# Nonlinear electromagnetic gyrokinetic equation for plasmas with large mean flows

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A new nonlinear electromagnetic gyrokinetic equation is derived for plasmas with large flow velocities on the order of the ion thermal speed. The gyrokinetic equation derived here retains a collision term and is given in the form which is valid for general magnetic geometries including the slab, cylindrical and toroidal configurations. The source term for the anomalous viscosity arising through the Reynolds stress is identified in the gyrokinetic equation. For the toroidally rotating plasma, particle, energy and momentum balance equations as well as the detailed definitions of the anomalous transport fluxes and the anomalous entropy production are shown. The quasilinear anomalous transport matrix connecting the conjugate pairs of the anomalous fluxes and the forces satisfies the Onsager symmetry.  [S1070-664X(98)02607-X]

## I. INTRODUCTION

Gyrokinetic equations[1–13] give a foundation for investigating microinstabilities, which cause the turbulent or anomalous transport in fusion plasmas. They describe fluctuations with short perpendicular wavelengths on the order of the ion gyroradius and frequencies much lower than the ion gyrofrequency. There are two types of methods to derive the gyrokinetic equation. The recursive technique[1–7] was used when the gyrokinetic equation was first obtained. The recursive method is also used for derivation of the drift kinetic equation[14,15] from which the neoclassical transport[16–18] is described. Another modern derivation is based on the perturbative Hamiltonian formalism.[8–13] The gyrokinetic equation obtained by the Hamiltonian method is written for the total distribution function, which is in contrast to the recursively derived form where the distribution function is separated into equilibrium and perturbed parts. The conservation of the phase space, the energy and the magnetic moment are systematically treated by the Hamiltonian formulation. However, the Hamiltonian method generally considers the collisionless case, due to the fact that its treatment of collisions does not yet seem to be systematically clear. In the recursive formulation, collisions are described by the gyrophase-averaged collision operator,[3] detailed structures of which have been given based on the Fokker-Planck collision model.[19–21]

In the present paper, we follow the recursive formulation with the ballooning representation to derive the nonlinear electromagnetic gyrokinetic equation for plasmas with large flows, because we also include collisional effects that are necessary for the unified description of the turbulent and collisional (classical and neoclassical) transport processes.[22–24] It is also known that the turbulent system with the finite transport fluxes requires finite collisionality to reach a steady state.[23,25] The gyrokinetic equation derived here is valid for general magnetic geometries with large flows on the order of the ion thermal speed. From this equation, the reduced forms for the slab, cylindrical and toroidal configurations are easily obtained. In recent years, the effects of large flows have been attracting much attention in relation to improved confinement such as high-confinement modes (H-modes)[26] and internal transport barriers (ITB) found in reversed shear configurations.[27,28] Artun and Tang[6,7] derived the gyrokinetic equations for the slab and toroidal system with large equilibrium flows by using the recursive method for the ballooning type of fluctuations. Hamiltonian derivation of the gyrokinetic equation for the toroidally rotating plasma was shown by Brizard.[12] In the slab and toroidal configurations, our gyrokinetic equation reduces to slightly different forms from those obtained by Artun and Tang.[6,7] It seems to be partly because they did not treat correctly the ballooning representation for the rotating system in which the temporal dependence of the radial wavenumber should be considered.[29] Instead, by using the correct ballooning representation, we see that our result for the toroidal case coincides with Brizard's result.[12] We elucidate which term in the gyrokinetic equation is responsible for the anomalous viscosity (or Reynolds stress). This term is important for rigorously describing the interaction between the background flow and the fluctuations with perpendicular wavelengths on the order of the thermal gyroradius.

Here, we assume that the large flow velocity is approximately balanced with the radial electric field in the same way as in Artun and Tang[6,7] and Brizard.[12] Recently, Hahm[13] presented the gyrokinetic equation that can treat the case where the steep pressure gradient produces the large radial electric field with relatively small flow velocity.

In the present work, we also show the particle, energy and momentum balance equations, as well as the detailed definitions of the anomalous transport fluxes and the anomalous entropy production for the toroidally rotating plasma. This work is an electromagnetic extension of our previous work.[24]

A basic kinetic equation for a turbulent plasma is written as

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\frac{e_a}{m_a}\left\{(\mathbf{E}+\hat{\mathbf{E}})+\frac{1}{c}\mathbf{v}\times(\mathbf{B}+\hat{\mathbf{B}})\right\}\cdot\frac{\partial}{\partial\mathbf{v}}\right](f_a+\hat{f}_a)$$
$$=C_a(f_a+\hat{f}_a),\tag{1}$$

where $C_a\equiv\Sigma_b C_{ab}$ denotes a collision term and the distribution function for species $a$ (the electromagnetic fields) is divided into the ensemble average part $f_a$ ($\mathbf{E}=-\nabla\Phi-c^{-1}\partial\mathbf{A}/\partial t,\mathbf{B}=\nabla\times\mathbf{A}$) and the fluctuating part $\hat{f}_a$ ($\hat{\mathbf{E}}=-\nabla\hat{\phi}-c^{-1}\partial\hat{\mathbf{A}}/\partial t,\hat{\mathbf{B}}=\nabla\times\hat{\mathbf{A}}$). Taking an ensemble average $\langle\cdot\rangle_{\text{ens}}$ of Eq. (1) gives the kinetic equation for $f_a$ as

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\frac{e_a}{m_a}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial\mathbf{v}}\right]f_a=\langle C_a\rangle_{\text{ens}}+\mathcal{D}_a,\tag{2}$$

where the right-hand side consists of the collision term and the fluctuation-particle interaction term $\mathcal{D}_a$ defined by

$$\mathcal{D}_a=-\frac{e_a}{m_a}\left\langle\left(\hat{\mathbf{E}}+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}\right)\cdot\frac{\partial\hat{f}_a}{\partial\mathbf{v}}\right\rangle_{\text{ens}}.\tag{3}$$

Subtracting Eq. (2) from Eq. (1) gives the equation for the $\hat{f}_a$ as

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\frac{e_a}{m_a}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial\mathbf{v}}\right]\hat{f}_a$$
$$=-\frac{e_a}{m_a}\left(\hat{\mathbf{E}}+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}\right)\cdot\frac{\partial(f_a+\hat{f}_a)}{\partial\mathbf{v}}+C_a-\langle C_a\rangle_{\text{ens}}-\mathcal{D}_a.\tag{4}$$

The drift kinetic equation describing the neoclassical transport and the gyrokinetic equation describing the anomalous transport are derived from Eqs. (2) and (4), respectively.

We employ the drift ordering parameter $\delta\equiv\rho_a/L$ [$\rho_a\equiv v_{Ta}/\Omega_a$: the thermal gyroradius, $v_{Ta}\equiv(2T_a/m_a)^{1/2}$: the thermal velocity, $\Omega_a\equiv e_aB/(m_ac)$: the gyrofrequency, $L$: the equilibrium scale length] to expand the distribution functions and the electromagnetic fields as

$$f_a=f_{a0}+f_{a1}+f_{a2}+\cdots,\quad \hat{f}_a=\hat{f}_{a1}+\hat{f}_{a2}+\cdots,$$
$$\mathbf{E}=\mathbf{E}_0+\mathbf{E}_1+\mathbf{E}_2+\cdots,\quad \hat{\mathbf{E}}=\hat{\mathbf{E}}_1+\hat{\mathbf{E}}_2+\cdots,\tag{5}$$
$$\mathbf{B}=\mathbf{B}_0,\quad \hat{\mathbf{B}}=\hat{\mathbf{B}}_1+\hat{\mathbf{B}}_2+\cdots,$$

where the fluctuating quantities are assumed to be $\mathcal{O}(\delta)$ of the ensemble-averaged values. Note that we can put $\mathbf{B}_1=\mathbf{B}_2=\cdots=0$ since $\mathbf{B}$ is used as the basis for defining the expansion parameter $\delta$. For the drift ordering, it is convenient to regard the electric charge $e_a$ (instead of $B$) as the parameter of $\mathcal{O}(\delta^{-1})$: $e_a=e_a^{(-1)}$.[30,31]

Here we allow the large mean flow on the order of the thermal velocity $v_{Ta}$ to exist and the lowest-order flow velocity is denoted by $\mathbf{V}_0[=\mathcal{O}(\delta^0)]$. We introduce the phase variables $(\mathbf{x},w,\mu,\xi)$, in which the particle position $\mathbf{x}$ is observed from the laboratory frame, while the particle kinetic energy $w$, the magnetic moment $\mu$, and the gyrophase $\xi$ are defined in terms of the velocity $\mathbf{v}'\equiv\mathbf{v}-\mathbf{V}_0$ in the moving frame as

$$w=\frac{1}{2}m_a(v')^2,\quad \mu=\frac{m_a(v'_\perp)^2}{2B},\quad \frac{\mathbf{v}'_\perp}{v'_\perp}=\mathbf{e}_1\cos\xi+\mathbf{e}_2\sin\xi,\tag{6}$$

where $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{b}\equiv\mathbf{B}/B)$ are unit vectors which forms a right-handed orthogonal system at each point, and $\mathbf{v}'=v'_\parallel\mathbf{b}+\mathbf{v}'_\perp$ with $v'_\parallel=\mathbf{v}'\cdot\mathbf{b}$.

From the lowest order $[=\mathcal{O}(\delta^{-1})]$ of Eq. (2) [or of Eq. (1)], we obtain

$$\mathbf{E}_0+\frac{1}{c}\mathbf{V}_0\times\mathbf{B}=0\tag{7}$$

and

$$\frac{\partial f_{a0}}{\partial\xi}=0.\tag{8}$$

Thus the lowest-order distribution function $f_{a0}$ is independent of the gyrophase $\xi$. We also assume that the temporal variation of the ensemble-averaged quantities is so slow that the transport ordering $\partial/\partial t=\mathcal{O}(\delta^2)$ is applicable for them. Then, the ensemble-averaged inductive field $\mathbf{E}^{(A)}\equiv-c\,\partial\mathbf{A}/\partial t$ is of $\mathcal{O}(\delta^2)$ while the $\mathcal{O}(\delta^0)$ and $\mathcal{O}(\delta)$ electric fields are electrostatic: $\mathbf{E}_0=-\nabla\Phi_0$, $\mathbf{E}_1=-\nabla\Phi_1$. The lowest-order electrostatic potential is written as $\Phi_{-1}$ in the paper by Hinton and Wong,[32] although it is denoted by $\Phi_0$ in the present work, since we regard the electric charge $e$ (instead of $\Phi$) as the parameter of $\mathcal{O}(\delta^{-1})$.

From the next order $[=\mathcal{O}(\delta^0)]$ of Eq. (2), we have

$$\left(\mathbf{v}\cdot\nabla+\frac{e_a}{m_a}\mathbf{E}_1\cdot\frac{\partial}{\partial\mathbf{v}}\right)f_{a0}-\Omega_a\frac{\partial f_{a1}}{\partial\xi}=C_a(f_{a0}).\tag{9}$$

Taking a gyrophase average $\overline{\cdot}\equiv\oint\cdot\,d\xi/2\pi$ of Eq. (9), we have

$$\bar{\mathcal{L}}_0 f_{a0}\equiv\left[(\mathbf{V}_0+v'_\parallel\mathbf{b})\cdot\nabla+\overline{\left(\frac{dw}{dt}\right)_0}\frac{\partial}{\partial w}+\overline{\left(\frac{d\mu}{dt}\right)_0}\frac{\partial}{\partial\mu}\right]f_{a0}$$
$$=C_a(f_{a0}),\tag{10}$$

where the time derivatives $\overline{(dw/dt)_0}$ and $\overline{(d\mu/dt)_0}$ along the lowest-order guiding center orbit are given by

$$\overline{\left(\frac{dw}{dt}\right)_0} = -m_a \mathbf{V}_0\cdot\nabla\mathbf{V}_0\cdot\mathbf{b}v'_\parallel + e_a\mathbf{E}_1\cdot\mathbf{b}v'_\parallel - m_a(v'_\parallel)^2\mathbf{b}\cdot\nabla\mathbf{V}_0\cdot\mathbf{b} - \frac{1}{2}m_a(v'_\perp)^2 \times(\nabla\cdot\mathbf{V}_0 - \mathbf{b}\cdot\nabla\mathbf{V}_0\cdot\mathbf{b}),$$

$$\overline{\left(\frac{d\mu}{dt}\right)_0} = -\mu\mathbf{V}_0\cdot\nabla\ln B - \mu(\nabla\cdot\mathbf{V}_0 - \mathbf{b}\cdot\nabla\mathbf{V}_0\cdot\mathbf{b}), \tag{11}$$

respectively. Taking a gyrophase average of the $\mathcal{O}(\delta)$ part of Eq. (2) gives the linearized drift kinetic equation, which is solved to derive the neoclassical transport fluxes for rotating plasmas.[32–34]

The rest of this work is organized as follows. In Sec. II, the new nonlinear electromagnetic gyrokinetic equation is derived for plasmas with general magnetic geometries and large flows. The reduced forms of the gyrokinetic equation are given for the cylindrical and slab configurations in Sec. III and for the axisymmetric toroidal configuration in Sec. IV. Definitions of the anomalous transport fluxes and the anomalous entropy production rate are given for the toroidally rotating plasma in Sec. IV. Also found are complete balance equations for particles, energy, and toroidal momentum which include the classical, neoclassical, and anomalous transport processes. In Sec. V, conclusions are given. In Appendix A, the charge neutrality condition and the Ampère's law are given as the constraints on the self-consistent electromagnetic fluctuations, from which the intrinsic ambipolarity for the anomalous particle fluxes is shown. Appendix B is presented for discussing the derivation of the flow shear terms in our gyrokinetic equation. Appendix C shows the Onsager symmetry of the quasilinear anomalous transport matrix, which connects the anomalous fluxes to the conjugate thermodynamic forces.

## II. NONLINEAR ELECTROMAGNETIC GYROKINETIC EQUATION FOR GENERAL GEOMETRY

In the present work, we assume that any fluctuating field $\hat{F}$ is written as a superposition of components in the WKB (or eikonal) form

$$\hat{F}(t,\mathbf{x},w,\mu,\xi) = \sum_{\mathbf{k}_\perp} \hat{F}(t,\mathbf{x},w,\mu,\xi;\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{x},t)], \tag{12}$$

where the eikonal $S_{\mathbf{k}_\perp}(\mathbf{x},t)$ represents the rapid variation in the directions perpendicular to the magnetic field lines. The wavenumber vector is given by $\mathbf{k}_\perp = \nabla S_{\mathbf{k}_\perp}$. The eikonal $S_{\mathbf{k}_\perp}(\mathbf{x},t)$ also contains the rapid Doppler shift frequency due to the large flow, which is given by $-\partial S_{\mathbf{k}_\perp}/\partial t = \mathbf{k}_\perp\cdot\mathbf{V}_0$. Then, we should note that, for the ballooning representation for the system with large sheared flows,[29] the wavenumber vector depends on the time as seen from $\partial\mathbf{k}_\perp/\partial t = \nabla(\partial S_{\mathbf{k}_\perp}/\partial t) = -\nabla(\mathbf{k}_\perp\cdot\mathbf{V}_0)$. The gyrokinetic ordering employed here for the turbulent fluctuations is written in terms of $\delta$ as

$$\frac{\hat{f}_a}{f_a}\sim\frac{e_a\hat{\phi}}{T_a}\sim\frac{e_a v_{T_a}|\hat{\mathbf{A}}|}{cT_a}\sim\frac{k_\parallel}{k_\perp}\sim\frac{(\omega-\mathbf{k}_\perp\cdot\mathbf{V}_0)}{\Omega_a}\sim\delta, \tag{13}$$

where $(\omega-\mathbf{k}_\perp\cdot\mathbf{V}_0)$ denotes the characteristic frequency observed in the moving frame. The characteristic parallel and perpendicular wavenumbers are given by $k_\parallel\sim L^{-1}$ and $k_\perp\sim\rho_a^{-1}$, respectively.

The lowest-order part of Eq. (4) in $\delta$ is written for the fluctuations in the WKB form of Eq. (12) as

$$\begin{aligned}\left(i\mathbf{k}_\perp\cdot\mathbf{v}' - \Omega_a\frac{\partial}{\partial\xi}\right)\hat{f}_{a1}(\mathbf{k}_\perp) &\equiv -\Omega_a e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}\frac{\partial}{\partial\xi}[e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}\hat{f}_{a1}(\mathbf{k}_\perp)]\\ &= ie_a(\mathbf{k}_\perp\cdot\mathbf{v}'_\perp)\left[\left(\hat{\phi}(\mathbf{k}_\perp) - \frac{1}{c}\mathbf{V}_0\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)\right)\times\left(\frac{\partial}{\partial w}+\frac{\partial}{B\partial\mu}\right) - \frac{v'_\parallel}{c}\hat{A}_\parallel(\mathbf{k}_\perp)\frac{\partial}{B\partial\mu}\right]f_{a0},\end{aligned} \tag{14}$$

where $\boldsymbol{\rho}_a\equiv(\mathbf{b}\times\mathbf{v}')/\Omega_a$ represents the gyroradius. Integrating Eq. (14) in $\xi$, we have

$$\hat{f}_{a1}(\mathbf{k}_\perp) = e_a\left[\left(\hat{\phi}(\mathbf{k}_\perp) - \frac{1}{c}\mathbf{V}_0\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)\right)\left(\frac{\partial}{\partial w}+\frac{\partial}{B\partial\mu}\right) - \frac{v'_\parallel}{c}\hat{A}_\parallel(\mathbf{k}_\perp)\frac{\partial}{B\partial\mu}\right]f_{a0} + \hat{g}_a(\mathbf{k}_\perp)e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}, \tag{15}$$

where $\hat{g}_a(\mathbf{k}_\perp)$ is independent of $\xi$.

From the $\mathcal{O}(\delta)$ part of Eq. (4), we have the equation for the second-order fluctuating function $\hat{f}_{a2}$ as

$$-\Omega_a e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}\frac{\partial}{\partial\xi}[e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}\hat{f}_{a2}(\mathbf{k}_\perp)] = -\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\frac{e_a}{m_a}\mathbf{E}_1\cdot\frac{\partial}{\partial\mathbf{v}}\right)\hat{f}_{a1}(\mathbf{k}_\perp)-\frac{e_a}{m_a}\left(\hat{\mathbf{E}}_1(\mathbf{k}_\perp)+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}_1(\mathbf{k}_\perp)\right)$$
$$\cdot\frac{\partial f_{a1}}{\partial\mathbf{v}}-\frac{e_a}{m_a}\left(\hat{\mathbf{E}}_2(\mathbf{k}_\perp)+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}_2(\mathbf{k}_\perp)\right)\cdot\frac{\partial f_{a0}}{\partial\mathbf{v}}-\frac{e_a}{m_a}\sum_{\mathbf{k}'_\perp+\mathbf{k}''_\perp=\mathbf{k}_\perp}$$
$$\times\left(\hat{\mathbf{E}}_1(\mathbf{k}'_\perp)+\frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}_1(\mathbf{k}'_\perp)\right)\cdot\frac{\partial\hat{f}_{a1}(\mathbf{k}''_\perp)}{\partial\mathbf{v}}+C_a^L[\hat{f}_{a1}(\mathbf{k}_\perp)], \tag{16}$$

where $\hat{\mathbf{E}}_1(\mathbf{k}_\perp)=-i\mathbf{k}_\perp\hat{\phi}(\mathbf{k}_\perp)$, $\hat{\mathbf{B}}_1(\mathbf{k}_\perp)=i\mathbf{k}_\perp\times\hat{\mathbf{A}}(\mathbf{k}_\perp)$, $\hat{\mathbf{E}}_2(\mathbf{k}_\perp)=-\nabla\hat{\phi}(\mathbf{k}_\perp)-c^{-1}\partial\hat{\mathbf{A}}(\mathbf{k}_\perp)/\partial t$, and $\hat{\mathbf{B}}_2(\mathbf{k}_\perp)=\nabla\times\hat{\mathbf{A}}(\mathbf{k}_\perp)$. Here $C_a^L$ denotes the linearized collision operator [see Eq. (4.24) in Ref. 17 for its definition]. The solvability condition of Eq. (16) is written as

$$\oint\frac{d\xi}{2\pi}e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}[\text{RHS of Eq. (16)}]=0 \tag{17}$$

which leads to the gyrokinetic equation.

From Eq. (17) with Eqs. (9), (10), (15) and (16), we obtain the nonlinear electromagnetic gyrokinetic equation after lengthy calculation as

$$\left(\frac{\partial}{\partial t}+\overline{\mathcal{L}}_0+i\mathbf{k}_\perp\cdot\mathbf{v}_{da}\right)\hat{h}_a(\mathbf{k}_\perp)-\oint\frac{d\xi}{2\pi}e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}C_a^L[\hat{f}_a(\mathbf{k}_\perp)]$$
$$=e_a\hat{\psi}_a(\mathbf{k}_\perp)\frac{i(\mathbf{b}\times\mathbf{k}_\perp)}{m_a\Omega_a}\cdot\left[-\nabla+\{m_a\mathbf{V}_0\cdot\nabla\mathbf{V}_0+m_a v'_\parallel[\mathbf{b}\cdot(\nabla\mathbf{V}_0)+(\nabla\mathbf{V}_0)\cdot\mathbf{b}]+e_a\nabla\Phi_1\}\frac{\partial}{\partial w}\right]f_{a0}$$
$$-\left[\left\{\frac{\partial}{\partial t}+\overline{\mathcal{L}}_0-v'_\parallel\mathbf{b}\cdot\nabla\right\}(e_a\hat{\psi}_a(\mathbf{k}_\perp))\right]\frac{\partial f_{a0}}{\partial w}-\frac{\partial(e_a\hat{\psi}_a(\mathbf{k}_\perp))}{\partial w}v'_\parallel\mathbf{b}\cdot\nabla f_{a0}-e_a\hat{\chi}_a(\mathbf{k}_\perp)$$
$$\times\left(\nabla\cdot\mathbf{V}_0-\mathbf{b}\cdot\nabla\mathbf{V}_0\cdot\mathbf{b}+\frac{\mathbf{V}_0\cdot\nabla B}{B}\right)\left(\frac{\partial}{\partial w}+\frac{\partial}{B\partial\mu}\right)f_{a0}-e_a\hat{\psi}_a(\mathbf{k}_\perp)\overline{\mathcal{L}}_0\frac{\partial f_{a0}}{\partial w}-J_0(\gamma_a)\frac{e_a}{c}v'_\parallel\hat{A}_\parallel(\mathbf{k}_\perp)\frac{\partial}{\partial w}\overline{\mathcal{L}}_0 f_{a0}$$
$$+J_1(\gamma_a)e_a\frac{v'_\perp}{c}\frac{\hat{B}_\parallel(\mathbf{k}_\perp)}{k_\perp}\left(\frac{\partial}{\partial w}+\frac{\partial}{B\partial\mu}\right)\overline{\mathcal{L}}_0 f_{a0}+\frac{c}{B}\sum_{\mathbf{k}'_\perp+\mathbf{k}''_\perp=\mathbf{k}_\perp}[\mathbf{b}\cdot(\mathbf{k}'_\perp\times\mathbf{k}''_\perp)]\hat{\psi}_a(\mathbf{k}'_\perp)\hat{h}_a(\mathbf{k}''_\perp), \tag{18}$$

where $\hat{h}_a(\mathbf{k}_\perp)$ is independent of $\xi$ and is related to $\hat{f}_a(\mathbf{k}_\perp)$ by

$$\hat{f}_a(\mathbf{k}_\perp)=e_a\left(\hat{\phi}(\mathbf{k}_\perp)-\frac{1}{c}\mathbf{V}_0\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)\right)\times\frac{\partial f_{a0}}{\partial w}+e_a\left[\left(\hat{\phi}(\mathbf{k}_\perp)-\frac{1}{c}\mathbf{V}_0\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)-\frac{1}{c}v'_\parallel\hat{A}_\parallel(\mathbf{k}_\perp)\right)-e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}\hat{\psi}_a(\mathbf{k}_\perp)\right]\times\frac{\partial f_{a0}}{B\partial\mu}+\hat{h}_a(\mathbf{k}_\perp)e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}. \tag{19}$$

In Eqs. (18) and (19), $\hat{\psi}_a(\mathbf{k}_\perp)$ is defined by

$$\hat{\psi}_a(\mathbf{k}_\perp)=\overline{e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}\left[\hat{\phi}(\mathbf{k}_\perp)-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)\right]}$$
$$=J_0(\gamma_a)\left(\hat{\phi}(\mathbf{k}_\perp)-\frac{\mathbf{V}_0}{c}\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)-\frac{v'_\parallel}{c}\hat{A}_\parallel(\mathbf{k}_\perp)\right)+J_1(\gamma_a)\frac{v'_\perp}{c}\frac{\hat{B}_\parallel(\mathbf{k}_\perp)}{k_\perp},$$

$$\hat{\chi}_a(\mathbf{k}_\perp)=\overline{e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}(i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a)\left[\hat{\phi}(\mathbf{k}_\perp)-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)\right]}$$
$$=-\gamma_a J_1(\gamma_a)\left(\hat{\phi}(\mathbf{k}_\perp)-\frac{\mathbf{V}_0}{c}\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)-\frac{v'_\parallel}{c}\hat{A}_\parallel(\mathbf{k}_\perp)\right)+[\gamma_a J_0(\gamma_a)-J_1(\gamma_a)]\frac{v'_\perp}{c}\frac{\hat{B}_\parallel(\mathbf{k}_\perp)}{k_\perp}, \tag{20}$$

where $\hat{A}_\parallel(\mathbf{k}_\perp)\equiv\mathbf{b}\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)$ and $\hat{B}_\parallel(\mathbf{k}_\perp)\equiv i\mathbf{b}\cdot\mathbf{k}_\perp\times\hat{\mathbf{A}}(\mathbf{k}_\perp)$. Here $J_0$ and $J_1$ are the zeroth and first-order Bessel functions of $\gamma_a\equiv k_\perp v'_\perp/\Omega_a$. In Eq. (18), the first-order guiding center drift velocity $\mathbf{v}_{da}$ is defined by

$$\begin{aligned}\mathbf{v}_{da}&\equiv\overline{\frac{d}{dt}\left(\frac{\mathbf{v}'\times\mathbf{b}}{\Omega_a}\right)}\\&=\frac{c\mu}{e_aB}(\nabla\times\mathbf{B})\cdot\mathbf{bb}+\frac{c}{e_aB}\mathbf{b}\times[\mu\nabla B+m_a(v'_\parallel)^2\mathbf{b}\cdot\nabla\mathbf{b}\\&\quad+e_a\nabla\Phi_1+m_a\mathbf{V}_0\cdot\nabla\mathbf{V}_0+m_av'_\parallel\mathbf{b}\cdot\nabla\mathbf{V}_0\\&\quad+m_av'_\parallel\mathbf{V}_0\cdot\nabla\mathbf{b}].\end{aligned}\tag{21}$$

Representation of the gyrokinetic equation in the real $\mathbf{x}$-space is useful. Following Eq. (12), multiplying Eq. (19) by $\exp[iS_{\mathbf{k}_\perp}(\mathbf{x},t)]$ and summing up with respect to $\mathbf{k}_\perp$, we have

$$\begin{aligned}\hat{f}_a(\mathbf{x})&=e_a\left(\hat{\phi}(\mathbf{x})-\frac{1}{c}\mathbf{V}_0\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\frac{\partial f_{a0}}{\partial w}\\&\quad+e_a\left[\left\{\hat{\phi}(\mathbf{x})-\frac{1}{c}(\mathbf{V}_0+v'_\parallel\mathbf{b})\cdot\hat{\mathbf{A}}(\mathbf{x})\right\}-\hat{\psi}_a(\mathbf{X})\right]\\&\quad\times\frac{\partial f_{a0}}{B\partial\mu}+\hat{h}_a(\mathbf{X}),\end{aligned}\tag{22}$$

where $\mathbf{X}\equiv\mathbf{x}-\boldsymbol{\rho}_a$ denotes the position of the guiding center. In deriving Eq. (22), we have used

$$S_{\mathbf{k}_\perp}(\mathbf{x},t)\simeq S_{\mathbf{k}_\perp}(\mathbf{X},t)+i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a,$$

$$\hat{h}_a(\mathbf{X})=\sum_{\mathbf{k}_\perp}\hat{h}_a(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})],$$

$$\begin{aligned}\hat{\psi}_a(\mathbf{X})&=\sum_{\mathbf{k}_\perp}\hat{\psi}_a(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})]\\&=\left\langle\hat{\phi}(\mathbf{X}+\boldsymbol{\rho}_a)-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{X}+\boldsymbol{\rho}_a)\right\rangle_{\mathbf{X}},\end{aligned}\tag{23}$$

where $\langle\cdot\rangle_{\mathbf{X}}$ represents the gyrophase average with $\mathbf{X}$ fixed. Multiplying the gyrokinetic equation (18) in the $\mathbf{k}_\perp$-space by $\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})]$ and summing up with respect to $\mathbf{k}_\perp$ gives the gyrokinetic equation for the nonadiabatic fluctuating distribution function $\hat{h}_a(\mathbf{X},w,\mu)$ as

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\overline{\mathcal{L}}_0+\left(\mathbf{v}_{da}-\frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X})\times\mathbf{b}\right)\cdot\nabla\right]\hat{h}_a(\mathbf{X})-\langle C_a^L[\hat{f}_a(\mathbf{X}+\boldsymbol{\rho}_a)]\rangle_{\mathbf{X}}\\&=\frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X})\times\mathbf{b}\cdot\left[\nabla-\{m_a\mathbf{V}_0\cdot\nabla\mathbf{V}_0+m_av'_\parallel[\mathbf{b}\cdot(\nabla\mathbf{V}_0)+(\nabla\mathbf{V}_0)\cdot\mathbf{b}]+e_a\nabla\Phi_1\}\frac{\partial}{\partial w}\right]f_{a0}-e_a\left[\left(\frac{\partial}{\partial t}+\overline{\mathcal{L}}_0-v'_\parallel\mathbf{b}\cdot\nabla\right)\hat{\psi}_a(\mathbf{X})\right]\\&\quad\times\frac{\partial f_{a0}}{\partial w}-e_a\frac{\partial\hat{\psi}_a(\mathbf{X})}{\partial w}v'_\parallel\mathbf{b}\cdot\nabla f_{a0}-e_a\hat{\chi}_a(\mathbf{X})\left(\nabla\cdot\mathbf{V}_0-\mathbf{b}\cdot\nabla\mathbf{V}_0\cdot\mathbf{b}+\frac{\mathbf{V}_0\cdot\nabla B}{B}\right)\left(\frac{\partial}{\partial w}+\frac{\partial}{B\partial\mu}\right)f_{a0}\\&\quad-e_a\hat{\psi}_a(\mathbf{X})\overline{\mathcal{L}}_0\frac{\partial f_{a0}}{\partial w}-\frac{e_a}{c}v'_\parallel\langle\hat{A}_\parallel(\mathbf{X}+\boldsymbol{\rho}_a)\rangle_{\mathbf{X}}\frac{\partial}{\partial w}\overline{\mathcal{L}}_0f_{a0}-\frac{e_a}{c}\langle\mathbf{v}'_\perp\cdot\hat{\mathbf{A}}(\mathbf{X}+\boldsymbol{\rho}_a)\rangle_{\mathbf{X}}\left(\frac{\partial}{\partial w}+\frac{\partial}{B\partial\mu}\right)\overline{\mathcal{L}}_0f_{a0},\end{aligned}\tag{24}$$

where the spatial gradient is taken with respect to $\mathbf{X}$ as $\nabla=\partial/\partial\mathbf{X}$ and we have used

$$\langle\hat{A}_\parallel(\mathbf{X}+\boldsymbol{\rho}_a)\rangle_{\mathbf{X}}=\sum_{\mathbf{k}_\perp}J_0(\gamma_a)\hat{A}_\parallel(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})],$$

$$\begin{aligned}\langle\mathbf{v}'_\perp\cdot\hat{\mathbf{A}}(\mathbf{X}+\boldsymbol{\rho}_a)\rangle_{\mathbf{X}}&=-\sum_{\mathbf{k}_\perp}J_1(\gamma_a)v'_\perp\frac{\hat{B}_\parallel(\mathbf{k}_\perp)}{k_\perp}\\&\quad\times\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})],\end{aligned}$$

$$\begin{aligned}\hat{\chi}_a(\mathbf{X})&=\sum_{\mathbf{k}_\perp}\hat{\chi}_a(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})]\\&=\left\langle\boldsymbol{\rho}_a\cdot\nabla\left[\hat{\phi}(\mathbf{X}+\boldsymbol{\rho}_a)\right.\right.\\&\quad\left.\left.-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{X}+\boldsymbol{\rho}_a)\right]\right\rangle_{\mathbf{X}}.\end{aligned}\tag{25}$$

We should note in Eq. (24) that differences of equilibrium quantities' values at the particle's position $\mathbf{x}$ from those at the guiding center's position $\mathbf{X}$ are neglected as $\mathcal{O}(\delta)$ smaller $[\mathbf{B}(\mathbf{x})\simeq\mathbf{B}(\mathbf{X})$, $\mathbf{V}_0(\mathbf{x})\simeq\mathbf{V}_0(\mathbf{X})$, and $f_{a0}(\mathbf{x})\simeq f_{a0}(\mathbf{X})]$

although that is not the case for the fluctuating quantities $\hat{\phi}$, $\hat{\mathbf{A}}$, and $\hat{h}_a$ because of small perpendicular wavelengths of $\mathcal{O}(\rho_a)$. In Appendix A, the charge neutrality condition and the Ampère's law are given as the constraints on the self-consistent electromagnetic fluctuations.

## III. CYLINDRICAL AND SLAB CONFIGURATIONS

Let us consider a cylindrical configuration in which the magnetic field and the mean flow velocity are given by

$$\mathbf{B} = B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}},$$

$$\mathbf{V}_0 = V_\theta(r)\hat{\boldsymbol{\theta}} + V_z(r)\hat{\mathbf{z}}, \tag{26}$$

respectively, where the cylindrical coordinates $(r,\theta,z)$ are used and the unit vectors in the $r$, $\theta$, and $z$ directions are denoted by $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\mathbf{z}}$, respectively. Surfaces defined by $r = \text{const}$ are regarded as magnetic flux surfaces. In Eq. (26), $B_\theta(r)$, $B_z(r)$, $V_\theta(r)$, and $V_z(r)$ are flux surface functions independent of $\theta$ and $z$. The lowest-order electric field is given from Eqs. (7) and (26) as $\mathbf{E}_0 = -[d\Phi_0(r)/dr]\hat{\mathbf{r}} = -c^{-1}[V_\theta(r)B_z(r) - V_z(r)B_\theta(r)]\hat{\mathbf{r}}$. The first-order electrostatic potential is also assumed to be a surface function as $\Phi_1 = \Phi_1(r)$. From Eq. (26), we easily find that

$$\mathbf{V}_0 \cdot \nabla B = \nabla \cdot \mathbf{V}_0 = \mathbf{b} \cdot \nabla \mathbf{V}_0 \cdot \mathbf{b} = \mathbf{V}_0 \cdot \nabla \mathbf{V}_0 \cdot \mathbf{b} = 0 \tag{27}$$

from which with Eqs. (10) and (11) we have

$$\overline{\left(\frac{dw}{dt}\right)_0} = \overline{\left(\frac{d\mu}{dt}\right)_0} = 0, \quad \bar{\mathcal{L}}_0 = (\mathbf{V}_0 + v_\parallel' \mathbf{b}) \cdot \nabla. \tag{28}$$

Assuming the lowest-order distribution function to be homogeneous in the $\theta$ and $z$ directions, we obtain from Eq. (10) with Eqs. (26)–(28)

$$\mathbf{V}_0 \cdot \nabla f_{a0} = \mathbf{b} \cdot \nabla f_{a0} = C_a(f_{a0}) = 0. \tag{29}$$

which requires $f_{a0}$ to be the Maxwellian distribution function $f_{a0} = n_a(m_a/2\pi T_a)^{3/2}\exp(-w/T_a)$ with the density $n_a = n_a(r)$ and the temperature $T_a = T_a(r)$. Thus we have $\partial f_{a0}/\partial\mu = 0$.

Using Eqs. (26)–(29), the gyrokinetic equation (18) is simplified for the cylindrical configuration as

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + (\mathbf{V}_0 + v_\parallel' \mathbf{b}) \cdot \nabla + i\mathbf{k}_\perp \cdot \mathbf{v}_{da}\right]\hat{h}_a(\mathbf{k}_\perp) - \oint \frac{d\xi}{2\pi} e^{i\mathbf{k}_\perp \cdot \boldsymbol{\rho}_a} C_a^L[\hat{f}_a(\mathbf{k}_\perp)] \\
&\quad = -e_a \hat{\psi}_a(\mathbf{k}_\perp) \frac{i(\mathbf{b}\times\mathbf{k}_\perp)}{m_a\Omega_a} \cdot \left(\nabla + \frac{e_a}{T_a}\nabla\Phi_1 + \frac{m_a}{T_a}\mathbf{V}_0 \cdot \nabla \mathbf{V}_0 + \frac{m_a v_\parallel'}{T_a}\{\mathbf{b}\cdot(\nabla\mathbf{V}_0) + (\nabla\mathbf{V}_0)\cdot\mathbf{b}\}\right) f_{a0} \\
&\qquad + \frac{e_a}{T_a}\left[\left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla\right)\hat{\psi}_a(\mathbf{k}_\perp)\right] f_{a0} + \frac{c}{B}\sum_{\mathbf{k}_\perp' + \mathbf{k}_\perp'' = \mathbf{k}_\perp} [\mathbf{b}\cdot(\mathbf{k}_\perp' \times \mathbf{k}_\perp'')]\hat{\psi}_a(\mathbf{k}_\perp')\hat{h}_a(\mathbf{k}_\perp''),
\end{aligned} \tag{30}$$

where $\hat{h}_a(\mathbf{k}_\perp)$ is related to $\hat{f}_a(\mathbf{k}_\perp)$ by

$$\hat{f}_a(\mathbf{k}_\perp) = -\frac{e_a}{T_a}\left(\hat{\phi}(\mathbf{k}_\perp) - \frac{1}{c}\mathbf{V}_0 \cdot \hat{\mathbf{A}}(\mathbf{k}_\perp)\right) f_{a0} + \hat{h}_a(\mathbf{k}_\perp) e^{-i\mathbf{k}_\perp \cdot \boldsymbol{\rho}_a}. \tag{31}$$

The gyrokinetic equation for the $\mathbf{X}$-space is obtained from Eq. (30) by applying the same procedure as in deriving Eq. (24) [or directly from Eq. (24) with using Eqs. (26)–(29)]

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \left(\mathbf{V}_0 + v_\parallel' \mathbf{b} + \mathbf{v}_{da} - \frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X}) \times \mathbf{b}\right) \cdot \nabla\right]\hat{h}_a(\mathbf{X}) - \langle C_a^L[\hat{f}_a(\mathbf{X} + \boldsymbol{\rho}_a)]\rangle_{\mathbf{X}} \\
&\quad = \frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X}) \times \mathbf{b} \cdot \left[\nabla + \frac{e_a}{T_a}\nabla\Phi_1 + \frac{m_a}{T_a}\mathbf{V}_0 \cdot \nabla\mathbf{V}_0 + \frac{m_a v_\parallel'}{T_a}\{\mathbf{b}\cdot(\nabla\mathbf{V}_0) + (\nabla\mathbf{V}_0)\cdot\mathbf{b}\}\right] f_{a0} \\
&\qquad + \frac{e_a}{T_a}\left[\left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla\right)\hat{\psi}_a(\mathbf{X})\right] f_{a0},
\end{aligned} \tag{32}$$

where $\hat{h}_a(\mathbf{X})$ is related to $\hat{f}_a(\mathbf{x} = \mathbf{X} + \boldsymbol{\rho}_a)$ by

$$\hat{f}_a(\mathbf{x}) = -\frac{e_a}{T_a}\left(\hat{\phi}(\mathbf{x}) - \frac{1}{c}\mathbf{V}_0\cdot\hat{\mathbf{A}}(\mathbf{x})\right)f_{a0} + \hat{h}_a(\mathbf{X}). \tag{33}$$

In a similar way as in the case of the cylindrical configuration, we can consider a slab configuration in which the magnetic field and the mean flow velocity are given by

$$\mathbf{B} = B_y(x)\hat{\mathbf{y}} + B_z(x)\hat{\mathbf{z}}, \quad \mathbf{V}_0 = V_y(x)\hat{\mathbf{y}} + V_z(x)\hat{\mathbf{z}}, \tag{34}$$

respectively, where the Cartesian coordinates $(x,y,z)$ are used and the unit vectors in the $x$, $y$, and $z$ directions are denoted by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, respectively. Planes defined by $x = \text{const}$ are regarded as magnetic flux surfaces. In Eq. (34), $B_y(x)$, $B_z(x)$,$V_y(x)$, and $V_z(x)$ are flux surface functions independent of $y$ and $z$. The lowest-order electric field is given from Eqs. (7) and (34) as $\mathbf{E}_0 = -[d\Phi_0(x)/dx]\hat{\mathbf{x}} = -c^{-1}[V_y(x)B_z(x) - V_z(x)B_y(x)]\hat{\mathbf{x}}$. The first-order electrostatic potential is also assumed to be a surface function as $\Phi_1 = \Phi_1(x)$. Then, for the slab configuration, Eqs. (27)–(29) are still valid and $f_{a0}$ is the Maxwellian distribution function $f_{a0} = n_a(m_a/2\pi T_a)^{3/2}\exp(-w/T_a)$ with the density $n_a = n_a(x)$ and the temperature $T_a = T_a(x)$. Thus we find that the gyrokinetic equation for the slab configuration is also given by Eq. (30) [or Eq. (32)] with Eq. (31) [or Eq. (33)] and $\mathbf{V}_0\cdot\nabla\mathbf{V}_0 = \mathbf{b}\cdot\nabla\mathbf{V}_0 = 0$. In Eq. (30), the flow shear term with $(\nabla\mathbf{V}_0)\cdot\mathbf{b}$ survives in the slab limit, in which $dV_y(x)/dx$ and $dV_z(x)/dx$ are contained.

We see from Appendix B that, in Eq. (30) [or in Eq. (32)], the term with $(\partial/\partial t + \mathbf{V}_0\cdot\nabla)\hat{\psi}_a$ also contains the flow shear $d(r^{-1}V_\theta)/dr$ and $dV_z/dr$ for the cylindrical case and $dV_y/dx$ and $dV_z/dx$ for the slab case [see Eqs. (B1)–(B4)]. These flow shear terms were missed by Artun and Tang[6] when they derived the linearized gyrokinetic equation describing electrostatic fluctuations in the slab system with sheared equilibrium flows [see Eq. (30) in Ref. 6 and Appendix B]. Compared to their gyrokinetic equation, our gyrokinetic equation (30) [or (32)] is not only nonlinear and electromagnetic, but also contains even in the linear electrostatic limit these new flow shear terms, which are deeply related to the anomalous viscosity (or Reynolds stress).

It is instructive to derive the Hasegawa-Mima equation[35] from the gyrokinetic equation (30) and see how the flow shear effects enter it. Following the same procedure as in Frieman and Chen,[5] we consider a collisionless plasma consisting of adiabatic electrons $(\hat{n}_e/n_e \simeq e\hat{\phi}/T_e)$ with $T_e = \text{const}$ and a single species of ions with charge $e_i \equiv Z_i e$ and low temperature $T_i \ll T_e$ $(k_\perp\rho_i < 1)$, and assume that the ion nonadiabatic distribution function has the form $\hat{h}_i(\mathbf{X}) \simeq f_{i0}\hat{n}_i^{(\text{nad})}(\mathbf{X})/n_i$. Then, from the electrostatic version of the gyrokinetic equation (30) [or (32)] for ions with the charge neutrality condition $\hat{n}_e = Z_i\hat{n}_i$, we obtain the generalized Hasegawa-Mima equation in the fluid limit as

$$\begin{aligned}
&\left[\frac{\partial}{\partial t_0} + \mathbf{V}_0\cdot\nabla + i\mathbf{k}_\perp\cdot\frac{c}{Z_i eB}\mathbf{b}\times(Z_i eB\nabla\Phi_1 + m_i\mathbf{V}_0\cdot\nabla\mathbf{V}_0)\right]\\
&\quad\times\left[(1 + k_\perp^2\rho_s^2)\frac{e\hat{\phi}(\mathbf{k}_\perp)}{T_e}\right] + i\mathbf{k}_\perp\cdot\frac{cT_e}{eB}\\
&\quad\times\left[-\mathbf{b}\times\nabla\ln n_e + B\nabla\times\left(\frac{\mathbf{b}}{B}\right)\right]\frac{e\hat{\phi}(\mathbf{k}_\perp)}{T_e}\\
&= \frac{1}{2}c_s\rho_s^3\sum_{\mathbf{k}_\perp' + \mathbf{k}_\perp'' = \mathbf{k}_\perp}[\mathbf{b}\cdot(\mathbf{k}_\perp'\times\mathbf{k}_\perp'')][(k_\perp'')^2\\
&\quad - (k_\perp')^2]\frac{e\hat{\phi}_a(\mathbf{k}_\perp')}{T_e}\,\frac{e\hat{\phi}_a(\mathbf{k}_\perp'')}{T_e},
\end{aligned} \tag{35}$$

where $c_s \equiv (Z_i T_e/m_i)^{1/2}$and $\rho_s \equiv c_s/\Omega_i$. Equation (35) is valid for the both slab and cylindrical configurations, and reduces to the Hasegawa-Mima equation derived by Frieman and Chen[5] in the limit of $\mathbf{V}_0 \to 0$. The energy balance equation is obtained from Eq. (35) as

$$\begin{aligned}
&\left(\frac{\partial}{\partial t_0} + \mathbf{V}_0\cdot\nabla\right)\sum_{\mathbf{k}_\perp}\left(\frac{1}{2}n_i m_i\langle|\hat{\mathbf{v}}_E(\mathbf{k}_\perp)|^2\rangle_{\text{ens}}\right.\\
&\quad\left. + \frac{1}{2}\frac{n_e e^2}{T_e}\langle|\hat{\phi}(\mathbf{k}_\perp)|^2\rangle_{\text{ens}}\right)\\
&\quad = -\sum_{\mathbf{k}_\perp} n_i m_i\langle\hat{\mathbf{v}}_E^*(\mathbf{k}_\perp)\hat{\mathbf{v}}_E(\mathbf{k}_\perp)\rangle_{\text{ens}} : (\nabla\mathbf{V}_0),
\end{aligned} \tag{36}$$

where $\hat{\mathbf{v}}_E(\mathbf{k}_\perp) \equiv -i(c/B)\hat{\phi}(\mathbf{k}_\perp)\mathbf{k}_\perp\times\mathbf{b}$ is the $\hat{\mathbf{E}}\times\mathbf{B}$ drift velocity due to the electrostatic fluctuations. Equations (35) and (36) are rewritten in the $\mathbf{x}$-space as

$$\begin{aligned}
&\left[\frac{\partial}{\partial t_0} + \left\{\mathbf{V}_0 + \frac{c}{Z_i eB}\mathbf{b}\times(Z_i eB\nabla\Phi_1 + m_i\mathbf{V}_0\cdot\nabla\mathbf{V}_0)\right\}\cdot\nabla\right]\\
&\quad\times\left[(1 - \rho_s^2\nabla_\perp^2)\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)\right] + \frac{cT_e}{eB}\left[-\mathbf{b}\times\nabla\ln n_e + B\nabla\right.\\
&\quad\left.\times\left(\frac{\mathbf{b}}{B}\right)\right]\cdot\nabla_\perp\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)\\
&= c_s\rho_s^3\mathbf{b}\times\nabla\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)\cdot\nabla\nabla_\perp^2\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla\right)\left(\frac{1}{2}n_i m_i\langle|\hat{\mathbf{v}}_E(\mathbf{x})|^2\rangle_{\text{ens}} + \frac{1}{2}\frac{n_e e^2}{T_e}\langle|\hat{\phi}(\mathbf{x})|^2\rangle_{\text{ens}}\right)\\
&\quad = -n_i m_i\langle\hat{\mathbf{v}}_E(\mathbf{x})\hat{\mathbf{v}}_E(\mathbf{x})\rangle_{\text{ens}} : (\nabla\mathbf{V}_0)
\end{aligned} \tag{38}$$

respectively, where $\nabla_\perp \equiv \nabla - \mathbf{bb}\cdot\nabla$ and $\hat{\mathbf{v}}_E(\mathbf{x}) \equiv -(c/B)\nabla\hat{\phi}(\mathbf{x})\times\mathbf{b}$. The right-hand side of Eq. (36) [or (38)] represents the energy transfer from the background sheared flow to the fluctuations through the Reynolds stress multiplied by the flow shear.

The reaction of the divergence of the Reynolds stress on the flow profile $\mathbf{V}_0$ is given by

$$\frac{\partial}{\partial t}(n_i m_i \mathbf{V}_0) = -\nabla\cdot(n_i m_i \langle \hat{\mathbf{v}}_E \hat{\mathbf{v}}_E\rangle_{\rm ens}) + \cdots. \tag{39}$$

Thus the change in the kinetic energy in the flow profile

$$\frac{\partial}{\partial t}\int \frac{1}{2} n_i m_i V_0^2 \, d^3x = \int n_i m_i \langle \hat{\mathbf{v}}_E \hat{\mathbf{v}}_E\rangle_{\rm ens} : \nabla \mathbf{V}_0 \, d^3x + \cdots \tag{40}$$

is equal and opposite to the change in the total energy in the turbulent fluctuations in the absence of sources and dissipations. The direction of the energy flow between the shear flow and the turbulent fluctuations depends on the phase relations, or equivalently, the tilting of the vortices.[36] When the turbulence is driven by the ion temperature gradient, the shear flow is typically generated. Now we consider how these processes appear in the axisymmetric torus where the zeroth order flow must be toroidal.

## IV. AXISYMMETRIC TOROIDAL CONFIGURATION

In this section, we consider an axisymmetric system, for which the magnetic field is given by

$$\mathbf{B} = I(\Psi)\nabla\zeta + \nabla\zeta\times\nabla\Psi, \tag{41}$$

where $\zeta$ is the toroidal angle, $\Psi$ represents the poloidal flux, and $I(\Psi) = RB_T$ denotes the covariant toroidal component of the magnetic field. Hinton and Wong[32] showed that, in the axisymmetric systems, the poloidal flow decays in a few transit or collision times and that the lowest-order flow velocity $\mathbf{V}_0$ is in the toroidal direction and is derived from $\mathbf{E}_0 + \mathbf{V}_0\times\mathbf{B}/c = 0$ as

$$\mathbf{V}_0 = V_0\hat{\boldsymbol{\zeta}}, \quad V_0 = RV^\zeta = -Rc\frac{\partial\Phi_0(\Psi)}{\partial\Psi}, \tag{42}$$

where the lowest-order electrostatic potential $\Phi_0(\Psi)$ is a flux-surface function and $\mathbf{E}_0 = -\nabla\Phi_0 = -(\partial\Phi_0/\partial\Psi)\nabla\Psi$. We should note that the toroidal angular velocity $V^\zeta = -c\,\partial\Phi_0/\partial\Psi$ is directly given by the radial electric field and is also a flux-surface quantity. Then, we easily find that $\mathbf{V}_0\cdot\nabla B = \nabla\cdot\mathbf{V}_0 = \mathbf{b}\cdot\nabla\mathbf{V}_0\cdot\mathbf{b} = 0$ and $\mathbf{V}_0\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{V}_0$. Here it is convenient to use independent phase space variables $(\mathbf{x},\varepsilon,\mu,\xi)$ instead of $(\mathbf{x},w,\mu,\xi)$ where the new energy variable $\varepsilon$ is defined by

$$\varepsilon = \tfrac{1}{2}m_a(v')^2 + \Xi_a, \quad \Xi_a \equiv e_a\tilde{\Phi}_1 - \tfrac{1}{2}m_a V_0^2. \tag{43}$$

In Eq. (43), $\tilde{\Phi}_1 \equiv \Phi_1 - \langle\Phi_1\rangle [=\mathcal{O}(\delta)]$ is the poloidal-angle-dependent part of the electrostatic potential and $-\frac{1}{2}m_a V_0^2$ represents the potential for the centrifugal force due to the toroidal rotation. The magnetic flux surface average is denoted by $\langle\cdot\rangle$. It is shown that $\varepsilon$ and $\mu$ are conserved along the lowest-order guiding center orbit: $\overline{(d\varepsilon/dt)_0} = \overline{(d\mu/dt)_0} = 0$. Thus we have $\overline{\mathcal{L}}_0 = (\mathbf{V}_0 + v_\parallel\mathbf{b})\cdot\nabla$ with the independent variables $(\mathbf{x},\varepsilon,\mu,\xi)$.

The lowest-order distribution function $f_{a0}$ is written in the Maxwellian form

$$\begin{aligned} f_{a0} &= n_a\left(\frac{m_a}{2\pi T_a}\right)^{3/2}\exp\left(-\frac{m_a(v')^2}{2T_a}\right) \\ &= N_a\left(\frac{m_a}{2\pi T_a}\right)^{3/2}\exp\left(-\frac{\varepsilon}{T_a}\right) \end{aligned} \tag{44}$$

which satisfies $\overline{\mathcal{L}}_0 f_{a0} = C_a(f_{a0}) = 0$. Here the temperature $T_a = T_a(\Psi)$ and $N_a = N_a(\Psi)$ are flux-surface functions, although generally the density $n_a$ depends on the poloidal angle $\theta$ through $\Xi_a$ and is given by $n_a = N_a\exp(-\Xi_a/T_a)$. The charge neutrality $\Sigma_a e_a n_a = 0$ imposes the constraints on $\tilde{\Phi}_1$ and $N_a$.

### A. Gyrokinetic equation

Using Eqs. (41)–(44) with the independent phase-space variables $(\mathbf{x},\varepsilon,\mu,\xi)$, we find that, for the axisymmetric configuration, the gyrokinetic equation (18) simplifies to

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + (\mathbf{V}_0 + v'_\parallel\mathbf{b})\cdot\nabla + i\mathbf{k}_\perp\cdot\mathbf{v}_{da}\right]\hat{h}_a(\mathbf{k}_\perp) - \frac{c}{B}\sum_{\mathbf{k}'_\perp+\mathbf{k}''_\perp=\mathbf{k}_\perp}[\mathbf{b}\cdot(\mathbf{k}'_\perp \\ &\quad\times\mathbf{k}''_\perp)]\hat{\psi}_a(\mathbf{k}'_\perp)\hat{h}_a(\mathbf{k}''_\perp) - \oint\frac{d\xi}{2\pi}e^{i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_a}C_a^L[\hat{f}_a(\mathbf{k}_\perp)] \\ &= -e_a\hat{\psi}_a(\mathbf{k}_\perp)\frac{i(\mathbf{b}\times\mathbf{k}_\perp)}{m_a\Omega_a}\cdot\left[\nabla + \left\{\frac{e_a}{T_a}\frac{\partial\langle\Phi_1\rangle}{\partial\Psi} + \frac{m_a}{T_a}\left(R^2V^\zeta\right.\right.\right. \\ &\quad\left.\left.\left.+\frac{I}{B}v'_\parallel\right)\frac{\partial V^\zeta}{\partial\Psi}\right\}\nabla\Psi\right]f_{a0} + \frac{e_a}{T_a}\left[\left(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla\right)\hat{\psi}_a(\mathbf{k}_\perp)\right]f_{a0} \\ &= f_{a0}[\hat{w}_{a1}(\mathbf{k}_\perp)X^A_{a1} + \hat{w}_{a2}(\mathbf{k}_\perp)X^A_{a2} + \hat{w}_{aV}(\mathbf{k}_\perp)X^A_{aV} \\ &\quad + \hat{w}_{aT}(\mathbf{k}_\perp)X^A_{aT}], \end{aligned} \tag{45}$$

which is written by the $\mathbf{X}$-space representation as

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \left(\mathbf{V}_0 + v'_\parallel\mathbf{b} + \mathbf{v}_{da} - \frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X})\times\mathbf{b}\right)\cdot\nabla\right]\hat{h}_a(\mathbf{X}) \\ &\quad - \langle C_a^L[\hat{f}_a(\mathbf{X}+\boldsymbol{\rho}_a)]\rangle_{\mathbf{X}} \\ &= \frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X})\times\mathbf{b}\cdot\left[\nabla + \left\{\frac{e_a}{T_a}\frac{\partial\langle\Phi_1\rangle}{\partial\Psi} + \frac{m_a}{T_a}\left(R^2V^\zeta\right.\right.\right. \\ &\quad\left.\left.\left.+\frac{I}{B}v'_\parallel\right)\frac{\partial V^\zeta}{\partial\Psi}\right\}\nabla\Psi\right]f_{a0} + \frac{e_a}{T_a}\left[\left(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla\right)\hat{\psi}_a(\mathbf{X})\right]f_{a0} \\ &= f_{a0}[\hat{w}_{a1}(\mathbf{X})X^A_{a1} + \hat{w}_{a2}(\mathbf{X})X^A_{a2} + \hat{w}_{aV}(\mathbf{X})X^A_{aV} \\ &\quad + \hat{w}_{aT}(\mathbf{X})X^A_{aT}]. \end{aligned} \tag{46}$$

The nonadiabatic part of the fluctuation distribution function $\hat{h}_a(\mathbf{k}_\perp)$ in Eq. (45) $[\hat{h}_a(\mathbf{X})$ in Eq. (46)] is related to $\hat{f}_{a1}(\mathbf{k}_\perp)$ $[\hat{f}_{a1}(\mathbf{x})]$ by Eq. (31) [Eq. (33)].

In the right-hand side of Eqs. (45) and (46), we have defined the thermodynamic forces $(X^A_{a1}, X^A_{a2}, X^A_{aV}, X^A_{aT})$ as

$$X_{a1}^A \equiv -\frac{\partial \ln(N_a T_a)}{\partial \Psi} - \frac{e_a}{T_a}\frac{\partial \langle \Phi_1 \rangle}{\partial \Psi}, \quad X_{a2}^A \equiv -\frac{\partial \ln T_a}{\partial \Psi},$$

$$X_{aV}^A \equiv -\frac{1}{T_a}\frac{\partial V^\zeta}{\partial \Psi} = \frac{c}{T_a}\frac{\partial^2 \Phi_0}{\partial \Psi^2}, \quad X_{aT}^A \equiv \frac{1}{T_a}, \tag{47}$$

and the fluctuating functions $(\hat{w}_{a1}, \hat{w}_{a2}, \hat{w}_{aV}, \hat{w}_{aT})$ as

$$\begin{aligned}
\hat{w}_{a1}(\mathbf{k}_\perp) &\equiv ic\mathbf{k}_\perp \cdot (R\hat{\boldsymbol{\zeta}})\hat{\psi}_a(\mathbf{k}_\perp), \\
\hat{w}_{a2}(\mathbf{k}_\perp) &\equiv ic\mathbf{k}_\perp \cdot (R\hat{\boldsymbol{\zeta}})\hat{\psi}_a(\mathbf{k}_\perp)\left(\frac{\varepsilon}{T_a} - \frac{5}{2}\right), \\
\hat{w}_{aV}(\mathbf{k}_\perp) &\equiv m_a c\left(R^2 V^\zeta + \frac{I}{B}v'_\parallel\right) i\mathbf{k}_\perp \cdot (R\hat{\boldsymbol{\zeta}})\hat{\psi}_a(\mathbf{k}_\perp) \\
&\quad + e_a \frac{1}{k_\perp^2}(\mathbf{k}_\perp\mathbf{k}_\perp):(R\hat{\boldsymbol{\zeta}})(\nabla\Psi)\hat{\chi}_a(\mathbf{k}_\perp), \\
\hat{w}_{aT}(\mathbf{k}_\perp) &\equiv e_a J_0(\gamma_a)\left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla\right)\left(\hat{\phi}(\mathbf{k}_\perp) - \frac{\mathbf{V}_0}{c}\cdot \hat{\mathbf{A}}(\mathbf{k}_\perp)\right. \\
&\quad \left. - \frac{v'_\parallel}{c}\hat{A}_\parallel(\mathbf{k}_\perp)\right) + e_a J_1(\gamma_a)\frac{v'_\perp}{c} \\
&\quad \times \left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla\right)\left(\frac{\hat{B}_\parallel(\mathbf{k}_\perp)}{k_\perp}\right),
\end{aligned} \tag{48}$$

which are written by the $\mathbf{X}$-space representation as

$$\begin{aligned}
\hat{w}_{a1}(\mathbf{X}) &\equiv \sum_{\mathbf{k}_\perp} \hat{w}_{a1}(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})] \\
&= -\frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X}) \times \mathbf{b}\cdot\nabla\Psi, \\
\hat{w}_{a2}(\mathbf{X}) &\equiv \sum_{\mathbf{k}_\perp} \hat{w}_{a2}(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})] \\
&= -\frac{c}{B}\nabla\hat{\psi}_a(\mathbf{X}) \times \mathbf{b}\cdot\nabla\Psi\left(\frac{\varepsilon}{T_a} - \frac{5}{2}\right), \\
\hat{w}_{aV}(\mathbf{X}) &\equiv \sum_{\mathbf{k}_\perp} \hat{w}_{aV}(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})] \\
&= \left\langle -\frac{c}{B}\nabla\left(\hat{\phi}(\mathbf{x}) - \frac{1}{c}(\mathbf{V}_0 + \mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\right. \\
&\quad \left. \times \mathbf{b}\cdot\nabla\Psi m_a(\mathbf{V}_0 + \mathbf{v}')\cdot(R\hat{\boldsymbol{\zeta}})\right\rangle_{\mathbf{X}}, \\
\hat{w}_{aT}(\mathbf{X}) &\equiv \sum_{\mathbf{k}_\perp} \hat{w}_{aT}(\mathbf{k}_\perp)\exp[iS_{\mathbf{k}_\perp}(\mathbf{X})] \\
&= e_a\left\langle \left(\frac{\partial}{\partial t} + \mathbf{V}_0(\mathbf{x})\cdot\frac{\partial}{\partial \mathbf{x}}\right)\right. \\
&\quad \left. \times \left(\hat{\phi}(\mathbf{x}) - \frac{1}{c}(\mathbf{V}_0 + \mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\right\rangle_{\mathbf{X}}.
\end{aligned} \tag{49}$$

In deriving Eqs. (48) and (49), we have used Eqs. (B1) and (B5) in Appendix B and $\mathbf{k}_\perp \cdot (R\hat{\boldsymbol{\zeta}}) = -B^{-1}(\mathbf{k}_\perp \times \mathbf{b})\cdot\nabla\Psi$.

The gyrokinetic equation (45) [or (46)] is slightly different from that derived by Artun and Tang[7] for the toroidally rotating plasma. Their equation (51) [or (56)] contains a different magnetic fluctuation term from ours and they do not seem to treat correctly the ballooning representation for the rotating system in which the temporal dependence of the radial wavenumber should be considered.[29] Interestingly, by the ballooning representation, we see that our gyrokinetic equation (45) [or (46)] coincides with Brizard's result[12] obtained by the Hamiltonian method for the collisionless case [see Eq. (C8) in Ref. 12]. The use of the correct ballooning representation is crucial for obtaining the formulas in Appendix B and the expression for $\hat{w}_{aV}$ which is deeply related to the definition of the anomalous momentum transport (or viscosity) as shown later.

### B. Entropy production by anomalous transport

In the same way as in Refs. 23 and 24, the contribution from the turbulent fluctuations to the entropy balance is represented by

$$\begin{aligned}
\langle \dot{S}_a^A \rangle &\equiv -\left\langle \int d^3v(\ln \overline{f}_a + 1)(\overline{\mathcal{D}}_a - \overline{\mathcal{L}\tilde{f}_a^A})\right\rangle \\
&= -\frac{1}{V'}\frac{\partial}{\partial \Psi}(V' J_{Sa}^A) + \langle \sigma_a^A \rangle,
\end{aligned} \tag{50}$$

where the surface-averaged radial anomalous entropy flux is given by

$$J_{Sa}^A = \left(\frac{S_a}{n_a} - \frac{\Xi_a}{T_a}\right)\Gamma_a^A + \frac{q_a^A}{T_a} \tag{51}$$

and the surface-averaged anomalous entropy production rate is written in the thermodynamic form as

$$\langle \sigma_a^A \rangle = \Gamma_a^A X_{a1}^A + \frac{1}{T_a}q_a^A X_{a2}^A + \Pi_a^A X_{aV}^A + Q_a^A X_{aT}. \tag{52}$$

The anomalous fluxes $(\Gamma_a^A, q_a^A/T_a, \Pi_a^A, Q_a^A)$ conjugate to the forces $(X_{a1}^A, X_{a2}^A, X_{aV}^A, X_{aT}^A)$ are given by the correlations between $\hat{h}_a$ and $(\hat{w}_{a1}, \hat{w}_{a2}, \hat{w}_{aV}, \hat{w}_{aT})$ as

$$\begin{aligned}
\Gamma_a^A &\equiv \left\langle\left\langle \int d^3v \sum_{\mathbf{k}_\perp} \hat{h}_a^*(\mathbf{k}_\perp)\hat{w}_{a1}(\mathbf{k}_\perp)\right\rangle\right\rangle \\
&= \left\langle\left\langle \int d^3v\ \hat{h}_a(\mathbf{X})\hat{w}_{a1}(\mathbf{X})\right\rangle\right\rangle, \\
\frac{q_a^A}{T_a} &\equiv \left\langle\left\langle \int d^3v \sum_{\mathbf{k}_\perp} \hat{h}_a^*(\mathbf{k}_\perp)\hat{w}_{a2}(\mathbf{k}_\perp)\right\rangle\right\rangle \\
&= \left\langle\left\langle \int d^3v\ \hat{h}_a(\mathbf{X})\hat{w}_{a2}(\mathbf{X})\right\rangle\right\rangle, \\
\Pi_a^A &\equiv \left\langle\left\langle \int d^3v \sum_{\mathbf{k}_\perp} \hat{h}_a^*(\mathbf{k}_\perp)\hat{w}_{aV}(\mathbf{k}_\perp)\right\rangle\right\rangle \\
&= \left\langle\left\langle \int d^3v\ \hat{h}_a(\mathbf{X})\hat{w}_{aV}(\mathbf{X})\right\rangle\right\rangle,
\end{aligned} \tag{53}$$

$$Q_a^A \equiv \left\langle \left\langle \int d^3v \sum_{\mathbf{k}_\perp} \hat{h}_a^*(\mathbf{k}_\perp) \hat{w}_{aT}(\mathbf{k}_\perp) \right\rangle \right\rangle$$
$$= \left\langle \left\langle \int d^3v \hat{h}_a(\mathbf{X}) \hat{w}_{aT}(\mathbf{X}) \right\rangle \right\rangle,$$

where $\langle\langle \cdot \rangle\rangle$ denotes a double average over the magnetic surface and the ensemble.

From the gyrokinetic equation (45), we obtain the balance equation for the fluctuation amplitude as

$$\frac{\partial}{\partial t}\left\langle \left\langle \int d^3v \frac{1}{2f_{a0}} \sum_{\mathbf{k}_\perp} \left| \hat{h}_a(\mathbf{k}_\perp) \right|^2 \right\rangle \right\rangle - \left\langle \left\langle \int d^3v \frac{1}{f_{a0}} \times \sum_{\mathbf{k}_\perp} \hat{h}_a^*(\mathbf{k}_\perp) e^{i\mathbf{k}_\perp \cdot \boldsymbol{\rho}_a} C_a^L[\hat{h}_a(\mathbf{k}_\perp) e^{-i\mathbf{k}_\perp \cdot \boldsymbol{\rho}_a}] \right\rangle \right\rangle$$
$$= \frac{\partial}{\partial t}\left\langle \left\langle \int d^3v \frac{1}{2f_{a0}} \hat{h}_a(\mathbf{X})^2 \right\rangle \right\rangle - \left\langle \left\langle \int d^3v \frac{1}{f_{a0}} \hat{h}_a(\mathbf{X}) C_a^L[\hat{h}_a(\mathbf{X})] \right\rangle \right\rangle = \langle \sigma_a^A \rangle. \tag{54}$$

Thus in the stationary turbulent states, the anomalous entropy production driven by the turbulent transport equals the collisional dissipation of the fluctuating distribution function, which results in the positive definiteness of the total anomalous entropy production: $\Sigma_a T_a \langle \sigma_a^A \rangle = -\Sigma_a T_a \langle\langle \int d^3v (1/f_{a0}) \times \hat{h}_a(\mathbf{X}) C_a^L[\hat{h}_a(\mathbf{X})] \rangle\rangle \geqslant 0$. The symmetry for the quasilinear anomalous transport matrix relating the anomalous fluxes $(\Gamma_a^A, q_a^A/T_a, \Pi_a^A, Q_a^A)$ to the conjugate forces $(X_{a1}^A, X_{a2}^A, X_{aV}^A, X_{aT}^A)$ is described in Appendix C.

### C. Balance equations for particles, energy, and toroidal momentum

Taking the velocity moment and the magnetic surface average of Eq. (2), we obtain the particle density equation:

$$\frac{\partial \langle n_a \rangle}{\partial t} + \frac{1}{V'} \frac{\partial}{\partial \Psi}(V' \Gamma_a) = \left\langle \int d^3v\ \mathcal{I}_a \right\rangle; \tag{55}$$

the energy balance equation:

$$\frac{3}{2} \frac{\partial}{\partial t} \langle p_a \rangle + \frac{1}{V'} \frac{\partial}{\partial \Psi}\left[ V' \left( q_a + \frac{5}{2} T_a \Gamma_a \right) \right]$$
$$= -\Pi_a \frac{\partial V^\zeta}{\partial \Psi} - e_a \Gamma_a \frac{\partial \langle \Phi_1 \rangle}{\partial \Psi} + \langle n_a e_a \mathbf{u}_{a1} \cdot \mathbf{E}^{(A)} \rangle - e_a \left\langle \tilde{\Phi}_1 \frac{\partial n_a}{\partial t} \right\rangle + \frac{1}{2} (V^\zeta)^2 \frac{\partial}{\partial t} \langle m_a n_a R^2 \rangle + \left\langle \int d^3v\ \varepsilon (C_a + \mathcal{D}_a + \mathcal{I}_a) \right\rangle; \tag{56}$$

the toroidal momentum balance equation:

$$\frac{\partial}{\partial t}\left\langle \left( \sum_a m_a n_a \right) \left( 1 + \frac{v_{PA}^2}{c^2} \right) R^2 V^\zeta \right\rangle + \frac{1}{V'} \frac{\partial}{\partial \Psi}\left( V' \sum_a \Pi_a \right)$$
$$= \sum_a \left\langle \int d^3v\ m_a v_\zeta (\mathcal{D}_a + \mathcal{I}_a) \right\rangle, \tag{57}$$

where $v_{PA} \equiv B_P/(4\pi \Sigma_a n_a m_a)^{1/2}$ is the poloidal Alfvén velocity, $v_\zeta \equiv R\hat{\boldsymbol{\zeta}} \cdot \mathbf{v} = R^2 V^\zeta + (I/B) v_\parallel' + R\hat{\boldsymbol{\zeta}} \cdot \mathbf{v}_\perp'$ is the covariant toroidal component of the particle velocity in the laboratory frame, $V' = 2\pi \oint d\theta \sqrt{g}$ ($\theta$: a poloidal angle) is the specific volume, and $\sqrt{g} \equiv (\nabla \Psi \cdot \nabla \theta \times \nabla \zeta)^{-1} = 1/B^\theta$ is the Jacobian. In the energy balance equation (56), $p_a \equiv n_a T_a$ is the pressure, $\mathbf{E}^{(A)} \equiv -c^{-1} \partial \mathbf{A}/\partial t$ is the inductive electric field, and $\mathbf{u}_{a1}$ is the first-order flow velocity [see Eq. (16) in Ref. 24]. Taken in the order of appearance, the right-hand side of Eq. (56) describes the viscous heating, the work done by the radial electric field during the transport $\Gamma_a$, the work done by the inductive electric field on the surface flows $\mathbf{u}_{a1}$, the cooling from a secular rise in the local density, the heating by the secular rise in the moment of inertia, the collisional and turbulent energy transfers in the plasma frame and the auxiliary injected power.

In the right-hand side of Eqs. (55)–(57), the terms with $\mathcal{I}_a$ are written to represent the case where the right-hand side of Eq. (2) contains external sources such as neutral beam injection. Since we assume that $\mathcal{I}_a = \mathcal{O}(\delta^2)$, the gyrokinetic equation (45), which is of $\mathcal{O}(\delta)$, is not affected by $\mathcal{I}_a$.

In Eqs. (55)–(57), the surface-averaged radial particle flux $\Gamma_a$, heat flux $q_a$, and toroidal momentum flux $\Pi_a$ are written as

$$\Gamma_a \equiv \left\langle \int d^3v\ f_a \mathbf{v} \cdot \nabla \Psi \right\rangle = \Gamma_a^{\mathrm{cl}} + \Gamma_a^{\mathrm{ncl}} + \Gamma_a^H + \Gamma_a^{(E)} + \Gamma_a^{\mathrm{anom}},$$
$$q_a \equiv T_a \left\langle \int d^3v f_a \left( \frac{\varepsilon}{T_a} - \frac{5}{2} \right) \mathbf{v} \cdot \nabla \Psi \right\rangle = q_a^{\mathrm{cl}} + q_a^{\mathrm{ncl}} + q_a^H + q_a^{(E)} + q_a^{\mathrm{anom}}, \tag{58}$$
$$\Pi_a \equiv \left\langle \int d^3v f_a m_a v_\zeta \mathbf{v} \cdot \nabla \Psi \right\rangle = \Pi_a^{\mathrm{cl}} + \Pi_a^{\mathrm{ncl}} + \Pi_a^H + \Pi_a^{(E)} + \Pi_a^{\mathrm{anom}}.$$

Here the superscripts "cl" and "ncl" represent the classical and neoclassical fluxes, respectively. The fluxes with the superscripts $H$ and $(E)$ are the gyroviscosity-driven and inductive-electric-field-driven parts, respectively [see Eqs. (20), (22)–(24), (A6), and (A7) in Ref. 24 for definitions of these fluxes]. The anomalous fluxes $(\Gamma_a^{\mathrm{anom}}, q_a^{\mathrm{anom}}/T_a, \Pi_a^{\mathrm{anom}})$ are rewritten in terms of the gyrophase-dependent part of the fluctuation-particle interaction operator $\tilde{\mathcal{D}}_a$ as

$$\Gamma_a^{\rm anom} = -\frac{m_a c}{e_a}\left\langle \int d^3v\ \tilde{\mathcal{D}}_a \mathbf{v}'_\perp \cdot (R\hat{\boldsymbol{\zeta}})\right\rangle,$$

$$\frac{1}{T_a} q_a^{\rm anom} = -\frac{m_a c}{e_a}\left\langle \int d^3v\ \tilde{\mathcal{D}}_a \left(\frac{\varepsilon}{T_a} - \frac{5}{2}\right)\mathbf{v}'_\perp \cdot (R\hat{\boldsymbol{\zeta}})\right\rangle,$$

$$\Pi_a^{\rm anom} = -\frac{m_a^2 c}{e_a}\left\langle \int d^3v\ \tilde{\mathcal{D}}_a \frac{1}{2}\,(\widetilde{v_\zeta^2})\right\rangle, \tag{59}$$

where $\mathbf{v}'_\perp \cdot (R\hat{\boldsymbol{\zeta}}) = -B^{-1}(\mathbf{v}' \times \mathbf{b})\cdot\nabla\Psi$ and $\frac{1}{2}\widetilde{(v_\zeta}^2 \equiv \frac{1}{2}[v_\zeta^2 - \overline{(v_\zeta^2)}] = [R^2 V^\zeta + (I/B)v'_\parallel]\mathbf{v}'_\perp \cdot (R\hat{\boldsymbol{\zeta}}) + \frac{1}{2}\widetilde{\mathbf{v}'_\perp \mathbf{v}'_\perp} : (R\hat{\boldsymbol{\zeta}})(R\hat{\boldsymbol{\zeta}})$.

Comparing the anomalous fluxes defined by Eqs. (53) with those defined by Eqs. (59), we find that

$$\Gamma_a^A \equiv \Gamma_a^{\rm anom},$$

$$q_a^A \equiv q_a^{\rm anom} + e_a \left\langle\left\langle \int d^3v\ \hat{f}_a\left(\hat{\phi} - \frac{1}{c}\mathbf{V}_0 \cdot \hat{\mathbf{A}}\right)\mathbf{v}\cdot\nabla\Psi\right\rangle\right\rangle, \tag{60}$$

$$\Pi_a^A \equiv \Pi_a^{\rm anom} + \frac{e_a}{c}\left\langle\left\langle \int d^3v\ \hat{f}_a \hat{\mathbf{A}}\cdot(R\hat{\boldsymbol{\zeta}})\mathbf{v}\cdot\nabla\Psi\right\rangle\right\rangle.$$

We see from Eq. (60) that the anomalous heat flux $q_a^A$ and the anomalous toroidal momentum flux $\Pi_a^A$ include the fluctuating potential energy transport $e_a\langle\langle \int d^3v\ \hat{f}_a(\hat{\phi} - c^{-1}\mathbf{V}_0 \cdot \hat{\mathbf{A}})\mathbf{v}\cdot\nabla\Psi\rangle\rangle$ and the toroidal momentum transport due to the fluctuating vector potential $(e_a/c)\langle\langle \int d^3v\ \hat{f}_a \hat{\mathbf{A}}\cdot(R\hat{\boldsymbol{\zeta}})\mathbf{v}\cdot\nabla\Psi\rangle\rangle$, respectively.

In the right-hand sides of Eqs. (56) and (57), we find the anomalous terms $\langle \int d^3v\ \varepsilon\mathcal{D}_a\rangle$ and $\Sigma_a\langle \int d^3v\ m_a v_\zeta \mathcal{D}_a\rangle$. The anomalous heat production term $\langle \int d^3v\ \varepsilon\mathcal{D}_a\rangle$ in Eq. (56) is rewritten as

$$\left\langle \int d^3v\ \varepsilon\mathcal{D}_a\right\rangle = Q_a^A - \frac{1}{V'}\frac{\partial}{\partial\Psi}[V'(q_a^A - q_a^{\rm anom})] + (\Pi_a^A - \Pi_a^{\rm anom})\left(-\frac{\partial V^\zeta}{\partial\Psi}\right) \tag{61}$$

which shows that, in the energy balance equation (56) with (58), $q_a^A$ and $\Pi^A$ replace $q_a^{\rm anom}$ and $\Pi_a^{\rm anom}$ and that $Q_a^A$ appear as the anomalous heat transfer term. Thus the definitions of the anomalous transport fluxes given by Eq. (53) are reasonable not only from the viewpoint of the thermodynamic expression for the entropy production in Eq. (52) but also from that of the energy balance equation. The anomalous toroidal momentum production term $\Sigma_a\langle \int d^3v\ m_a v_\zeta \mathcal{D}_a\rangle$ in Eq. (57) is given by

$$\sum_a \left\langle \int d^3v\ m_a v_\zeta \mathcal{D}_a\right\rangle = \sum_a e_a \left\langle\left\langle (R\hat{\boldsymbol{\zeta}})\cdot \int d^3v\ \hat{f}_a\left(\hat{\mathbf{E}} + \frac{1}{c}\mathbf{v}\times\hat{\mathbf{B}}\right)\right\rangle\right\rangle. \tag{62}$$

Using Eq. (60) and the Maxwell equations for the fluctuating electromagnetic fields, Eq. (62) is rewritten as

$$\sum_a \left\langle \int d^3v\ m_a v_\zeta \mathcal{D}_a\right\rangle = \frac{1}{4\pi V'}\frac{\partial}{\partial\Psi}(V'\langle\langle \hat{E}_\zeta \hat{E}^\Psi + \hat{B}_\zeta \hat{B}^\Psi\rangle\rangle)$$
$$= -\frac{1}{V'}\frac{\partial}{\partial\Psi}\left[V'\left\{\frac{1}{4\pi}\sum_{\mathbf{k}_\perp}\langle\langle(\mathbf{k}_\perp\mathbf{k}_\perp):(R\hat{\zeta})(\nabla\Psi) \times |\mathbf{A}(\mathbf{k}_\perp)|^2\rangle\rangle + \sum_a (\Pi_a^A - \Pi_a^{\rm anom})\right\}\right], \tag{63}$$

where the ordering $\hat{\phi} \sim \mathbf{V}_0 \cdot \hat{\mathbf{A}}/c$ is used and terms of $\mathcal{O}(V_0^2/c^2)$ are neglected. We see from Eqs. (63) that the anomalous momentum production from $\mathcal{D}_a$ is due to the Maxwell stress of the fluctuating electromagnetic fields. When Eq. (63) is substituted into the momentum balance equation (57), the Maxwell stress gives $-(4\pi V')^{-1}\ (\partial/\partial\Psi)\ [V'\ \{\Sigma_{\mathbf{k}_\perp}\langle\langle(\mathbf{k}_\perp\mathbf{k}_\perp):(R\hat{\zeta})(\nabla\Psi)|\mathbf{A}(\mathbf{k}_\perp)|^2\rangle\rangle]$ and replaces $\Pi_a^{\rm anom}$ to $\Pi_a^A$.

Using the charge neutrality condition and the Ampère's law for the self-consistent fluctuations (see Appendix A), we have the ambipolarity of the anomalous particle fluxes $\Sigma_a e_a \Gamma_a^A = 0$ and the cancellation of the total anomalous heat transfer $\Sigma_a Q_a^A = 0$, which shows that the self-consistent fluctuations cause no net heating of the total particles but result in the anomalous heat exchange between different species of particles.

## V. CONCLUSIONS

In this work, we have presented the nonlinear electromagnetic gyrokinetic equation for plasmas with general magnetic geometries and large flow velocities on the order of the ion thermal speed. In the derivation, we have used the recursive formulation to give the relation of the perturbed distribution function to the equilibrium distribution and the electromagnetic fluctuations, since it is useful to retain collisional effects and synthetically formulate the turbulent and collisional (classical and neoclassical) transport processes. The reduced forms of the gyrokinetic equation for the slab, cylindrical, and toroidal configurations were obtained from the general one [see Eqs. (18), (24), (30), (32), (45), and (46)].

We specified the source terms in the gyrokinetic equation, which is related to the anomalous momentum transport [see Eqs. (B3)–(B5) and $\hat{w}_{aV}$ in Eqs. (48) and (49)]. We also derived the generalized Hasegawa-Mima equation (35) [or (37)] which correctly describes the energy transfer between the background sheared flow and the turbulent energy through the Reynolds stress tensor contracted into the flow shear tensor [see Eqs. (36) and (38)].

Based on the gyrokinetic equation for the toroidally rotating system, we have defined the conjugate pairs of the anomalous transport fluxes in Eq. (53) and the thermodynamic forces in Eq. (47), the inner product of which gives the anomalous entropy production rate in Eq. (52). The Onsager symmetry of the quasilinear matrix relating the anomalous fluxes to the conjugate forces is shown. Also given are complete balance equations for particles, energy and toroidal momentum including the classical, neoclassical, and anomalous transport fluxes [see Eqs. (55)–(57)].

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## APPENDIX A: SELF-CONSISTENT ELECTROMAGNETIC FIELDS AND AMBIPOLARITY CONDITION

Here we show the self-consistent constraints on the turbulent fields, which are given by the charge neutrality condition

$$\lambda_D^{-2}\left(\hat{\phi}(\mathbf{k}_\perp)-\frac{\mathbf{V}_0}{c}\cdot\hat{\mathbf{A}}(\mathbf{k}_\perp)\right)=4\pi\sum_a e_a\int d^3v\; J_0(\gamma_a)\hat{h}_a(\mathbf{k}_\perp) \tag{A1}$$

and, the parallel and perpendicular components of the Ampère's law

$$k_\perp^2\hat{A}_\parallel(\mathbf{k}_\perp)=\frac{4\pi}{c}\sum_a e_a\int d^3v\; J_0(\gamma_a)\hat{h}_a(\mathbf{k}_\perp)v'_\parallel, \tag{A2}$$

$$-k_\perp\hat{B}_\parallel(\mathbf{k}_\perp)=\frac{4\pi}{c}\sum_a e_a\int d^3v\; J_1(\gamma_a)\hat{h}_a(\mathbf{k}_\perp)v'_\perp, \tag{A3}$$

where Eq. (31) and the Debye length $\lambda_D\equiv(4\pi\Sigma_a n_a e_a^2/T_a)^{-1/2}$ are used. The use of the Ampère's law is justified, since the displacement current is neglected due to the gyrokinetic ordering. Equations (A1)–(A3) are rewritten in the **x**-space as

$$\lambda_D^{-2}\left(\hat{\phi}(\mathbf{x})-\frac{\mathbf{V}_0}{c}\cdot\hat{\mathbf{A}}(\mathbf{x})\right)=4\pi\sum_a e_a\int d^3v\;\hat{h}_a(\mathbf{x}-\boldsymbol{\rho}_a), \tag{A4}$$

$$-\nabla_\perp^2\hat{A}_\parallel(\mathbf{x})=\frac{4\pi}{c}\sum_a e_a\int d^3v\;\hat{h}_a(\mathbf{x}-\boldsymbol{\rho}_a)v'_\parallel, \tag{A5}$$

$$\nabla\hat{B}_\parallel(\mathbf{x})\times\mathbf{b}=\frac{4\pi}{c}\sum_a e_a\int d^3v\;\hat{h}_a(\mathbf{x}-\boldsymbol{\rho}_a)\mathbf{v}'_\perp. \tag{A6}$$

Substituting Eqs. (A1)–(A3) into the definition of the anomalous fluxes in Eq. (53), we find that the anomalous particle fluxes are intrinsically ambipolar

$$\sum_a e_a\Gamma_a^A=0. \tag{A7}$$

From Eqs. (20), (48), (53), and (A1)–(A3) with the quasi-steady state ordering $\partial\langle\cdot\rangle_{\rm ens}/\partial t=\mathcal{O}(\delta^2)$, it is shown that the species summation of the anomalous heating $Q_a^A$ vanishes

$$\sum_a Q_a^A=0. \tag{A8}$$

The self-consistent fluctuations cause no net heating of the total particles since the source of the anomalous heating is the energy of the fluctuating electromagnetic fields, which cannot be a stationary energy supplier unless the fluctuations are externally driven.

## APPENDIX B: ON THE SOURCE TERMS IN THE GYROKINETIC EQUATION RELATING TO THE ANOMALOUS MOMENTUM TRANSPORT

In the right-hand side of the gyrokinetic equation (30) [or (32)], we easily find the shear flow term which is proportional to $v'_\parallel\{\mathbf{b}\cdot(\nabla\mathbf{V}_0)+(\nabla\mathbf{V}_0)\cdot\mathbf{b}\}$. We also see that the term with $(\partial/\partial t+\mathbf{V}_0\cdot\nabla)\hat{\psi}_a$ contains other contributions from the flow shear by noticing the following formula:

$$e_a\left(\frac{\partial}{\partial t}+\mathbf{V}_0(\mathbf{X})\cdot\frac{\partial}{\partial\mathbf{X}}\right)\hat{\psi}_a(\mathbf{X})=e_a\left\langle\left(\frac{\partial}{\partial t}+\mathbf{V}_0(\mathbf{x})\cdot\frac{\partial}{\partial\mathbf{x}}\right)\left(\hat{\phi}(\mathbf{x})-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\right\rangle_{\mathbf{x}}+\hat{\mathcal{P}}_a, \tag{B1}$$

where the first group of terms in the right-hand side represents the increase of the fluctuation potential energy and $\hat{\mathcal{P}}_a$ is given by

$$\hat{\mathcal{P}}_a=e_a\left\langle(\mathbf{V}_0\cdot\nabla\boldsymbol{\rho}_a-\boldsymbol{\rho}_a\cdot\nabla\mathbf{V}_0)\cdot\nabla\times\left(\hat{\phi}(\mathbf{x})-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\right\rangle_{\mathbf{x}}. \tag{B2}$$

In deriving Eqs. (B1) and (B2), we have used the ballooning representation (12) for the system with the large flow.

For the slab and cylindrical cases considered in Sec. III, $\hat{\mathcal{P}}_a$ is rewritten as

$$\hat{\mathcal{P}}_a=-\frac{\partial V_y}{\partial x}\left\langle m_a\mathbf{v}'_\perp\cdot\hat{\mathbf{y}}\left(-\frac{c}{B}\right)\nabla\left(\hat{\phi}(\mathbf{x})-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\times\mathbf{b}\cdot\hat{\mathbf{x}}\right\rangle_{\mathbf{x}}-\frac{\partial V_z}{\partial x}\left\langle m_a\mathbf{v}'_\perp\cdot\hat{\mathbf{z}}\left(-\frac{c}{B}\right)\nabla\left(\hat{\phi}(\mathbf{x})-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\times\mathbf{b}\cdot\hat{\mathbf{x}}\right\rangle_{\mathbf{x}} \tag{B3}$$

and

$$\hat{\mathcal{P}}_a=-\frac{\partial(r^{-1}V_\theta)}{\partial r}\left\langle m_a\mathbf{v}'_\perp\cdot(r\hat{\boldsymbol{\theta}})\left(-\frac{c}{B}\right)\nabla\left(\hat{\phi}(\mathbf{x})-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\times\mathbf{b}\cdot\hat{\mathbf{r}}\right\rangle_{\mathbf{x}}-\frac{\partial V_z}{\partial r}\left\langle m_a\mathbf{v}'_\perp\cdot\hat{\mathbf{z}}\left(-\frac{c}{B}\right)\nabla\left(\hat{\phi}(\mathbf{x})-\frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\times\mathbf{b}\cdot\hat{\mathbf{r}}\right\rangle_{\mathbf{x}}, \tag{B4}$$

respectively. Thus $\hat{\mathcal{P}}_a$ is given by the product of the flow shear and the perpendicular momentum transport due to the fluctuations.

Equation (B1) is still valid for the axisymmetric case considered in Sec. IV if we note that the spatial gradient $\partial/\partial\mathbf{X}$ should be taken with the energy variable $\varepsilon$ fixed (not with the kinetic energy $w$ fixed as in the slab and cylindrical cases). Then, the first group of terms in the right-hand side of Eq. (B1) is written as $\hat{w}_{aT}(\mathbf{X})$ in Eq. (49) and $\hat{\mathcal{P}}_a$ is given by the product of the toroidal flow shear and the anomalous transport of the toroidal component of the perpendicular momentum as

$$\hat{\mathcal{P}}_a = -\frac{\partial V_\zeta}{\partial \Psi}\left\langle m_a \mathbf{v}'_\perp \cdot (R\hat{\boldsymbol{\zeta}})\left(-\frac{c}{B}\right)\nabla\left(\hat{\phi}(\mathbf{x}) - \frac{1}{c}(\mathbf{V}_0+\mathbf{v}')\cdot\hat{\mathbf{A}}(\mathbf{x})\right)\times\mathbf{b}\cdot\nabla\Psi\right\rangle_{\mathbf{X}} \tag{B5}$$

which is used to obtain the expression for $\hat{w}_{aV}$ in Eq. (49).

If we use the same fluid approximation as made to derive the generalized Hasegawa-Mima equation (35) [or (37)], the velocity-space integral of the nonadiabatic ion distribution function $\hat{h}_{i1}$ multiplied by $\hat{\mathcal{P}}_i$ gives

$$\int d^3v \langle \hat{h}_i(\mathbf{X})\hat{\mathcal{P}}_i(\mathbf{X})\rangle_{\text{ens}} \simeq -n_i m_i \langle \hat{\mathbf{v}}_E(\mathbf{X})\hat{\mathbf{v}}_E(\mathbf{X})\rangle_{\text{ens}} : (\nabla\mathbf{V}_0), \tag{B6}$$

where $\nabla_\perp \equiv \nabla - \mathbf{bb}\cdot\nabla$ and $\hat{\mathbf{v}}_E(\mathbf{X}) \equiv -(c/B)\nabla\hat{\phi}(\mathbf{X})\times\mathbf{b}$. Equation (B6) represents the energy transfer from the background sheared flow to the fluctuations through the Reynolds stress multiplied by the flow shear, which coincides with the right-hand side of Eq. (38)]. Thus $\hat{\mathcal{P}}_a$ is deeply related to the anomalous momentum transport (or viscosity), which reduces to the Reynolds stress in the fluid limit.

Artun and Tang derived the linearized gyrokinetic equation describing electrostatic fluctuations in the slab system with sheared equilibrium flows.[6] However, their gyrokinetic equation misses the contribution from $\mathcal{P}_a$ in Eq. (B3), which is included in our gyrokinetic equation (32) and is written as

$$\frac{1}{T_a}\mathcal{P}_a f_{a0}. \tag{B7}$$

Now, we will see that these terms can be derived from the last term in Eq. (27) of their paper (Ref. 6), which is written in our notation as

$$\frac{1}{\Omega_a}\frac{m_a}{T_a}[\hat{\mathbf{x}}\cdot(\nabla\mathbf{V}_0)\cdot(\mathbf{b}\times\hat{\mathbf{x}})][(\mathbf{b}\times\hat{\mathbf{x}})\cdot\mathbf{v}'_\perp](\mathbf{b}\times\hat{\mathbf{x}})f_{a0}, \tag{B8}$$

where $\mathbf{V}_0$ is given by Eq. (34). It seems that Artun and Tang did not retain this term's contribution to their resulting gyrokinetic equation [see Eq. (30) in Ref. 6]. Following the procedure in Eqs. (24)–(28) of Ref. 6 and retaining the term in Eq. (B8), we find that the following additional term should appear in the right-hand side of the gyrokinetic equation:

$$\begin{aligned}\frac{e_a}{m_a}&\langle\nabla\hat{\phi}(\mathbf{x}=\mathbf{X}+\boldsymbol{\rho}_a)\cdot[\text{Eq. (B8)}]\rangle_{\mathbf{X}} \\ &= \frac{m_a c}{BT_a}[\hat{\mathbf{x}}\cdot(\nabla\mathbf{V}_0)\cdot(\mathbf{b}\times\hat{\mathbf{x}})]\langle[(\mathbf{b}\times\hat{\mathbf{x}})\cdot\mathbf{v}'_\perp] \\ &\quad\times(\mathbf{b}\times\hat{\mathbf{x}})\cdot\nabla\hat{\phi}(\mathbf{x})\rangle_{\mathbf{X}}\, f_{a0}\end{aligned} \tag{B9}$$

which is found to be the same as Eq. (B7) by using Eq. (B3) for the electrostatic slab case and $(\mathbf{b}\times\hat{\mathbf{x}})(\mathbf{b}\times\hat{\mathbf{x}}) = \mathbf{I}-\mathbf{bb}-\hat{\mathbf{x}}\hat{\mathbf{x}}$.

Artun and Tang also derived the nonlinear electromagnetic gyrokinetic equation for the toroidally rotating axisymmetric system.[7] In fact, $(\partial/\partial t+\mathbf{V}_0\cdot\nabla)\hat{\psi}_a$ appears in their gyrokinetic equation (56) in Ref. 7. However, the term in the form of Eq. (B7) is still missed in their gyrokinetic equation because they did not use the ballooning formalism taking account of the temporal dependence of the radial wavenumber. Actually, our gyrokinetic equation (45) for the toroidally rotating system is found to coincide with Brizard's result[12] by the correct ballooning representation. Here, we will see again that the term in Eq. (B7) can be derived also from their procedure in Ref. 7. To show this briefly, we only explain how to derive the electrostatic part of Eq. (B7). Artun and Tang seem to have missed the contributions of the third group of terms in the right-hand of Eq. (B1) in Ref. 7, which are written in our notation as

$$\begin{aligned}-\frac{1}{4\Omega_a}&(\mathbf{b}\times\nabla\mathbf{V}_0\cdot\mathbf{v}'_\perp - \mathbf{v}'_\perp\cdot\nabla\mathbf{V}_0\times\mathbf{b} \\ &+\mathbf{v}'_\perp\times\mathbf{b}\cdot\nabla\mathbf{V}_0 + \nabla\mathbf{V}_0\cdot\mathbf{v}'_\perp\times\mathbf{b}) \\ &= \frac{1}{\Omega_a}\frac{\partial V^\zeta}{\partial\Psi}\mathbf{v}'_\perp\cdot(R\hat{\boldsymbol{\zeta}})\left(\frac{1}{4}I\mathbf{b}+\nabla\Psi\times\mathbf{b}\right),\end{aligned} \tag{B10}$$

where Eqs. (41) and (42) are used. Then, we find from Eqs. (44) and (50) in Ref. 7 that the terms in Eq. (B10) lead to the following additional term in the right-hand side of the gyrokinetic equation

$$\begin{aligned}-\frac{e_a}{T_a}&f_{a0}\langle\nabla_\perp\hat{\phi}(\mathbf{x}=\mathbf{X}+\boldsymbol{\rho}_a)\cdot[\text{Eq. (B10)}]\rangle_{\mathbf{X}} \\ &= \frac{m_a c}{BT_a}\frac{\partial V^\zeta}{\partial\Psi}\langle\mathbf{v}'_\perp\cdot(R\hat{\boldsymbol{\zeta}})\nabla\hat{\phi}(\mathbf{x})\times\mathbf{b}\cdot\nabla\Psi\rangle_{\mathbf{X}}\, f_{a0}\end{aligned} \tag{B11}$$

which is found to be the same as the electrostatic part of Eq. (B7) by recalling Eq. (B5).

## APPENDIX C: ONSAGER SYMMETRY OF QUASILINEAR ANOMALOUS TRANSPORT EQUATIONS

Here we assume that the spectra of the electrostatic fluctuations $\hat{\phi}(\mathbf{k}_\perp)$ are given *a priori* and that the nonlinear term in the gyrokinetic equation (45) [or (46)] is negligible. Then, using the definitions in Eq. (53) with the solution of the linearized gyrokinetic equation, we obtain the quasilinear anomalous transport equations

$$\begin{bmatrix} \Gamma_a^A \\ q_a^A/T_a \\ \Pi_a^A \\ Q_a^A \end{bmatrix} = \sum_b \begin{bmatrix} (L^A)_{11}^{ab} & (L^A)_{12}^{ab} & (L^A)_{1V}^{ab} & (L^A)_{1T}^{ab} \\ (L^A)_{21}^{ab} & (L^A)_{22}^{ab} & (L^A)_{2V}^{ab} & (L^A)_{2T}^{ab} \\ (L^A)_{V1}^{ab} & (L^A)_{V2}^{ab} & (L^A)_{VV}^{ab} & (L^A)_{VT}^{ab} \\ (L^A)_{T1}^{ab} & (L^A)_{T2}^{ab} & (L^A)_{TV}^{ab} & (L^A)_{TT}^{ab} \end{bmatrix} \times \begin{bmatrix} X_{a1}^A \\ X_{a2}^A \\ X_{aV}^A \\ X_{aT}^A \end{bmatrix}. \tag{C1}$$

Here the anomalous transport coefficients $(L^A)_{rs}^{ab}$ $(r,s=1,2,V,T)$ are functionals of the fluctuation spectra, and they also contain the equilibrium fields $\mathbf{B}$ and $\mathbf{V}_0$ as parameters

$$(L^A)_{rs}^{ab} = (L^A)_{rs}^{ab}[\mathbf{B},\mathbf{V}_0,\{\hat{\phi},\widehat{A_\parallel},\widehat{B_\parallel}\}]. \tag{C2}$$

In the same way as in Refs. 23 and 24, we can show that the quasilinear anomalous transport coefficients satisfy the following Onsager symmetry:

$$\begin{aligned} T_a(L^A)_{mn}^{ab}&[\mathbf{B},\mathbf{V}_0,\{\hat{\phi}(t),\widehat{A_\parallel}(t),\widehat{B_\parallel}(t)\}] \\ &= T_b(L^A)_{nm}^{ba}[-\mathbf{B},-\mathbf{V}_0,\{\hat{\phi}(-t),\widehat{A_\parallel}(-t), \\ &\quad \widehat{B_\parallel}(-t)\}] \quad (m,n=1,2), \\ T_a(L^A)_{MN}^{ab}&[\mathbf{B},\mathbf{V}_0,\{\hat{\phi}(t),\widehat{A_\parallel}(t),\widehat{B_\parallel}(t)\}] \\ &= T_b(L^A)_{NM}^{ba}[-\mathbf{B},-\mathbf{V}_0,\{\hat{\phi}(-t),\widehat{A_\parallel}(-t), \\ &\quad \widehat{B_\parallel}(-t)\}] \quad (M,N=V,T), \\ T_a(L^A)_{mM}^{ab}&[\mathbf{B},\mathbf{V}_0,\{\hat{\phi}(t),\widehat{A_\parallel}(t),\widehat{B_\parallel}(t)\}] \\ &= -T_b(L^A)_{Mm}^{ba}[-\mathbf{B},-\mathbf{V}_0,\{\hat{\phi}(-t),\widehat{A_\parallel}(-t), \\ &\quad \widehat{B_\parallel}(-t)\}] \quad (m=1,2;M=V,T), \end{aligned} \tag{C3}$$

where $\{\hat{\phi}(-t),\widehat{A_\parallel}(-t),\widehat{B_\parallel}(-t)\}$ represents the fluctuation spectra obtained by the time reversal of the original spectra $\{\hat{\phi}(t),\widehat{A_\parallel}(t),\widehat{B_\parallel}(t)\}$.


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