (1), we obtain

$$f^{(2)}_{\alpha,1} = f_0\phi^{(2)}_1 + \frac{e_\alpha}{2m_\alpha c}\left(e^{i\theta} f_-(1) A^{(2)}_{-,1} + e^{-i\theta} f_+(1) A^{(2)}_{+,1}\right)$$
$$- \frac{ie_\alpha}{2m_\alpha c}\left(e^{i\theta} \mathcal{f}_{1,-} A^{(1)}_{-,1} + e^{-i\theta} \mathcal{f}_{1,+} A^{(1)}_{+,1}\right), \tag{20}$$

where $\mathcal{f}_{1,\pm}$ is a differential operator with respect to the slow variables η and σ , defined as

$$\mathcal{f}_{1,\pm} = d_{\pm}^{-1}(1)[\mathcal{S} d_{\pm}^{-1}(1)\mathcal{D} - \mathcal{S}\frac{\partial}{\partial v_{\perp}} + \frac{\partial}{\partial \eta} v_{\perp} \frac{\partial}{\partial v_z}]F_{\alpha}^{(0)} \quad (21)$$

with

$$\mathcal{S} = \frac{\partial}{\partial \sigma} + v_z \frac{\partial}{\partial \eta} \quad . \quad (22)$$

Substituting (20) into eqs. (2) and (3) with $n = 2$, $\ell = 1$, we obtain

$$-k^2 \varepsilon_{\pm}(\omega,k) A_{\pm,1}^{(2)} + i\{2(k\frac{\partial}{\partial \eta} + \frac{\omega}{c^2}\frac{\partial}{\partial \sigma}) - (\frac{2\pi}{c})^2 \sum_{\alpha} \frac{e_{\alpha}^2}{m_{\alpha}}$$

$$\int dv_z \int dv_{\perp} v_{\perp}^2 \mathcal{f}_{1,\pm}\} A_{\pm,1}^{(1)} = 0, \quad (23)$$

and

$$-k^2[1 - (\omega/kc)^2]\varepsilon_{\parallel}(\omega, k)\phi_1^{(2)} = 0. \quad (24)$$

Since $\varepsilon_{\parallel}(\omega, k)$ does not vanish, (24) determines that $\phi_1^{(2)}$ is equal to zero. In (23), since the first term vanishes for a given linear dispersion relation of the carrier wave, the second term specifies dependence of $A_{\pm,1}^{(1)}$ on the slow variables η and σ. Substituting (21) into (23), we observe that the second term of (23) is reduced to a simple expression

$$\left[\frac{\partial}{\partial\sigma} + \lambda\frac{\partial}{\partial\eta}\right] A^{(1)}_{\pm,1} = 0, \tag{25}$$

where λ is equal to the group velocity

$$\lambda \equiv \frac{B}{A} = \frac{d\omega}{dk} \tag{26}$$

with

$$A = \omega/kc^2\{1 - \frac{2\pi^2}{\omega} \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int dv_z \int dv_\perp v_\perp^2 d_\pm^{-1}(1)[d_\pm^{-1}(1)\mathscr{D} - \frac{\partial}{\partial v_\perp}]F_\alpha^{(0)}\} , \tag{27}$$

$$B = 1 - \frac{2\pi^2}{kc^2} \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int dv_z \int dv_\perp\, v_\perp^2 d_\pm^{-1}(1)[\mathbf{v}_z d_\pm^{-1}(1)\mathscr{D}$$

$$- v_z \frac{\partial}{\partial v_\perp} + v_\perp \frac{\partial}{\partial v_z}]F_\alpha^{(0)} . \tag{28}$$

The equality (26) can be seen by differenciating the equation $k^2\varepsilon_\pm(\omega,k) = 0$ with respect to k. Hence, (η,σ) dependence of $A^{(1)}_{\pm,1}$ is determined as a general solution of (25) as follows,

$$A^{(1)}_{\pm,1} (\eta,\sigma;\zeta) = A_{\pm,1} (\eta-\lambda\sigma,\zeta) . \tag{29}$$

The second order terms with $\ell = 2$ are reduced as follows: from (1), (4) and (9), we get

$$f_{\alpha,2}^{(2)} = f_0\phi_2{}^{(2)} + \frac{e_\alpha}{2m_\alpha c}\{e^{i\theta}\ f_-(2)A_{-,2}^{(2)} + e^{-i\theta}f_+(2)A_{+,2}^{(2)}\}$$

$$+ (\frac{e_\alpha}{2m_\alpha c})^2\ \{e^{2i\theta}\ f_{2,-}\ A_{-,1}^{(1)2} + e^{-2i\theta}\ f_{2,+}\ A_{+,1}^{(1)2}\}\ , \tag{30}$$

where $f_\pm(2)$ is given by (11) with $n = 2$, and

$$f_{2,\pm} = \frac{1}{2}\, d_\pm^{-1}(1)[\ \mathcal{D}\ - (\omega - kv_z)v_\perp^{-1}]f_\pm(1). \tag{31}$$

Then, with the substitution of (30) into (2) and (3), we have

$$(2k)^2\varepsilon_\pm(2\omega,2k)A_{\pm,2}^{(2)} = 0, \tag{32}$$

$$(2k)^2[1 - (\omega/kc)^2]\varepsilon_\parallel(2\omega,2k)\phi_2^{(2)} = 0. \tag{33}$$

Since the coefficients of $A_{\pm,2}^{(2)}$ and $\phi_2^{(2)}$ do not vanish, the second order $\ell = 2$ components of the electromagnetic field are not induced in the system.

To the third order, we first consider the fundamental components with $\ell = 1$. With the help of relations obtained so far, eqs. (1) ~ (4) are reduced to

$$- k^2\varepsilon_\pm(\omega,k)A_{\pm,1}^{(3)} + \mathcal{T} A_{\pm,1}^{(2)} + (2ik\frac{\partial}{\partial\zeta} + \frac{\partial^2}{\partial\eta^2}$$

$$- \frac{1}{c^2}\frac{\partial^2}{\partial\sigma^2})A_{\pm,1}^{(1)} + (\frac{2\pi}{c})^2\sum_\alpha \frac{e_\alpha^2}{m_\alpha}\int dv_z\int dv_\perp v_\perp^2 d^{-1}(1)$$

$$\{ -\mathcal{S}\mathcal{F}_{1,\pm} - i[v_z f_\pm - (v_z \frac{\partial}{\partial v_\perp} - v_\perp \frac{\partial}{\partial v_z})F_\alpha^{(0)}]\frac{\partial}{\partial \zeta} - (\frac{e_\alpha}{2m_\alpha c})^2$$

$$[\mathcal{D} + 2(\omega - kv_z)v_\perp^{-1}]f_{2,\pm}|A_{\pm,1}^{(1)}|^2 + \mathcal{D}f_{\alpha,0}^{(2)}\} A_{\pm,1}^{(1)} = 0, \quad (34)$$

and

$$k^2[1 - (\omega/kc)^2]\varepsilon_{\parallel}(\omega,k)\phi_1^{(3)} = 0 , \quad (35)$$

where an operator $\mathcal{T}$ is the same one that operating on $A_{\pm,1}^{(1)}$ in (23). Therefore, the second term of (34) is eliminated by requiring that $A_{\pm,1}^{(2)}$ depends on (η,σ) through the combination of $\eta - \lambda\sigma$, while the first term of (34) vanishes because of the linear dispersion relation of the carrier wave. Eq. (35) determines that $\phi_1^{(3)}$ should vanish identically. Thus, defining a set of new variables (ξ,τ) as

$$\xi = \eta - \lambda\sigma , \quad (36)$$

$$\tau = \zeta/\lambda , \quad (37)$$

we obtain a generalized nonlinear Schrödinger equation for $A_{\pm,1}$

$$i\alpha\frac{\partial}{\partial\tau} A_{\pm,1} + \beta\frac{\partial^2}{\partial\xi^2} A_{\pm,1} + \gamma_2 |A_{\pm,1}|^2 A_{\pm,1}$$

$$+ \frac{2\pi^2\lambda}{kc^2} \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int dv_z \int dv_\perp \, v_\perp^2 d_\pm^{-1}(1)\mathcal{D} f_{\alpha,0}^{(2)} A_{\pm,1} = 0 . \quad (38)$$

Here, the coefficients are defined as follows;

$$\alpha = 1 - \frac{2\pi^2}{kc^2} \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int dv_z \int dv_\perp v_\perp^2 d_\pm^{-2}(1) [(\omega \mp \Omega_\alpha) v_\perp \frac{\partial}{\partial v_z} \pm \Omega_\alpha v_z \frac{\partial}{\partial v_\perp}] F_\alpha^{(0)}, \tag{39}$$

$$\beta = \frac{\lambda}{2k}\{1 - (\frac{\lambda}{c})^2 - (\frac{2\pi}{c})^2 \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int dv_z \int dv_\perp v_\perp^2 d_\pm^{-1}(1)(v_z - \lambda) f_{1,\pm}\} , \tag{40}$$

$$\gamma_2 = - \frac{\lambda}{2k} (\frac{2\pi}{c})^2 \sum_\alpha \frac{e^2_\alpha}{m_\alpha} (\frac{e_\alpha}{2m_\alpha c})^2 \int dv_z \int dv_\perp v_\perp^2 d_\pm^{-1}(1)$$

$$[\mathcal{D} + 2(\omega - kv_z) v_\perp^{-1}] f_{2,\pm} , \tag{41}$$

where $f_{1,\pm}$ is a function of v and v_z defined by

$$f_{1,\pm} = [(v_z - \lambda) d_\pm^{-2}(1) \mathcal{D} - (v_z - \lambda) d_\pm^{-1}(1) \frac{\partial}{\partial v_\perp} + v_\perp d_\pm^{-1}(1) \frac{\partial}{\partial v_z}] F_\alpha^{(0)} . \tag{42}$$

Now, we come to a stage to determine the slow mode contribution $f^{(2)}_{\alpha,0}$ from the zeroth harmonic components of (1) and (4) with their third order terms of ε. Eq. (1) gives rise to an equation having following structure,

$$- \Omega_\alpha \frac{\partial}{\partial \theta} f^{(3)}_{\alpha,0} + \mathcal{S} f^{(2)}_{\alpha,0} = (\theta\text{-independent terms})$$

$$+ (\text{periodic terms in } \theta). \tag{43}$$

Since $f_{\alpha,0}^{(2)}$ is shown to be independent of θ by (18), and at the same time $f_{\alpha,0}^{(3)}$ should be a periodic function of θ , eq. (43) can be decomposed into the following equations,

$$- \Omega_\alpha \frac{\partial}{\partial \theta} f_{\alpha,0}^{(3)} = (\text{periodic terms in } \theta), \qquad (44)$$

and explicitly

$$\mathcal{S} f_{\alpha,0}^{(2)} = \frac{e_\alpha}{m_\alpha} \left(\frac{\partial}{\partial \eta} \phi_0^{(2)} + c^{-1} \frac{\partial}{\partial \sigma} A_{z,0}^{(2)}\right) \frac{\partial}{\partial v_z} F_\alpha^{(0)}$$
$$+ \left(\frac{e_\alpha}{2m_\alpha c}\right)^2 \left\{ \left[\mathcal{S} \left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right) - \frac{\partial}{\partial \eta} v_\perp \frac{\partial}{\partial v_z}\right] f_\pm |A_{\pm,1}^{(1)}|^2 \right.$$
$$\left. + \{\mathcal{D} + (\omega - kv_z) v_\perp^{-1}\} f_{1,\pm} |A_{\pm,1}^{(1)}|^2 \right\} . \qquad (45)$$

Eq. (4) yields

$$c^{-1} \frac{\partial}{\partial \sigma} \phi_0^{(2)} + \frac{\partial}{\partial \eta} A_{z,0}^{(2)} = 0 . \qquad (46)$$

In solving eqs. (45) and (46), we expand the slow mode into the Fourier-Laplace integral with respect to the variables η and σ as

$$X(\eta,\sigma) = (2\pi)^{-2} \int_C d\Omega \int_{-\infty}^{+\infty} dK \exp[i(K\eta - \Omega\sigma)] \tilde{X}(K,\Omega) , \qquad (47)$$

where the contour C is lying in the convergent region parallel to the real axis. Then, we obtain from (45),

$$\tilde{f}^{(2)}_{\alpha,0}(v_\perp,v_z;K,\Omega) = -\frac{e_\alpha}{m_\alpha}\frac{1-(\Omega/cK)^2}{\Omega-Kv_z} K \frac{\partial}{\partial v_z} F^{(0)}_\alpha \tilde{\phi}^{(2)}_0$$

$$- \left(\frac{e_\alpha}{2m_\alpha c}\right)^2 \frac{N_\pm(v_\perp,v_z)}{\Omega-Kv_z} K\Psi(K,\Omega), \qquad (48)$$

where $\Psi(K,\Omega)$ is a transformation of $|A^{(1)}_{\pm,1}|^2$,

$$\Psi(K,\Omega) = \delta(\Omega-K\lambda)\int dK' A_{\pm,1}(K')A^*_{\pm,1}(K'-K), \qquad (49)$$

and $N_\pm(v_\perp,\ v_z)$ is defined as

$$N_\pm(v_\perp,v_z) = \left[(v_z - \lambda)\left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right) - v_\perp \frac{\partial}{\partial v_z}\right] f_\pm$$

$$+ \left(\mathcal{D} + \frac{\omega-kv_z}{v_\perp}\right) f_{1,\pm} \qquad (50)$$

The electro-static component of the slow mode $\tilde{\phi}^{(2)}_0$ can be determined from the Fourier-Laplace transformation of (19) with substitution of (48) as follows

$$[1 - (\Omega/cK)^2] \sum_\alpha \frac{e^2_\alpha}{m_\alpha}\int dv_z \int dv_\perp \frac{v_\perp}{\Omega-Kv_z} \frac{\partial}{\partial v_z} F^{(0)}_\alpha \tilde{\phi}^{(2)}_0$$

$$= -\sum_\alpha e_\alpha \left(\frac{e_\alpha}{2m_\alpha c}\right)^2 \int dv_z \int dv_\perp v_\perp \frac{N_\pm(v_\perp,\ v_z)}{\Omega-Kv_z} \Psi(K,\Omega). \qquad (51)$$

Eliminating $\tilde{\phi}^{(2)}_0$ from (48) with the help of (51), we determine $\tilde{f}^{(2)}_{\alpha,0}$ in terms of $\Psi(K,\Omega)$.

Returning to (38), we evaluate the coefficient of $A^{(1)}_{\pm,1}$ of the last term as

$$(2\pi)^{-2} \int_C d\Omega \int dK \exp[i(K\eta - \Omega\sigma)]\{\cdots\cdots \tilde{f}^{(2)}_{\alpha,0}\}$$

$$= (2\pi)^{-2} \int_C d\Omega \int dK \exp[i(K\eta - \Omega\sigma)]\left(\frac{W^2}{\Gamma} + C\right)\Psi(K,\Omega), \tag{52}$$

where

$$C = -\frac{2\pi^2\lambda}{kc^2} \sum_\alpha \frac{e_\alpha}{m_\alpha}\left(\frac{e_\alpha}{2m_\alpha c}\right)^2 \int dv_z \int dv_\perp v_\perp^2 d_\pm^{-1}(1)\mathscr{D} \frac{K}{\Omega - Kv_z} N_\pm \quad , \tag{53}$$

$$\Gamma = -\frac{8\pi^2\lambda}{k} \sum_\alpha e_\alpha \int dv_z \int dv_\perp v_\perp \frac{K}{\Omega - Kv_z} G_\alpha \; , \tag{54}$$

$$W = \frac{2\pi^2\lambda}{kc^2} \sum_\alpha \frac{e_\alpha^2}{m_\alpha} \int dv_z \int dv_\perp v_\perp^2 d_\pm^{-1}(1)\mathscr{D} \frac{K}{\Omega - Kv_z} G_\alpha$$

$$= -\frac{8\pi^2\lambda}{k} \sum_\alpha e_\alpha \left(\frac{e_\alpha}{2m_\alpha c}\right)^2 \int dv_z \int dv_\perp v_\perp \frac{K}{\Omega - Kv_z} N_\pm \; , \tag{55}$$

with the abbreviation of

$$G_\alpha = \frac{e_\alpha}{m_\alpha} \frac{\partial}{\partial v_z} F_\alpha^{(0)} \quad . \tag{56}$$

The equality between the second line and the last term of (55) can be proved by carrying out the integration by parts with respect to $v_\perp$ and v_z. Deforming the contour C to the Landau contour with real Ω, we use the relation

$$\frac{K}{\Omega - Kv_z} = \frac{P}{(\Omega/K) - v_z} - i\pi \mathrm{sgn}(K)\delta(v_z - (\Omega/K)), \tag{57}$$

where P stands for the principal value and $\mathrm{sgn}(K) = K/|K|$. Referring to (49), we may replace Ω/K by the group velocity λ in various terms of (52). Hence, the integrand of (52) is reduced to a simple form

$$\left(\frac{W^2}{\Gamma} + C\right)\Psi(K,\Omega) = [\,Ⓗ_1 - i\,\mathrm{sgn}(K)\,Ⓗ_2\,]\Psi(K,\Omega), \tag{58}$$

in which $Ⓗ_1$ and $Ⓗ_2$ are independent of K and Ω . They are given as

$$Ⓗ_1 = C_1 + \frac{W^2{}_1}{\Gamma_1} - \frac{1}{\Gamma_1}|\Gamma|^{-2}(\Gamma_2 W_1 - \Gamma_1 W_2)^2 , \tag{59}$$

$$Ⓗ_2 = C_2 + \frac{W_2^2}{\Gamma_2} - \frac{1}{\Gamma_2}|\Gamma|^{-2}(\Gamma_2 W_1 - \Gamma_1 W_2)^2 , \tag{60}$$

where C_1, C_2 , Γ_1, Γ_2 and W_1 , W_2 are defined by decompositions of the quantities C, Γ and W as

$$C = C_1 - i\,\mathrm{sgn}(K)C_2 , \tag{61}$$

$$\Gamma = \Gamma_1 - i\,\mathrm{sgn}(K)\Gamma_2 , \tag{62}$$

$$W = W_1 - i\,\mathrm{sgn}(K)W_2 , \tag{63}$$

respectively. Remembering that

$$\mathrm{sgn}(K) = \frac{P}{i\pi}\int_{-\infty}^{+\infty} d\xi\, \exp[iK\xi]\,\frac{1}{\xi} , \tag{64}$$

we obtain a modified nonlinear Schrödinger equation for a small but finite complex amplitude $A_\pm$,

$$i\alpha \frac{\partial}{\partial\tau} A_\pm + \beta \frac{\partial^2}{\partial\xi^2} A_\pm + (\gamma^2 + \textcircled{H}_1)|A_\pm|^2 A_\pm + \textcircled{H}_2 \frac{P}{\pi}\int_{-\infty}^{+\infty} \frac{|A_\pm(\xi',\tau)|^2}{\xi - \xi'} d\xi' A_\pm = 0 . \qquad (65)$$

To conclude this section, we show that the coefficient β of the dispersion term is actually proportional to the second derivative of $\omega(k)$ with respect to k. We have, after straightforward manipulations,

$$\frac{d^2}{dk^2}[k^2\varepsilon_\pm(\omega,k)] = 2(\frac{2k}{\lambda})\beta - 2\frac{d\lambda}{dk}(\frac{k}{\lambda})\alpha = 0, \qquad (66)$$

which yields that

$$\beta = \frac{\alpha}{2}\frac{d^2\omega}{dk^2} \qquad (67)$$

Furthermore, we add a remark on effects of the cyclotron damping. The linear cyclotron damping rate is given as

$$\gamma_c = -\frac{\pi}{\partial\varepsilon_\pm/\partial\omega}(\frac{2\pi}{kc})^2\sum_\alpha \frac{e_\alpha^2}{m_\alpha}\int dv_z \int dv_\perp v_\perp^2 \delta(\omega - kv_z \mp \Omega_\alpha)\mathcal{D} F_\alpha^{(0)} . \qquad (68)$$

If $\gamma_c/\omega(k) = 0(\varepsilon^2)$, a replacement of

$$\frac{\partial}{\partial\tau} \rightarrow \frac{\partial}{\partial\tau} + \varepsilon^{-2}\gamma_c$$

is consistent to accommodate the linear cyclotron damping process into the present analysis. Thus, we obtain a modified nonlinear Schrödinger equation for the cyclotron waves propagating through collisionless plasmas as

$$i\left(\frac{\partial}{\partial\tau} A_\pm + \gamma A_\pm\right) + p \frac{\partial^2}{\partial\xi^2} A_\pm + q|A_\pm|^2 A_\pm + s\,\frac{P}{\pi}\int_{-\infty}^{+\infty} \frac{|A_\pm(\xi',\tau)|^2}{\xi-\xi'}\, d\xi' A_\pm = 0 \quad , \qquad (69)$$

where $\gamma = \gamma_c/\varepsilon^2$, $p = (1/2)d^2\omega/dk^2$, $q = (\gamma_2 + \textcircled{H}_1)/\alpha$ and $s = \textcircled{H}_2/\alpha$. Eq.(65) is in agreement with the one dimensional version of a generalized nonlinear Schrödinger equation derived by Suzuki and Ichikawa[8)] on the basis of Fourier-expansion method for separation of slow processes in nonlinear systems.

§3. Cyclotron Waves in a Maxwellian Plasma

We investigate nonlinear modulation of the cyclotron waves in a two-componenet plasma by applying the result of the preceeding section. We assume that the unperturbed state of the system is characterized by isotropic Maxwellian distribution functions,

$$F^{(0)} = n_0 F_z^{(0)} F_\perp^{(0)} \quad , \tag{70}$$

with

$$F_z^{(0)} = (m/2\pi T)^{1/2} \exp(-mv_z^2/2T) \quad , \tag{71}$$

$$F_\perp^{(0)} = (m/2\pi T)\exp(-mv_\perp^2/2T) \quad , \tag{72}$$

where T represents a reduced temperature of the particles and the suffix for particle species is omitted and n_0 is the number density of the particles. The linear dispersion relation is, after integrating over $v_\perp$ of (16), given as

$$\varepsilon_\pm(\omega,k) = 1 - (\frac{\omega}{kc})^2 + \Sigma\omega\ (\frac{\omega_p}{kc})^2 \int_{-\infty}^{+\infty} dv_z d^{-1}(1) F_z^{(0)}(v_z) = 0. \tag{73}$$

Similarly, the coefficient α given by (39) is calculated to be

$$\alpha = 1 + \frac{\omega}{2}\ \Sigma\ (\frac{\omega_p}{kc})^2 \int dv_z\ kv_z\ d_\pm^{-2}\ (1) F_z^{(0)}(v_z). \tag{74}$$

After being integrated over $v_\perp$, the expression of γ_2, (41) is reduced to

$$\gamma_2 = -\frac{k\lambda\omega}{4c^2}\ \Sigma\ \omega_p^2 (\frac{e}{mc})^2 \int dv_z\ F_z^{(0)} \{ -\frac{m\omega}{kT}\ v_z\ d_\pm^{-3}(1)$$

$$+ (2\omega + kv_z) d_\pm^{-4}(1) - \frac{3k^2 T}{m}\ d_\pm^{-5}(1) \}, \tag{75}$$

which represents the coupling between the carrier wave and its second harmonic beat wave. In order to obtain an explicit result, we consider the low temperature limit of (75) by taking the following limits for the v_z-integration;

$$F_z^{(0)} \rightarrow \delta(v_z), \quad \frac{m}{T} v_z F_z^{(0)} \rightarrow -\delta'(v_z), \quad \frac{T}{m} F_z^{(0)} \rightarrow 0, \tag{76}$$

Then, (75) is evaluated as

$$\gamma_2 = \frac{k\lambda}{4c^2} \omega^2 \Sigma \omega_p^2 \left(\frac{e}{mc}\right)^2 (\omega \mp \Omega)^{-4} . \tag{77}$$

Now, turning to the evaluation of Ⓗ_1 and Ⓗ_2, we calculate the real parts of various quantities such as C, Γ and W under the same approximation of (76). We get

$$C_1 = -\frac{k\lambda}{4c^2} \Sigma \omega_p^2 \left(\frac{e}{mc}\right)^2 (\omega\mp\Omega)^{-4} \{\omega^2 + \Omega^2[1 - \frac{\omega}{k\lambda}(1\mp\frac{\omega}{\Omega})^2]\}, \tag{78}$$

$$\Gamma_1 = (k\lambda)^{-1} \Sigma \omega_p^2 , \tag{79}$$

$$W_1 = \Sigma \omega_p^2 \frac{e}{2mc^2}(\pm\Omega)(\omega\mp\Omega)^{-2}[1 - \frac{\omega}{k\lambda_0}(1 \mp \frac{\omega}{\Omega})] . \tag{80}$$

On the contrary, the imaginary parts of these quantities are calculated, by taking advantage of the factor of $\delta(v_z - \lambda)$, as follows,

$$C_2 = -\pi\frac{k\lambda}{4c^2} \sum (\frac{k_d}{k})^2 (\frac{e}{mc})^2 (\frac{\omega-k\lambda}{\omega-k\lambda\mp\Omega})^2 \lambda F_z^{(0)}(\lambda) , \tag{81}$$

$$\Gamma_2 = \pi\frac{\lambda^2}{k} \sum k_d^2 F_z^{(0)}(\lambda) , \tag{82}$$

$$W_2 = \pi \frac{\lambda^2}{2kc^2} \sum k_d^2 \frac{e}{m} \frac{\omega-k\lambda}{\omega-k\lambda\mp\Omega} F_z^{(0)}(\lambda) . \tag{83}$$

with the abbreviation of $k_d^2 = 4\pi e^2 n_0/T$. In other words, we notice that relevant temperature effects manifest through the contribution of resonant particles with the group velocity.

Examining the structure of $f_{\alpha,0}^{(2)}$ given by (48) and (51), we notice that the term C represents the nonlinear coupling between the carrier wave and the virtual electromagnetic slow mode. Hence, adding C_1 to γ_2, we have

$$\gamma_{EM} \equiv \gamma_2 + C_1 = -\frac{k\lambda}{4c^2} \sum (\frac{e}{mc})^2 \frac{\omega_p^2\Omega^2}{(\omega\mp\Omega)^4} [1 - \frac{\omega}{k\lambda} \{1 \mp \frac{\omega}{\Omega}\}]^2. \tag{84}$$

At the same time, we have also the contributions of the electrostatic slow mode, expressed by the second and the third terms of (59). In the limit of vanishing temperature, since Γ_2 and W_2 approach zero very rapidly, we have for the second term of $\textcircled{H}_1$,

$$\gamma_{ES} \equiv \Gamma_1^{-1} W_1^2 = \frac{k\lambda}{4c^2} (\Sigma\omega_p^2)^{-1} \{\sum \frac{e\Omega}{mc} \frac{\omega_p^2}{(\omega\mp\Omega)^2} [1 - \frac{\omega}{k\lambda}(1 \mp \frac{\omega}{\Omega})]\}^2. \tag{85}$$

We also notice that the coefficient α is equal to unity in the limit of small temperature. Hence, we get the local non-linear coupling coefficient q of (69) as

$$q = \gamma_{EM} + \gamma_{ES} = -\frac{k\lambda}{4c^2}(\Sigma\omega_p^2)^{-1}\omega_{pi}^2\omega_{pe}^2$$

$$\cdot\{\Sigma\,\frac{e\Omega}{mc}(\omega \mp \Omega)^{-2}[1 - \frac{\omega}{k\lambda}(1 \mp \frac{\omega}{\Omega})]\}^2 \quad . \qquad (86)$$

Using the linear dispersion relation in the cold limit,

$$(\frac{kc}{\omega})^2 \equiv n^2 = 1 - \Sigma\,\frac{\omega_p^2}{\omega(\omega\mp\Omega)} , \qquad (87)$$

and a relation obtained by taking a derivative with respect to k of (87),

$$\frac{\omega}{k\lambda} = n^{-2}\;[1 - \Sigma\omega_p^2(\mp\Omega)/2\omega(\omega\mp\Omega)^2] \quad , \qquad (88)$$

we can express (86) in a form without summation,

$$q^* = -\frac{a^2k^3\lambda}{4n^2B_0^2}\;[2 - \frac{\omega}{k\lambda}(1 + n^2)]^2 , \qquad (89)$$

where $a^2 = B_0^2/4\pi n_0(m_i + m_e)c^2$ is a squared ratio of the Alfvén velocity to the light velocity. The coefficient q^*, given by (89), is in agreement with the one obtained by Kako[7] on the basis of the fluid equations for a cold plasma.

Now, turning to effects of the resonant particles at the group velocity, we observe that the third term of $Ⓗ_1$ is negative definite and expect that it will represent relevant temperature effects besides temperature dependent terms coming from the various factors γ_2 , C_1 , Γ_1 and W_1 . Furthermore, essential contributions of the resonant particles at the group velocity give rise to the following expression for the non-local nonlinear coupling coefficient s,

$$s = - \Gamma_2^{-1} \left(\frac{\pi\lambda^2}{2kc^2}\right)^2 \Big[e^2 \frac{F_i F_e}{T_i T_e} \Big(\sum_\alpha \frac{(\omega-k\lambda)\omega_{p\alpha}^2}{\omega-k\lambda\mp\Omega_\alpha}\Big)^2$$

$$+ (\Gamma_1^2 + \Gamma_2^2)^{-1} \Big\{ \sum_\alpha (k\lambda)^{-1} k_{d\alpha}^2 \sum_\beta \omega_{p\beta}^2 \Big[\frac{e_\beta}{m_\beta} \frac{\omega(\omega\mp\Omega_\beta) \pm k\lambda\Omega_\beta}{(\omega\mp\Omega_\beta)^2}$$

$$- \frac{e_\alpha}{m_\alpha} \frac{\omega-k\lambda}{\omega-k\lambda\mp\Omega_\alpha} \Big] F_\alpha \Big\}^2 \Big] \quad , \qquad (90)$$

where $F_\alpha = F_{z,\alpha}^{(0)}(\lambda)$. This expression is in agreement with the result obtained by Suzuki and Ichikawa[8]. Here, however, we reserve ourselves from estimating these quantities explicitly for plasma existing in nature. In a separate paper, we will discuss in detail the nonlinear wave modulation of whistler modes in the solar wind.

§4. Discussion

Employing the reductive perturbation method extended by Ichikawa and Taniuti[3] to take into account the effect of resonant particles at the group velocity, we have studied the nonlinear wave modulation in a magnetized plasma. The result obtained in the preceding sections is in agreement with one dimensional form of the result obtained by Suzuki and Ichikawa[8), who have applied a heuristic Fourier expansion method to separate the slow modulation process from a nonlinear system. Our rigorous derivation of the generalized nonlinear Schrödinger equation confirms that their heuristic introduction of the group velocity was indeed a plausible procedure.

In the present analysis, we have disregarded the resonant particles having the velocities of the order of $(\omega - \omega_c)/k$. These resonant particles will give rise to the trapping effects. Recently, Karpman and his collaborators[9) have been investigating the trapping effect of these resonant particles on the whistler modulation phenomena. Although we admit their relevant importance, we have forcussed our attention to the nonlinear wave modulation with account of the resonance effect at the group velocity, restricting ourselves to low frequency cyclotron waves propagating through low temperature plasmas.

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References

1) V.I. Karpman and E.M. Krushkal: Zh. Eksp. Teor. Fiz. 55 (1968) 530 [Sov. Phys. JETP 28 (1969) 277].

2) T. Taniuti and N. Yajima: J. Math. Phys. 10 (1969) 1369.

3) Yoshi H. Ichikawa and T. Taniuti: J. Phys. Soc. Japan 34 (1973) 513.

4) A. Hasegawa: J. Geophys. Res. 77 (1972) 84.

5) A. L. Brinca: Stanford University, Institute for Plasma Research, Report No. 464 (1972).

6) A. Hasegawa: Phys. Fluids 15 (1972) 870.

7) M. Kako: J. Phys. Soc. Japan 33 (1972) 1678.

8) T. Suzuki and Yoshi H. Ichikawa: Submitted for publications in Prog. Theor. Phys.

9) V.I. Karpman, Ja. N. Isotomin : Institute of Terrestrical Magnetism, Ionosphere and Radiowave Propagation, preprint No.5 (1972).