S30. The Range of Validity of the Rayleigh Hypothesis¹⁾

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The parameter range over which the Rayleigh hypothesis (RH) for optical gratings might be validly applied to analysis of high power backward wave oscillators (BWOs) has been investigated numerically. The field patterns and eigen frequency for the slow wave structure (SWS) are solved numerically by using code HIDM (Higher Order Implicit Difference Method) that is free from RH. For a deep corrugation, $h K_0 = 5 \times 0.448$, using RH is still valid for obtaining the dispersion relation, although the Floquet Harmonic Expansion (FHE) fails to correctly represent the field patterns inside the corrugation, here, h and K_0 are, respectively, the amplitude and wave number of the axisymmetric SWS. It has been pointed out that from rigorous mathematical viewpoint, RH was only valid for shallow corrugations of SWS, $h K_0 < 0.448$.

New analytical Frame for Electromagnetic Field

Magnetic field $\mathbf{B}$ can be expressed by vector potential $\mathbf{A} = (\Phi(r, \theta, z, t)/r, \Psi(r, \theta, z, t)/r, 0)$ without loss of generality, because of $\nabla \cdot \mathbf{B} = 0$. We choose $A_z = 0$. Faraday' law, $0 = \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}$, gives $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \lambda$. Scalar potential λ is possible to be solved from the z component of the Ampere' law,

$$\epsilon_0 \mu_0 \frac{\partial \lambda}{\partial t} = - \left(\frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial \Psi}{\partial \theta} \right) + \mu_0 \int J_z dz. \quad (1)$$

If we substitute this gauge expression for the scalar potential λ into the (r, θ) component of Ampere' law, we can get the following equations for the numerical computation.

$$\begin{aligned} \epsilon_0 \mu_0 \frac{\partial^2 \Phi}{\partial t^2} &= \tilde{\Delta} \Phi - \frac{2}{r^2} \frac{\partial \Psi}{\partial \theta} - \mu_0 r \int \left(\frac{\partial J_z}{\partial r} - \frac{\partial J_r}{\partial z} \right) dz \\ \epsilon_0 \mu_0 \frac{\partial^2 \Psi}{\partial t^2} &= \tilde{\Delta} \Psi + \frac{2}{r^2} \frac{\partial \Phi}{\partial \theta} - \mu_0 r \int \left(\frac{1}{r} \frac{\partial J_z}{\partial \theta} - \frac{\partial J_\theta}{\partial z} \right) dz \end{aligned}$$

The operator $\tilde{\Delta}$ is defined by the following expression.

$$\tilde{\Delta} f(r, \theta, z, t) = r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \quad (2)$$

For the vacuum field case, $\mathbf{J} = 0$. Boundary conditions on SWS with inner radius $r = X(z)$ given by $\mathbf{E} \times \nabla (r - X(z)) = 0$ are reduced to the following equations,

$$X'(z) \left\{ \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{r} \frac{\partial^2 \Psi}{\partial \theta \partial r} \right\} = \left\{ \frac{1}{r} \frac{\partial^2 \Psi}{\partial \theta \partial z} + \frac{\partial^2 \Phi}{\partial r \partial z} \right\},$$

$$\epsilon_0 \mu_0 \frac{\partial^2 \Psi}{\partial t^2} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} - \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} = 0.$$

The boundary condition for the normal component of magnetic field $\mathbf{B} \cdot \nabla (r - X(z)) = 0$ is automatically satisfied.

Numerical Treatment for Eigemode Analysis

We solve eigenmodes of electromagnetic field in SWSs with periodic inner radius numerically (periodicity = L , $X(z + L) = X(z)$).

$$\begin{Bmatrix} \Phi \\ \Psi \end{Bmatrix} = \begin{Bmatrix} \chi(r, z) \\ \eta(r, z) \end{Bmatrix} \exp(i\ell\theta + ik_{\parallel}z - i\omega t) \quad (3)$$

where $(k_{\parallel}, \ell, \omega)$ are wave numbers and a frequency of eigenmode (ℓ is arbitrary integer). $\chi(r, z)$ and $\eta(r, z)$ are periodic function in z with periodicity L .

$$\frac{\partial \Phi}{\partial z} = \left(ik_{\parallel} \chi + \frac{\partial \chi}{\partial z} \right) \exp(ik_{\parallel}z) \quad (4)$$

$$\frac{\partial^2 \Phi}{\partial z^2} = \left(-k_{\parallel}^2 \chi + 2ik_{\parallel} \frac{\partial \chi}{\partial z} + \frac{\partial^2 \chi}{\partial z^2} \right) \exp(ik_{\parallel}z) \quad (5)$$

Similar equations hold for Ψ . To treat the boundary conditions on the SWS in high accuracy, we introduce a curvilinear coordinate system ($\xi \equiv r/X(z), z$) instead of (r, z) . The inner wall of SWS is specified by $\xi = 1$. For z -differentiation, we introduce very high order difference scheme. Reduced simultaneous ordinary differential equation system in ξ direction is numerically solved by HIDM. Numerical example is shown in Fig.1.

Fig.1 Field pattern of quasi-TM mode ($\ell = 1$) in SWS.

Reference

- 1) Watanabe, T., et.al. : Physical Review E, 69 (2004) 056606.