Physics of Plasmas

Polarization and magnetization in collisional and turbulent transport processes

Cite as: Phys. Plasmas **29**, 052509 (2022); https://doi.org/10.1063/5.0080636
Submitted: 02 December 2021 • Accepted: 29 April 2022 • Published Online: 19 May 2022

H. Sugama, S. Matsuoka and M. Nunami

Physics of Plasmas ARTICLE scitation.org/journal/php

Polarization and magnetization in collisional and turbulent transport processes

Cite as: Phys. Plasmas **29**, 052509 (2022); doi: 10.1063/5.0080636
Submitted: 2 December 2021 · Accepted: 29 April 2022 ·
Published Online: 19 May 2022

View Online Export Citation CrossMark

H. Sugama,[1,2,a)] S. Matsuoka,[1,3] and M. Nunami[1,4]

AFFILIATIONS

[1]National Institute for Fusion Science, Toki 509-5292, Japan
[2]Department of Advanced Energy, University of Tokyo, Kashiwa 277-8561, Japan
[3]Department of Fusion Science, SOKENDAI (The Graduate University for Advanced Studies), Toki 509-5292, Japan
[4]Department of Physics, Nagoya University, Nagoya 464-8602, Japan

[a)]**Author to whom correspondence should be addressed**: sugama.hideo@nifs.ac.jp

ABSTRACT

Expressions of polarization and magnetization in magnetically confined plasmas are derived, which include full expansions in the gyroradius to treat effects of both equilibrium and microscopic electromagnetic turbulence. Using the obtained expressions, densities and flows of particles are related to those of gyrocenters. To the first order in the normalized gyroradius expansion, the mean part of the particle flow is given by the sum of the gyrocenter flow and the magnetization flow, which corresponds to the so-called magnetization law in drift kinetics, while the turbulent part contains the polarization flow as well. Collisions make an additional contribution to the second-order particle flow. The mean particle flux across the magnetic surface is of the second-order, and it contains classical, neoclassical, and turbulent transport processes. The Lagrangian variational principle is used to derive the gyrokinetic Poisson and Ampère equations, which properly include mean and turbulent parts so as to be useful for full-f global electromagnetic gyrokinetic simulations. It is found that the second-order Lagrangian term given by the inner product of the turbulent vector potential and the drift velocity consisting of the curvature drift and the ∇B drift should be retained in order for the derived Ampère equation to correctly include the diamagnetic current, which is necessary especially for the full-f high-beta plasma simulations. The turbulent parts of these gyrokinetic Poisson and Ampère equations are confirmed to agree with the results derived from the WKB representation in earlier works.

I. INTRODUCTION

Global simulations of collisional and turbulent plasma transport[1–9] are now vigorously conducted based on gyrokinetic equations using the gyrocenter coordinates that are derived from the Lie transformation method.[10,11] Conservation properties possessed by such gyrokinetic equations are suitable for global and long-time transport simulations, and they have been extensively investigated based on Lagrangian and Hamiltonian formulations.[11–22] It is well known that the finite gyroradius representing the distance between particle and gyrocenter positions generates so-called polarization and magnetization,[11,12] in terms of which the relations of the density and mean velocity of particles to those of gyrocenters are expressed. These relations are important for using gyrokinetic simulation results to correctly evaluate particle transport, as well as to accurately calculate the charge density and the electric current in Poisson and Ampère equations, which are required to self-consistently determine electromagnetic fields in the simulation.

In a general framework of macroscopic electromagnetism for material media consisting of molecules, polarization and magnetization are formulated for evaluating the macroscopic charge density and current by spatially averaging the microscopic density and current of the point charge around the center of mass of the molecule.[23] Then, the resultant expressions of the macroscopic charge density and current are given by the series expansion associated with multipole moments due to the finite distance of each point charge from the center of mass of the molecule. The local spatial average and the finite distance described above for the system of molecules are replaced by the phase-space integral including the distribution function and the finite gyroradius of the particle motion around the gyrocenter, respectively, for formulating the polarization and magnetization in the gyrokinetic system considered in the present study. In the drift kinetic system without microturbulence, the particle flow is represented by the sum of the gyrocenter flow and the magnetization flow, which is called the magnetization law.[24] In this work, we use the gyrocenter phase-space coordinates obtained from the particle phase-space coordinates using the Lie transformation, by which the effects of turbulent electromagnetic fields are included in definitions of the gyrocenter position and the gyroradius vector. This gyroradius vector is used for infinite series expansion to express the polarization and magnetization in magnetically confined plasmas with gyroradius scale fluctuations.

The polarization and magnetization are also derived from taking the variational derivative of the field–particle interaction part of the Lagrangian for the system with respect to the electric and magnetic fields, respectively.[11] This derivation is not commonly used in conventional gyrokinetic studies[25–27] where scalar and vector potentials are used instead of electromagnetic fields to formulate basic equations describing plasma microturbulence. In some recent studies,[22,28] the Lagrangian of the gyrokinetic system is expressed in terms of perturbed electromagnetic fields instead of perturbed scalar and vector potentials, so that the gyrokinetic polarization and magnetization can be obtained by the derivative of the Lagrangian. The scalar and vector potentials are used in our study where conventional studies' results on gyrokinetic Poisson and Ampère equations with the polarization and magnetization effects due to turbulent fields are consistently incorporated. In addition, the magnetization law in drift kinetics is reproduced from taking the ensemble average of the expression for the particle flow obtained in this paper. To the second order in the normalized gyroradius, the effect of the collision term, which is not described in the magnetization law, appears as the classical transport[29–31] in the ensemble-averaged particle flow. Then it is confirmed that in toroidal confinement systems, the average particle flux across the magnetic surface is given by the second-order flows in which the classical, neoclassical,[29–31] and turbulent transport[32] are included.

The rest of this paper is organized as follows. In Sec. II, the densities and flows of the particles and gyrocenters are defined using velocity–space integrals of the distribution functions in the particle and gyrocenter phase-space coordinates. Then, the gyrocenter and particle transport equations derived from the Boltzmann kinetic equations in the two coordinate systems are used to obtain the relation between the particle and gyrocenter flows, in which effects of polarization, magnetization, and collisions are included. The detailed expressions of the polarization and magnetization are presented in Sec. III. In Sec. IV, the particle flows due to gyrocenter motion, polarization, magnetization, and collisions are separately treated using expansion in the normalized gyroradius parameter and decomposition into the ensemble average and turbulent parts. There it is shown that the first-order ensemble-averaged particle flow obeys the so-called magnetization law in drift kinetics, while the mean particle flux across the magnetic surface is of the second order and contains classical, neoclassical, and turbulent transport processes. The Lagrangian for variational derivation of the gyrokinetic Vlasov equation, Poisson's equation, and Ampère's law is presented in Sec. V, where the linear polarization-magnetization approximation[12] is employed. Finally, conclusions are given in Sec. VI. In addition, Appendix A presents the transformation formulas from the particle coordinates to the gyrocenter coordinates, and the gyrocenter Lagrangian, from which the gyrocenter equations of motion are derived. In Appendix B, the gyrocenter velocity and the time derivative of the gyroradius vector are expanded in the normalized gyroradius parameter to obtain useful formulas for derivation of the results given in Sec. IV. The zeroth and first-order distribution functions and the conditions satisfied by them are described in Appendix C. It is verified in Appendix D that the turbulent parts of Poisson and Ampère equations obtained in the present work agree with the results derived in earlier works using the WKB representation.[25,26]

II. DENSITIES AND FLOWS OF PARTICLES AND GYROCENTERS

The gyrokinetic Boltzmann equation for the gyrocenter distribution function $f_a(\mathbf{Z}, t)$ of the particle species a is given by

$$\frac{df_a}{dt} \equiv \left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}}{dt}\cdot\frac{\partial}{\partial \mathbf{Z}}\right) f_a = C_a^{(g)}, \tag{1}$$

where the gyrocenter phase-space coordinates $\mathbf{Z} \equiv (\mathbf{X}, U, \mu, \xi)$ are defined in terms of the particle phase-space coordinates $\mathbf{z} \equiv (\mathbf{x}, v_\parallel, \mu_0, \xi_0)$ as shown in Appendix A [see Eqs. (A6)–(A9)]. In Eq. (1), $d\mathbf{Z}/dt$ is regarded as a function of $(\mathbf{Z}, t)$, which is given by the gyrocenter motion equations, Eqs. (A26)–(A30). The collision term $C_a^{(g)}$ in the gyrocenter coordinates is written as[33,34]

$$C_a^{(g)} \equiv \sum_b C_{ab}^{(g)}[f_a, f_b] \equiv \sum_b \mathcal{T}_a^{-1*} C_{ab}^{(p)}\left[\mathcal{T}_a^* f_a, \mathcal{T}_b^* f_b\right], \tag{2}$$

where the subscripts a and b represent species of colliding particles and $C_{ab}^{(p)}[f_a^{(p)}, f_b^{(p)}]$ is the Landau collision operator[29–31] for the distribution functions $f_a^{(p)} = \mathcal{T}_a^* f_a$ and $f_b^{(p)} = \mathcal{T}_b^* f_b$ in the particle coordinates, which are obtained by the pull-back operators $\mathcal{T}_a^*$ and $\mathcal{T}_b^*$ acting on the gyrocenter distribution functions f_a and f_b, respectively. It is noted that, for the function $f_a(\mathbf{Z})$ of the gyrocenter coordinates $\mathbf{Z}$, $\mathcal{T}_a^* f_a$ is defined by $(\mathcal{T}_a^* f_a)(\mathbf{z}) \equiv f_a(\mathcal{T}_a(\mathbf{z}))$, where $\mathbf{Z} = \mathcal{T}_a(\mathbf{z})$ represents the transformation from the particle coordinates to the gyrocenter coordinates. The detailed expressions of the coordinate transformation are shown in Appendix A. The push-forward operator $\mathcal{T}_a^{-1*}$ is used to obtain the expression of the collision term in the gyrocenter coordinates from that in the particle coordinates. For the function $C_a^{(p)}(\mathbf{z})$ of the particle coordinates $\mathbf{z}$, $\mathcal{T}_a^{-1*} C_a^{(p)}$ is defined by $(\mathcal{T}_a^{-1*} C_a^{(p)})(\mathbf{Z}) \equiv C_a^{(p)}(\mathcal{T}_a^{-1}(\mathbf{Z}))$ where $\mathbf{z} = \mathcal{T}_a^{-1}(\mathbf{Z})$ represents the transformation from the gyrocenter coordinates to the particle coordinates.

The gyrophase average of an arbitrary function Q of the gyrocenter phase-space coordinates $\mathbf{Z} \equiv (\mathbf{X}, U, \mu, \xi)$ is represented by

$$\langle Q \rangle_\xi \equiv \frac{1}{2\pi} \oint Q \, d\xi \tag{3}$$

and the gyrophase-dependent part of Q is written as

$$\widetilde{Q} \equiv Q - \langle Q \rangle_\xi. \tag{4}$$

From Eq. (1), we obtain

$$\frac{d\langle f_a \rangle_\xi}{dt} \equiv \left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}}{dt}\cdot\frac{\partial}{\partial \mathbf{Z}}\right) \langle f_a \rangle_\xi = \langle C_a^{(g)} \rangle_\xi \tag{5}$$

and

$$\frac{d\widetilde{f}_a}{dt} \equiv \left(\frac{\partial}{\partial t} + \frac{d\mathbf{Z}}{dt}\cdot\frac{\partial}{\partial \mathbf{Z}}\right) \widetilde{f}_a = \widetilde{C}_a^{(g)}. \tag{6}$$

Noting that the pull-back $\mathcal{T}_a^* f_a$ included in the definition of the gyrocenter collision operator $C^{(g)}$ has a gyrophase dependence different from what f_a has, we find that the gyrocenter collision term depends on the gyrophase angle ξ even when the operator $C_a^{(g)}$ acts on the gyrocenter distributions which are independent of ξ. Therefore, $\widetilde{C}_a^{(g)}$ does not vanish generally, and Eq. (6) yields the nonzero gyrophase-dependent part $\widetilde{f}$ of the gyrocenter distribution function. Using the gyrofrequency $\Omega_a \equiv e_a B/(m_a c)$ to approximately write $d\widetilde{f}_a/dt \simeq \Omega_a \partial \widetilde{f}_a/\partial\xi$, we have

$$\widetilde{f}_a \simeq \frac{1}{\Omega_a}\int^{\xi} \widetilde{C}_a^{(g)} d\xi = \mathscr{O}(\varepsilon^2 f_{a0}), \tag{7}$$

where $\widetilde{C}_a^{(g)}[f_a] \simeq \widetilde{C}_a^{(g)}[f_{a1}] = \mathscr{O}(\nu_a \varepsilon f_{a0})$ and $\nu_a/\Omega_a = \varepsilon \nu_a/\omega_{Ta} = \mathscr{O}(\varepsilon)$ are used. Here, f_{a0} and f_{a1} are the zeroth and first-order distribution functions [see Eq. (29)] in the expansion with respect to the normalized gyroradius parameter ε given by the ratio of the gyroradius ρ_a to the equilibrium scale length L. As for the ratio ν_a/ω_{Ta} of the collision frequency ν_a to the transit frequency $\omega_{Ta} \sim L/v_{Ta}$ [$v_{Ta} \equiv (2T_a/m_a)^{1/2}$: the thermal velocity], we do not consider a subsidiary ordering such as those used in the Pfirsh-Schlüter, plateau, and banana regimes.[29–31]

The Boltzmann equation for the distribution function $f_a^{(p)}$ of the particle species a in the particle coordinates $\mathbf{z} \equiv (\mathbf{x}, v_\parallel, \mu_0, \xi_0)$ is written as

$$\frac{df_a^{(p)}}{dt} \equiv \left(\frac{\partial}{\partial t} + \frac{d\mathbf{z}}{dt}\cdot\frac{\partial}{\partial \mathbf{z}}\right) f_a^{(p)} = C_a^{(p)}. \tag{8}$$

The particle density $n_a^{(p)}$ and the particle flow $\Gamma_a^{(p)}$ are defined as functions of the position $\mathbf{x}$ and the time t by

$$n_a^{(p)}(\mathbf{x}, t) = \int d^6z'\, \delta^3(\mathbf{x}' - \mathbf{x}) D_a^{(p)}(\mathbf{x}', t) f_a^{(p)}(\mathbf{z}', t) \tag{9}$$

and

$$\Gamma_a^{(p)}(\mathbf{x}, t) = \int d^6z'\, \delta^3(\mathbf{x}' - \mathbf{x}) D_a^{(p)}(\mathbf{x}', t) f_a^{(p)}(\mathbf{z}', t)\mathbf{v}', \tag{10}$$

respectively, where the Jacobian $D_a^{(p)}(\mathbf{x}, t) \equiv B(\mathbf{x}, t)/m_a$ is used.

Multiplying Eq. (8) by $D_a^{(p)}$ and integrating it with respect to the velocity space variables $v_\parallel$, $\mu_0 \equiv m_a v_\perp^2/(2B)$, and ξ_0, we obtain the continuity equation

$$\frac{\partial n_a^{(p)}(\mathbf{x}, t)}{\partial t} + \nabla\cdot\Gamma_a^{(p)}(\mathbf{x}, t) = 0, \tag{11}$$

where the particle number conservation in collisions, $\int d^6z'\,\delta(\mathbf{x} - \mathbf{x}') \times D_a^{(p)}(\mathbf{x}', t) C_a^{p}(\mathbf{z}', t) = 0$, is used. Similarly, multiplying Eq. (1) by $D_a(\mathbf{Z}, t) \equiv B_{a\parallel}^*(\mathbf{Z}, t)/m_a$ [see Eq. (A31) for the definition of $B_{a\parallel}^*$] and integrating it with respect to the velocity space variables U, μ, and ξ, we obtain

$$\begin{aligned}\frac{\partial n_a^{(g)}(\mathbf{x}, t)}{\partial t} + \nabla\cdot\Gamma_a^{(g)}(\mathbf{x}, t) &= \int d^6Z'\, \delta^3(\mathbf{X}' - \mathbf{x}) D_a(\mathbf{Z}', t) C_a^{(g)}(\mathbf{Z}', t)\\ &= -\nabla\cdot\Gamma_a^C(\mathbf{x}, t),\end{aligned} \tag{12}$$

where the gyrocenter density $n_a^{(g)}$ and the gyrocenter flow $\Gamma_a^{(g)} \equiv n_a^{(g)}\mathbf{u}_a^{(g)}$ are defined by

$$n_a^{(g)}(\mathbf{x}, t) = \int d^6Z\, D_a(\mathbf{Z}, t) f_a(\mathbf{Z}, t)\delta^3(\mathbf{X} - \mathbf{x}) \tag{13}$$

and

$$\begin{aligned}\Gamma_a^{(g)}(\mathbf{x}, t) &\equiv n_a^{(g)}\mathbf{u}_a^{(g)}(\mathbf{x}, t)\\ &\equiv \int d^6Z\, D_a(\mathbf{Z}, t) f_a(\mathbf{Z}, t)\delta^3(\mathbf{X} - \mathbf{x})\frac{d\mathbf{X}}{dt},\end{aligned} \tag{14}$$

respectively. The gyrocenter velocity $d\mathbf{X}/dt$ which enters the integrand in Eq. (14) is regarded as a function of $(\mathbf{Z}, t)$ using Eq. (A27). As shown in Ref. 34, Γ_a^C on the right-hand side of Eq. (12) is given by

$$\begin{aligned}\Gamma_a^C(\mathbf{x}, t) \equiv \sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\Bigg(\int d^6z'\,\delta^3(\mathbf{x}' - \mathbf{x})\\ \cdot D_a^{(p)}\sum_b C_{ab}^{(p)}\left[\mathscr{T}_a^* f_a, \mathscr{T}_b^* f_b\right]\Delta\mathbf{x}_a\,\Delta x_a^{j_1}\cdots\Delta x_a^{j_l}\Bigg),\end{aligned} \tag{15}$$

where $\Delta\mathbf{x}_a \equiv \mathbf{X} - \mathbf{x}$ is defined as a function of $\mathbf{z}$ using Eq. (A6) and Δx_a^j is its jth component. As seen later in Eq. (60), the classical particle transport is derived from Γ_a^C.

The particle density $n_a^{(p)}$ and the gyrocenter density $n_a^{(g)}$ are related to each other by

$$e_a\, n_a^{(p)} = e_a\, n_a^{(g)} - \nabla\cdot\mathbf{P}_a, \tag{16}$$

where $\mathbf{P}_a$ is the polarization vector due to the particle species a, and its detailed expression is presented later in Eq. (22). The polarization current due to the particle species a is given by

$$\mathbf{J}_a^{\rm pol} \equiv e_a\,\Gamma_a^{\rm pol} \equiv \frac{\partial\mathbf{P}_a}{\partial t}, \tag{17}$$

where $\Gamma_a^{\rm pol}$ represents the polarization particle flow of the species a.

It is shown in Sec. III that the particle flow $\Gamma_a^{(p)}$ is written as

$$\Gamma_a^{(p)} = \Gamma_a^{(g)} + \Gamma_a^{\rm pol} + \Gamma_a^{\rm mag} + \Gamma_a^{C*}, \tag{18}$$

where Γ_a^{C*} is defined later in Eq. (25) and it satisfies $\nabla\cdot\Gamma_a^{C*} = \nabla\cdot\Gamma_a^C$. Here, $\Gamma_a^{\rm mag}$ represents the particle flow due to the magnetization which is defined by

$$\mathbf{J}_a^{\rm mag} \equiv e_a\,\Gamma_a^{\rm mag} \equiv c\,\nabla\times\mathbf{M}_a, \tag{19}$$

where $\mathbf{M}_a$ and $\mathbf{J}_a^{\rm mag}$ are the magnetization vector and the magnetization current density due to the particle species a, respectively. The detailed expression of $\mathbf{M}_a$ is shown later in Eq. (28). Subtracting Eq. (12) from Eq. (11), we have

$$\frac{\partial(n_a^{(p)} - n_a^{(g)})}{\partial t} + \nabla\cdot(\Gamma_a^{(p)} - \Gamma_a^{(g)}) = \nabla\cdot\Gamma_a^C. \tag{20}$$

We can easily verify that Eq. (20) is satisfied by Eqs. (16)–(19).

III. POLARIZATION AND MAGNETIZATION FLOWS

Performing the transformation from the particle coordinates to the gyrocenter coordinates for the integration in Eq. (9), we obtain

$$\begin{aligned}n_a^{(p)}(\mathbf{x}, t) &= \int d^6Z\, D_a(\mathbf{Z}, t) f_a(\mathbf{Z}, t)\delta^3[\mathbf{X} + \rho_a(\mathbf{Z}, t) - \mathbf{x}]\\ &= n_a^{(g)}(\mathbf{x}, t) - \nabla\cdot\left[e_a^{-1}\mathbf{P}_a(\mathbf{x}, t)\right],\end{aligned} \tag{21}$$

where the gyroradius vector ρ_a is defined by Eqs. (A10)–(A17) in Appendix A and the polarization vector $\mathbf{P}_a(\mathbf{x}, t)$ is given by

$$\frac{1}{e_a}\mathbf{P}_a(\mathbf{x},t) = \sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\left(\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x}) \times D_a(\mathbf{Z},t)f_a(\mathbf{Z},t)\,\rho_a\,\rho_a^{j_1}\cdots\rho_a^{j_l}\right). \tag{22}$$

The jth components of the vectors $\mathbf{x}$ and ρ_a are denoted by x^j and ρ_a^j, respectively. Here and hereafter, we employ the summation convention that the same symbol used for a pair of indices in upper and lower positions within a term [such as in Eq. (22)] indicates summation over the range {1, 2, 3} of the symbol index. In deriving Eqs. (21) and (22), the Taylor expansion

$$\delta^3(\mathbf{X}+\rho_a-\mathbf{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\rho_a^{j_1}\cdots\rho_a^{j_l}\frac{\partial^l\delta^3(\mathbf{X}-\mathbf{x})}{\partial x^{j_1}\cdots\partial x^{j_l}} \tag{23}$$

is used and partial integrations are performed. Taking the partial time derivative of Eq. (22) and using Eq. (1), we find that the polarization flow $\Gamma_a^{\rm pol}$ due to the particle species a is given by

$$\begin{aligned}\Gamma_a^{\rm pol} &\equiv \frac{1}{e_a}\frac{\partial\mathbf{P}_a(\mathbf{x},t)}{\partial t} \\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\left[\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})\right. \\ &\quad\times\left\{D_af_a\left(\frac{d\rho_a}{dt}\rho_a^{j_1}\cdots\rho_a^{j_l}+l\,\rho_a\frac{d\rho_a^{j_1}}{dt}\rho_a^{j_2}\cdots\rho_a^{j_l}\right)\right. \\ &\quad\left.\left.-\frac{\partial}{\partial\mathbf{X}}\cdot\left(D_af_a\frac{d\mathbf{X}}{dt}\rho_a\,\rho_a^{j_1}\cdots\rho_a^{j_l}\right)\right\}\right]-\Gamma_a^{C*},\end{aligned} \tag{24}$$

where Γ_a^{C*} is defined by

$$\Gamma_a^{C*}(\mathbf{x},t) \equiv \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{(l+1)!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\left(\int d^6Z'\,\delta^3(\mathbf{X}'-\mathbf{x}) \cdot D_a\sum_b C_{ab}^{(g)}[f_a,f_b]\,\rho_a\,\rho_a^{j_1}\cdots\rho_a^{j_l}\right). \tag{25}$$

It can be shown from Eqs. (15) and (25) that $\nabla\cdot\Gamma_a^{C*}-\nabla\cdot\Gamma_a^{C} = \int d^6z'\delta^3(\mathbf{x}'-\mathbf{x})D_a^{(p)}C_a^{(p)} = 0$ and accordingly $\nabla\cdot\Gamma_a^{C*} = \nabla\cdot\Gamma_a^{C}$. In addition, as seen in Sec. IV, both Γ_a^{C*} and Γ_a^{C} are of $\mathscr{O}(\varepsilon^2)$ and their ensemble averages coincide with each other and represent the classical particle transport.

The particle flow $\Gamma_a^{(p)}$ defined in Eq. (10) is also given by the integration in the gyrocenter coordinates as

$$\begin{aligned}\Gamma_a^{(p)}(\mathbf{x},t) &\equiv n_a^{(p)}\mathbf{u}_a^{(p)}(\mathbf{x},t) \\ &\equiv \int d^6Z D_a(\mathbf{Z},t)f_a(\mathbf{Z},t)\delta^3(\mathbf{X}+\rho_a-\mathbf{x})\left(\frac{d\mathbf{X}}{dt}+\frac{d\rho_a}{dt}\right),\end{aligned} \tag{26}$$

where the particle velocity is represented by $d\mathbf{X}/dt + d\rho_a/dt$, which is regarded as a function of $(\mathbf{Z}, t)$, using Eq. (A27) in Appendix A and Eqs. (B6)–(B9) in Appendix B. Then we can use Eqs. (14), (15), and (22)–(24) to derive Eq. (18) which is written here as

$$\Gamma_a^{(p)}(\mathbf{x},t) = \Gamma_a^{(g)}(\mathbf{x},t)+\frac{1}{e_a}\frac{\partial\mathbf{P}_a(\mathbf{x},t)}{\partial t}+\frac{c}{e_a}\nabla\times\mathbf{M}_a(\mathbf{x},t)+\Gamma_a^{C*}(\mathbf{x},t), \tag{27}$$

where $\Gamma_a^{\rm mag} \equiv (c/e_a)\nabla\times\mathbf{M}_a(\mathbf{x},t)$ is the particle flow due to the magnetization vector $\mathbf{M}_a$ defined by

$$\frac{c}{e_a}\mathbf{M}_a(\mathbf{x},t) \equiv \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\left[\int d^6Z\,D_af_a\delta^3(\mathbf{X}-\mathbf{x}) \cdot\rho_a^{j_1}\cdots\rho_a^{j_l}\rho_a\times\left(\frac{1}{(l+2)}\frac{d\rho_a}{dt}+\frac{1}{(l+1)}\frac{d\mathbf{X}}{dt}\right)\right]. \tag{28}$$

IV. EXPANSION OF PARTICLE FLOWS IN THE NORMALIZED GYRORADIUS PARAMETER ε

We here first expand the gyrocenter distribution function in the normalized gyroradius parameter ε as

$$f_a(\mathbf{Z},t) = f_{a0}(\mathbf{Z},t)+f_{a1}(\mathbf{Z},t)+f_{a2}(\mathbf{Z},t)+\cdots, \tag{29}$$

where the subscripts $n = 0, 1, 2, \ldots$ represent the terms of $\mathscr{O}(\varepsilon^n)$. More precisely speaking, $f_{an} = \mathscr{O}(\varepsilon^n)$ implies that the magnitude of f_n is represented by $f_{an} = \mathscr{O}(\varepsilon^n f_{a0})$.

The gyrocenter distribution function is also written as the sum of the ensemble average part and the fluctuation part

$$f_a = \langle f_a\rangle_{\rm ens}+\hat{f}_a. \tag{30}$$

We denote the average and fluctuation parts of the magnetic field by $\mathbf{B} = \nabla\times\mathbf{A}$ and $\hat{\mathbf{B}} = \nabla\times\hat{\mathbf{A}}$, respectively. The ensemble average is used as the basic method of statistical mechanics to obtain the macroscopic mean values of physical valuables. For the case of gyrokinetic turbulence simulation, an ensemble literally corresponds to a group of a large number of simulations performed using many different sets of randomly given initial perturbations while being done for the same macroscopic state (characterized by the same conditions for background profiles of densities, temperatures, and electromagnetic fields), and the ensemble average of some variable is defined by the average of its values obtained from the repeatedly performed simulations. However, assuming that a single typical nonlinear gyrokinetic simulation shows ergodic behavior wandering among a large number of microscopic turbulent states which will be produced by the ensemble of simulations, the ensemble average is considered to equal the local space–time average obtained in the single simulation. This local space–time averaging of the distribution and other field functions in gyrokinetic systems is in detail described in Ref. 35, which shows the same results as given in Ref. 36 using the notation of the ensemble average.

We note here that the gyrophase average should be clearly distinguished from the local space average related to the ensemble average. The ensemble average can be replaced by the space–time average over scales, which are much smaller than macroscopic scales but sufficiently larger than microscopic fluctuation scales.[35] For example, for the fluctuation potential $\phi(\mathbf{x}) = \phi_{\mathbf{k}_\perp}\exp(i\mathbf{k}_\perp\cdot\mathbf{x})$ with the perpendicular wavenumber vector $\mathbf{k}_\perp(k_\perp\sim\rho^{-1})$, the local space average of $\phi(\mathbf{x})$ over the scale $l\ (\rho\ll l\ll L)$ in the plane perpendicular to the background magnetic field vanishes. On the other hand, the gyrophase average $\langle\cdots\rangle_\xi$ of the fluctuating potential is given by $\langle\phi(\mathbf{X}+\rho)\rangle_\xi = J_0(k_\perp\rho)\phi_{\mathbf{k}}\exp(i\mathbf{k}_\perp\cdot\mathbf{X})$ which shows that the gyrophase average does not completely remove the fluctuation but weakens it by the factor $J_0(k_\perp\rho) = \langle\exp(i\mathbf{k}_\perp\cdot\rho)\rangle_\xi$ [which is derived from the formula, $(2\pi)^{-1}\oint\exp(ix\sin\theta)d\theta = J_0(x)$].

As seen in Eq. (30), the fluctuation part of the distribution function is given as the deviation from the ensemble average. We now recall that, in the present work using the modern gyrokinetic formulation, the gyrocenter coordinates $\mathbf{Z}$ in $f_a = f_a(\mathbf{Z}, t)$ are defined from the particle coordinates $\mathbf{z}$ with effects of the electromagnetic fluctuations taken into account [see Eqs. (A6)–(A9)]. On the other hand, in the classical gyrokinetic formulation[25–27] using the WKB representation (see Appendix C) for the fluctuating parts of the distribution function and electromagnetic fields, the particle phase-space coordinates used as independent variables of the distribution function are defined without including effects of the fluctuations. Then, due to the difference between the two sets of the phase-space coordinates, the fluctuation part of the distribution function in the modern gyrokinetic formulation differs from that in the classical formulation [see Eq. (C23) in Appendix C where $\widehat{f}^{(p)}_{a1}$ and $\widehat{f}_{a1}$ correspond to the fluctuation parts of the distribution functions in the classical and modern formulations, respectively].

In the rest of this section, the expansion in ε [Eq. (29)] and the decomposition into the average and fluctuation parts [Eq. (30)] are employed to analyze various components, which compose the particle flow [Eq. (27)]. It is noted that, even in the case without microscopic fluctuations, the expansion of the distribution function in Eq. (29) is used in the drift kinetic theory[29–31] where the neoclassical transport fluxes are calculated from the first-order distribution function given as the solution of the drift kinetic equation [see Eq. (C5)]. In the gyrokinetic theory, small amplitudes of fluctuations of $\mathscr{O}(\varepsilon)$ are assumed so that the fluctuation parts appear from the first order as seen below.

A. Zeroth-order flows

The zeroth-order part f_{a0} of the distribution function f_a in ε is considered to represent the equilibrium part which contains no fluctuations, and we accordingly write

$$f_{a0} = \langle f_{a0}\rangle_{\rm ens}, \quad \widehat{f}_{a0} = 0. \tag{31}$$

The zeroth-order density $n^{(g)}_{a0}$ is given by

$$n^{(g)}_{a0}(\mathbf{x}, t) \equiv \int d^6Z\, D_{a0}(\mathbf{X}, t) f_{a0}(\mathbf{Z}, t)\delta^3(\mathbf{X}-\mathbf{x}), \tag{32}$$

where D_{a0} represents the zeroth-order Jacobian given by

$$D_{a0}(\mathbf{X}, t) = \frac{B(\mathbf{X}, t)}{m_a}. \tag{33}$$

The zeroth-order part $(d\mathbf{X}/dt)_0$ of the gyrocenter velocity $d\mathbf{X}/dt$ is given by Eq. (B1), and it has only the component parallel to the background magnetic field.

Noting that f_{a0} is independent of the gyrophase angle ξ and using Eqs. (24), (28), (B1), and (B6), we have

$$\left(\frac{1}{e_a}\frac{\partial \mathbf{P}_a(\mathbf{x}, t)}{\partial t}\right)_0 = \int d^6Z\, D_{a0} f_{a0}\delta^3(\mathbf{X}-\mathbf{x})\left(\frac{d\rho_a}{dt}\right)_0 = 0, \tag{34}$$

and

$$\left(\frac{c}{e_a}\mathbf{M}_a(\mathbf{x}, t)\right)_0 = 0. \tag{35}$$

Thus, the polarization and magnetization never produce particle flows of $\mathscr{O}(n_{a0}v_{Ta})$. From Eqs. (15) and (25), we also have

$$\Gamma^C_{a0}(\mathbf{x}, t) = \Gamma^{C*}_{a0}(\mathbf{x}, t) = 0. \tag{36}$$

In the present work, we use the low-flow ordering in which the lowest-order flow velocity is on the order of $\mathscr{O}(\varepsilon v_{Ta})$. This means that the zeroth-order particle flow vanishes

$$\Gamma^{(p)}_{a0}(\mathbf{x}, t) = 0 \tag{37}$$

and the zeroth-order gyrocenter flow given by f_{a0} also vanishes

$$\begin{aligned}\Gamma^{(g)}_{a0}(\mathbf{x}, t) &\equiv n^{(g)}_{a0}\mathbf{u}^{(g)}_{a0}(\mathbf{x}, t)\\ &\equiv \int d^6Z\, D_{a0}(\mathbf{Z}, t) f_{a0}(\mathbf{Z}, t)\delta^3(\mathbf{X}-\mathbf{x})\left(\frac{d\mathbf{X}}{dt}\right)_0 = 0.\end{aligned} \tag{38}$$

B. First-order flows

In the first-order in ε, the gyrocenter distribution function generally consists of ensemble average and fluctuation parts

$$f_{a1} = \langle f_{a1}\rangle_{\rm ens} + \widehat{f}_{a1}. \tag{39}$$

In the same way, the first-order particle and gyrocenter flows are written as

$$\Gamma^{(p)}_{a1}(\mathbf{x}, t) = \langle \Gamma^{(p)}_{a1}(\mathbf{x}, t)\rangle_{\rm ens} + \widehat{\Gamma}^{(p)}_{a1}(\mathbf{x}, t) \tag{40}$$

and

$$\Gamma^{(g)}_{a1}(\mathbf{x}, t) = \langle \Gamma^{(g)}_{a1}(\mathbf{x}, t)\rangle_{\rm ens} + \widehat{\Gamma}^{(g)}_{a1}(\mathbf{x}, t), \tag{41}$$

respectively. As explained in Appendix C, the collision term vanishes to the zeroth order in ε, and it is regarded as of the first order. Then we see from Eqs. (15) and (25) that $\mathbf{\Gamma}^C_a$ and $\mathbf{\Gamma}^{C*}_a$ are of $\mathscr{O}(\varepsilon^2 n_{a0}v_{Ta})$ (n_{a0}: the background particle density) and

$$\mathbf{\Gamma}^C_{a1} = \mathbf{\Gamma}^{C*}_{a1} = 0. \tag{42}$$

1. Ensemble-averaged part

The first-order ensemble-averaged gyrocenter flow is written as

$$\begin{aligned}\langle\Gamma^{(g)}_{a1}(\mathbf{x}, t)\rangle_{\rm ens} = \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})\Bigg[&D_{a0}f_{a0}\left\langle\left(\frac{d\mathbf{X}}{dt}\right)_1\right\rangle_{\rm ens}\\ &+(D_{a0}\langle f_{a1}\rangle_{\rm ens} + D_{a1}f_{a0})\left(\frac{d\mathbf{X}}{dt}\right)_0\Bigg],\end{aligned} \tag{43}$$

where $(d\mathbf{X}/dt)_0$ and $\langle(d\mathbf{X}/dt)_1\rangle_{\rm ens}$ are given as functions of $(\mathbf{Z}, t)$ as shown in by Eqs. (B1) and (B3), respectively. It is found from Eq. (22) that the first-order polarization flow vanishes

$$\langle\Gamma^{\rm pol}_{a1}(\mathbf{x}, t)\rangle_{\rm ens} \equiv \left\langle\left(\frac{1}{e_a}\frac{\partial \mathbf{P}_a(\mathbf{x}, t)}{\partial t}\right)_1\right\rangle_{\rm ens} = 0. \tag{44}$$

From Eq. (28), we obtain

$$\left\langle\left(\frac{c}{e_a}\mathbf{M}_a(\mathbf{x}, t)\right)_1\right\rangle_{\rm ens} = -\frac{c}{e_aB}(P_{a\perp})_0\mathbf{b} \tag{45}$$

and the first-order magnetization flow

$$\langle \Gamma_{a1}^{\rm mag}(\mathbf{x},t)\rangle_{\rm ens} \equiv \nabla\times\left\langle\left(\frac{c}{e_a}\mathbf{M}_a\right)_1\right\rangle_{\rm ens} = -\nabla\times\left(\frac{c}{e_aB}(P_{a\perp})_0\mathbf{b}\right), \tag{46}$$

where

$$(P_{a\perp})_0 \equiv \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}f_{a0}\,\mu B. \tag{47}$$

Using Eqs. (27), (42)–(44), and (46), the total first-order ensemble-averaged particle flow is written as

$$\begin{aligned}\langle\Gamma_{a1}^{(p)}(\mathbf{x},t)\rangle_{\rm ens} &= \langle\Gamma_{a1}^{(g)}\rangle_{\rm ens}+\langle\Gamma_{a1}^{\rm mag}\rangle_{\rm ens}\\ &= \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\langle f_{a1}\rangle_{\rm ens}U\mathbf{b}+\frac{n_{a0}}{B}\langle\mathbf{E}_1\rangle_{\rm ens}\times\mathbf{b}\\ &\quad+\frac{c}{e_aB}\left[\mathbf{b}\times\nabla(P_{a\perp})_0+\{(P_{a\parallel})_0-(P_{a\perp})_0\}(\nabla\times\mathbf{b})\right]\\ &= \mathbf{b}\left[\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\langle f_{a1}\rangle_{\rm ens}U\right.\\ &\quad\left.+\frac{c}{e_aB}\{(P_{a\parallel})_0-(P_{a\perp})_0\}(\mathbf{b}\cdot\nabla\times\mathbf{b})\right]\\ &\quad+\frac{c}{e_aB}\left[n_{a0}e_a\langle\mathbf{E}_1\rangle_{\rm ens}-\nabla\cdot\{(P_{a\parallel})_0\mathbf{bb}\right.\\ &\quad\left.+(P_{a\perp})_0(\mathbf{I}-\mathbf{bb})\}\right]\times\mathbf{b},\end{aligned} \tag{48}$$

where

$$(P_{a\parallel})_0 \equiv \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}f_{a0}\frac{1}{2}m_aU^2. \tag{49}$$

In a case where, as described in Appendix C1, f_{a0} takes the form of the local Maxwellian distribution with no mean flow, the zeroth-order pressure is isotropic so that we can write $(P_{a\parallel})_0=(P_{a\perp})_0=P_{a0}$. Equation (48) agrees with the magnetization law in drift kinetics.[24]

Within accuracy up to $\mathscr{O}(\varepsilon n_{a0}v_{Ta})$, Eq. (48) is rewritten more compactly as

$$\langle\Gamma_{a1}^{(p)}(\mathbf{x},t)\rangle_{\rm ens} = \int d^6Z\,\delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})\times\left[D_a\langle f_a\rangle_{\rm ens}\mathbf{v}_c+D_{a0}f_{a0}\mathbf{v}_{da}\right], \tag{50}$$

where $\mathbf{v}_c$ and $\mathbf{v}_{da}$ are given by Eqs. (A13) and (B3), respectively, and ρ_{a1} represents the lowest-order (or first-order) expression of the gyroradius vector shown in Eq. (A12). In the first term of the integrand on the right-hand side of Eq. (50), we need to use $D_a=D_{a0}+D_{a1}$ and $f_a=f_{a0}+f_{a1}$ in order to keep the validity up to $\mathscr{O}(\varepsilon n_{a0}v_{Ta})$.

2. Turbulent part

The first-order turbulent gyrocenter flow is given from Eq. (14) as

$$\widehat{\Gamma}_{a1}^{(g)}(\mathbf{x},t)=\int d^6Z\,D_0\delta^3(\mathbf{X}-\mathbf{x})(\widehat{f}_{a1}U\mathbf{b}+f_{a0}\widehat{\mathbf{v}}_{ga}), \tag{51}$$

where the first-order turbulent gyrocenter velocity $\widehat{\mathbf{v}}_{ga}$ is given by Eq. (B4). The first-order turbulent polarization flow is derived from Eq. (24) as

$$\begin{aligned}\widehat{\Gamma}_{a1}^{\rm pol}(\mathbf{x},t) &= \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{(l+1)!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\left[\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\right.\\ &\quad\cdot\left\{\rho_{a1}^{j_1}\cdots\rho_{a1}^{j_l}f_{a0}\left(\frac{e_a}{m_ac}\widehat{\mathbf{A}}_\perp+\frac{c}{B}\mathbf{b}\times\nabla\widehat{\psi}_a\right)\right.\\ &\quad\left.\left.-\rho_{a1}^{j_1}\cdots\rho_{a1}^{j_{l-1}}\frac{e_a}{B}\widetilde{\widehat{\psi}}_a\frac{\partial f_{a0}}{\partial\mu}\left(\rho_{a1}^{j_l}\mathbf{v}_{c\perp}+l(v_{c\perp})^{j_l}\rho_{a1}\right)\right\}\right],\end{aligned} \tag{52}$$

where ρ_{a1}^j is the jth component of ρ_{a1}. On the right-hand side of Eq. (52), $\widetilde{\widehat{\psi}}_a\equiv\widehat{\psi}_a-\langle\widehat{\psi}_a\rangle_\xi$ is the gyrophase-dependent part of $\widehat{\psi}_a\equiv\psi_a-\langle\psi_a\rangle_{\rm ens}\equiv\widehat{\phi}-c^{-1}\mathbf{v}_c\cdot\widehat{\mathbf{A}}$ where $\widehat{\phi}$ and $\widehat{\mathbf{A}}$ should be evaluated at $\mathbf{X}+\rho_{a1}$. The first-order turbulent magnetization flow is derived from Eq. (28) as

$$\begin{aligned}\widehat{\Gamma}_{a1}^{\rm mag}(\mathbf{x},t) &= \left(\frac{c}{e_a}\nabla\times\mathbf{M}_a\right)_1\\ &= \sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\left(\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\right.\\ &\quad\cdot\left[\rho_{a1}^{j_1}\cdots\rho_{a1}^{j_l}\left\{\widehat{f}_{a1}\mathbf{v}_c-f_{a0}\frac{e_a}{m_ac}\left(\widehat{A}_\parallel\mathbf{b}+\frac{l}{(l+1)}\widehat{\mathbf{A}}_\perp\right)\right.\right.\\ &\quad\left.+\frac{1}{(l+1)}f_{a0}\frac{c}{B}\mathbf{b}\times\nabla\widehat{\psi}_a\right\}+\rho_{a1}^{j_1}\cdots\rho_{a1}^{j_{l-1}}\frac{e_a}{B}\widetilde{\widehat{\psi}}_a\frac{\partial f_{a0}}{\partial\mu}\\ &\quad\left.\left.\cdot\left\{\rho_{a1}^{j_l}U\mathbf{b}+\frac{l}{l+1}\left(\rho_{a1}^{j_l}\mathbf{v}_{c\perp}-(v_{c\perp})^{j_l}\rho_{a1}\right)\right\}\right]\right).\end{aligned} \tag{53}$$

Then, using Eqs. (51)–(53), the first-order turbulent particle flux is written as

$$\begin{aligned}\widehat{\Gamma}_{a1}^{(p)}(\mathbf{x},t) &\equiv \widehat{\Gamma}_{a1}^{(g)}+\widehat{\Gamma}_{a1}^{\rm pol}+\widehat{\Gamma}_{a1}^{\rm mag}\\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{\partial^l}{\partial x^{j_1}\cdots\partial x^{j_l}}\left[\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\right.\\ &\quad\left.\cdot\rho_{a1}^{j_1}\cdots\rho_{a1}^{j_l}\left\{\widehat{f}_{a1}\mathbf{v}_c+\left(-f_{a0}\frac{e_a}{m_ac}\widehat{\mathbf{A}}+\frac{e_a\widetilde{\widehat{\psi}}_a}{B}\frac{\partial f_{a0}}{\partial\mu}\mathbf{v}_c\right)\right\}\right]\\ &= \int d^6Z\,\delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})D_{a0}\\ &\quad\cdot\left[\widehat{f}_{a1}\mathbf{v}_c+\left(-f_{a0}\frac{e_a}{m_ac}\widehat{\mathbf{A}}+\frac{e_a\widetilde{\widehat{\psi}}_a}{B}\frac{\partial f_{a0}}{\partial\mu}\mathbf{v}_c\right)\right]+\mathscr{O}(\varepsilon^2n_{a0}v_{Ta}).\end{aligned} \tag{54}$$

Summing up Eqs. (50) and (54), we obtain the expression of the first-order particle flow, which is valid up to $\mathscr{O}(\varepsilon n_{a0}v_{Ta})$, as

$$\begin{aligned}\Gamma_a^{(p)}(\mathbf{x},t) &= \langle\Gamma_a^{(p)}(\mathbf{x},t)\rangle_{\rm ens}+\widehat{\Gamma}_a^{(p)}(\mathbf{x},t)\\ &= \int d^6Z\,\delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})\left[D_a(\mathbf{Z},t)f_a(\mathbf{Z},t)\mathbf{v}_c\right.\\ &\quad\left.+D_{a0}\left\{f_{a0}\left(\mathbf{v}_{Ba}-\frac{e_a}{m_ac}\widehat{\mathbf{A}}\right)+\frac{e_a\widetilde{\psi}_a}{B}\frac{\partial f_{a0}}{\partial\mu}\mathbf{v}_c\right\}\right],\end{aligned} \tag{55}$$

where $\mathbf{v}_{Ba}$ is defined by Eq. (A25). In the same way as in Eq. (50), $D_a=D_{a0}+D_{a1}$ and $f_a=f_{a0}+f_{a1}$ should be used in the first term of the integrand on the right-hand side of Eq. (55), in order to keep the validity up to $\mathscr{O}(\varepsilon n_{a0} v_{Ta})$.

C. Second-order flows

When considering particle confinement of magnetically confined plasmas on the transport time scale of $(\varepsilon^2\omega_{Ta})^{-1}$, it is important to evaluate the ensemble-averaged or mean particle flux across the surface formed by field lines. We find from Eq. (22) that the second-order ensemble-averaged polarization flow vanishes

$$\left\langle \Gamma_{a2}^{\rm pol}(\mathbf{x}, t)\right\rangle_{\rm ens} \equiv \left\langle \left(\frac{1}{e_a}\frac{\partial \mathbf{P}_a}{\partial t}\right)_2\right\rangle_{\rm ens} = 0 \tag{56}$$

as well as the zeroth- and first-order parts shown in Eqs. (34) and (44).

For plasmas confined in the toroidal magnetic configuration where the zeroth-order equilibrium distribution function F_{a0} is given by the Maxwellian with no mean flow, we see from Eq. (28) that the second-order ensemble-averaged magnetization flow is given by

$$\langle \Gamma_{a2}^{\rm mag}(\mathbf{x}, t)\rangle_{\rm ens} \equiv \left\langle \left(\frac{c}{e_a}\nabla\times\mathbf{M}_a\right)_2\right\rangle_{\rm ens} = -\nabla\times\left(\frac{c}{e_aB}(P_\perp)_{a1}\mathbf{b}\right), \tag{57}$$

where $(P_\perp)_{a1} \equiv \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\langle f_{a1}\rangle_{\rm ens}\mu B$. For this Maxwellian equilibrium distribution function f_{a0}, we have the scalar equilibrium pressure $P_{a0} = (P_{a\parallel})_0 = (P_{a\perp})_0$ and the average electrostatic potential $\langle\phi\rangle_{\rm ens}$, which are given as flux surface functions, as explained after Eq. (C4) in Appendix C. Then the first-order ensemble-averaged particle flow in Eq. (50) is rewritten as

$$\langle \Gamma_{a1}^{(p)}(\mathbf{x}, t)\rangle_{\rm ens} = \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\langle f_{a1}\rangle_{\rm ens}U\mathbf{b} + \frac{c}{e_aB}(n_{a0}e_a\langle\mathbf{E}_1\rangle_{\rm ens} - \nabla P_{a0})\times\mathbf{b}, \tag{58}$$

which has no component in the radial direction perpendicular to the magnetic flux surface, because $\langle\mathbf{E}_1\rangle_{\rm ens} = -\nabla\langle\phi\rangle_{\rm ens}$ and ∇P_{a0} are both perpendicular to the surface. Therefore, the mean radial particle flow is of $\mathscr{O}(\varepsilon^2 n_{a0}v_{Ta})$, which is consistent with the ordering of the transport timescale given by $(\varepsilon^2\omega_{Ta})^{-1}$.

The second-order ensemble-averaged gyrocenter flow is obtained from Eq. (14) as

$$\langle \Gamma_{a2}^{(g)}(\mathbf{x}, t)\rangle_{\rm ens} = \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\left[f_{a0}\mathbf{v}_{da2} + \langle f_{a1}\rangle_{\rm ens}\mathbf{v}_{da} + \langle \widehat{f}_{a1}\,\widehat{\mathbf{v}}_{ga}\rangle_{\rm ens} + \left\langle f_{a2} + \frac{D_{a1}}{D_{a0}}f_{a1}\right\rangle_{\rm ens}U\mathbf{b}\right], \tag{59}$$

where $\mathbf{v}_{da}$, $\widehat{\mathbf{v}}_{ga}$, and $\mathbf{v}_{da2}$ are given by Eqs. (B3), (B4), and (B5), respectively.

The remaining part of the second-order ensemble-averaged particle flow is derived using Eqs. (15) and (25) as

$$\begin{aligned}\langle \Gamma_{a2}^{C}(\mathbf{x}, t)\rangle_{\rm ens} &= \langle \Gamma_{a2}^{C*}(\mathbf{x}, t)\rangle_{\rm ens} \\ &= \int d^6z'\,D_a^{(p)}\langle(\widetilde{C}_a^{(p)})_1\rangle_{\rm ens}\delta^3(\mathbf{x}'-\mathbf{x})\frac{\mathbf{v}'\times\mathbf{b}}{\Omega_a} \\ &= \frac{c}{e_aB}\mathbf{F}_{a1}\times\mathbf{b} = \int d^6z'\,D_a^{(p)}\langle\widetilde{f}_{a2}\rangle_{\rm ens}\delta^3(\mathbf{x}'-\mathbf{x})\mathbf{v}'_\perp,\end{aligned} \tag{60}$$

where $\langle(\widetilde{C}_a^{(p)})_1\rangle_{\rm ens}$ is defined by

$$\langle(\widetilde{C}_a^{(p)})_1\rangle_{\rm ens} \equiv \sum_b\left\{C_{ab}^{(p)}\left[\left\langle\widetilde{f}_{a1}^{(p)}\right\rangle_{\rm ens}, f_{b0}\right] + C_{ab}^{(p)}\left[f_{a0}, \left\langle\widetilde{f}_{b1}^{(p)}\right\rangle_{\rm ens}\right]\right\}. \tag{61}$$

$\mathbf{F}_{a1}$ is the collisional friction force defined by

$$\mathbf{F}_{a1} \equiv \int d^6z'\,D_a^{(p)}\delta^3(\mathbf{x}'-\mathbf{x})\langle(C_a^{(p)})_1\rangle_{\rm ens}m_a\mathbf{v}' \tag{62}$$

and $\widetilde{f}_{a2}$ is obtained using Eq. (7). It is verified from Eq. (60) that $\langle\Gamma_{a2}^{C}\rangle_{\rm ens} = \langle\Gamma_{a2}^{C*}\rangle_{\rm ens}$ represents the classical collisional particle flow.[29–31]

As seen from Eqs. (18) and (56), the total second-order particle flow is given by the sum of the gyrocenter, magnetization, and classical particle flows

$$\langle\Gamma_{a2}^{(p)}(\mathbf{x}, t)\rangle_{\rm ens} = \langle\Gamma_{a2}^{(g)}\rangle_{\rm ens} + \langle\Gamma_{a2}^{\rm mag}\rangle_{\rm ens} + \langle\Gamma_{a2}^{C}\rangle_{\rm ens}. \tag{63}$$

It is recalled here that the tangential component of the mean particle flow to the magnetic flux surface is dominated by the first-order flow $\langle\Gamma_{a1}^{(p)}\rangle_{\rm ens}$ given in Eq. (48) although the normal component is of the second order. Now using Eqs. (28), (59), (60), and (63), the component of the second-order particle flow $\langle\Gamma_{a2}^{(p)}\rangle_{\rm ens}$ perpendicular to the background magnetic field line is given by

$$\begin{aligned}\langle\Gamma_{a\perp2}^{(p)}(\mathbf{x}, t)\rangle_{\rm ens} &= \langle\Gamma_{a\perp a2}^{(g)}\rangle_{\rm ens} + \langle\Gamma_{a\perp2}^{\rm mag}\rangle_{\rm ens} + \langle\Gamma_{a2}^{C}\rangle_{\rm ens} \\ &= \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\left[\langle f_{a1}\rangle_{\rm ens}\mathbf{v}_{da} + \langle\widehat{f}_{a1}\,(\widehat{\mathbf{v}}_{ga})_\perp\rangle_{\rm ens}\right] \\ &\quad -\left[\nabla\times\left(\frac{c}{eB}(P_{a\perp})_1\mathbf{b}\right)\right]_\perp + \frac{c}{e_aB}\left[n_{a0}e\langle\mathbf{E}_2\rangle_{\rm ens} + \mathbf{F}_{a1}\right]\times\mathbf{b} \\ &= \frac{c}{e_aB}\left[-\nabla\cdot\{(P_{a\parallel})_1\mathbf{bb} + (P_{a\perp})_1(\mathbf{I}-\mathbf{bb})\}\right. \\ &\quad + n_{a1}e_a\langle\mathbf{E}_1\rangle_{\rm ens} + n_{a0}e\langle\mathbf{E}_2\rangle_{\rm ens} + \mathbf{F}_{a1} \\ &\quad \left.-\frac{c}{B}\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\langle\widehat{f}_{a1}\,\nabla\langle\widehat{\psi}_a\rangle_\xi\rangle_{\rm ens}\right]\times\mathbf{b},\end{aligned} \tag{64}$$

where $n_{a1} \equiv \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\langle f_{a1}\rangle_{\rm ens}$, $(P_{a\parallel})_1 \equiv \int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})\times D_{a0}\langle f_{a1}\rangle_{\rm ens}m_aU^2$, $\langle\mathbf{E}_1\rangle_{\rm ens} = -\nabla\langle\phi_1\rangle_{\rm ens}$, and $\langle\mathbf{E}_2\rangle_{\rm ens} = -\nabla\langle\phi_2\rangle_{\rm ens} - c^{-1}\partial\mathbf{A}/\partial t$ are used. In toroidal confinement systems, the lowest-order ensemble-averaged electrostatic potential $\langle\phi_1\rangle_{\rm ens}$ is considered to be uniform over the magnetic flux surface. On the right-hand side of Eq. (64), the part including the anisotropic pressure tensor represents the neoclassical particle transport[29–31] while the turbulent particle transport is given by the last term including the correlation between the fluctuating distribution function and the gradient of the gyrophase-averaged fluctuating potential field.[37]

V. LAGRANGIAN FOR VARIATIONAL DERIVATION OF POISSON'S EQUATION AND AMPÈRE'S LAW

The action integral for the gyrokinetic Vlasov–Poisson–Ampère system is given by

$$I \equiv \int_{t_1}^{t_2} dt\,L_{GKF} \equiv \int_{t_1}^{t_2} dt\,(L_{GK} + L_F), \tag{65}$$

where the Lagrangian L_{GK} is written as

$$L_{GK} \equiv L_{GK0} + L_{GK1} + L_{GK2}. \tag{66}$$

Here, we use the gyrocenter distribution function f_a to define L_{GK0} and L_{GK1} by

$$\begin{aligned}\begin{bmatrix} L_{GK0} \\ L_{GK1} \end{bmatrix} &\equiv \sum_a \int d^6 Z_0\, D_a(\mathbf{Z}_0, t_0) f_a(\mathbf{Z}_0, t_0) \\ &\quad \times \begin{bmatrix} L_{GYa0}(\mathbf{Z}_a(t), \dot{\mathbf{Z}}_a(t), t) \\ L_{GYa1}(\mathbf{Z}_a(t), t) \end{bmatrix} \\ &\equiv \sum_a \int d^6 Z\, D_a(\mathbf{Z}, t) f_a(\mathbf{Z}, t) \begin{bmatrix} L_{GYa0}(\mathbf{Z}, \dot{\mathbf{Z}}, t) \\ L_{GYa1}(\mathbf{Z}, t) \end{bmatrix}, \end{aligned} \tag{67}$$

where the gyrocenter phase-space orbit for the particle species a is represented by $\mathbf{Z}_a(t)$, which satisfies the initial condition $\mathbf{Z}_a(t_0) = \mathbf{Z}_0$. The gyrocenter Lagrangian L_{GYa0} appearing in Eq. (67) is defined by

$$\begin{aligned} L_{GYa0}(\mathbf{Z}, \dot{\mathbf{Z}}, t) \equiv{}& \frac{e_a}{c} \mathbf{A}_a^*(\mathbf{X}, U, t) \cdot \dot{\mathbf{X}} + \frac{m_a c}{e_a} \mu \dot{\xi} \\ & - \left(\frac{1}{2} m_a U^2 + \mu B(\mathbf{X}, t) \right), \end{aligned} \tag{68}$$

which describes the gyrocenter motion for the case where the electrostatic potential ϕ and the vector potential fluctuation $\widehat{\mathbf{A}}$ vanish. In this section, we use the modified vector potential $\mathbf{A}_a^*(\mathbf{X}, U, t) \equiv \mathbf{A}(\mathbf{X}, t) + (m_a c/e_a) U \mathbf{b}(\mathbf{X}, t)$, which is obtained from Eq. (A21) with the second-order small term neglected. The gyrocenter Lagrangian L_{GYa1} is the part, which linearly depends on ϕ and $\widehat{\mathbf{A}}$

$$\begin{aligned} L_{GYa1}(\mathbf{Z}, t) &\equiv -e_a \langle \psi_a(\mathbf{Z}, t) \rangle_\xi \\ &\equiv -e_a \left\langle \phi(\mathbf{X} + \rho_{a1}, t) - \frac{\mathbf{v}_c}{c} \cdot \widehat{\mathbf{A}}(\mathbf{X} + \rho_{a1}, t) \right\rangle_\xi. \end{aligned} \tag{69}$$

The second-order Lagrangian L_{GK2} is given by

$$L_{GK2} \equiv \sum_a \int d^6 Z\, D_{a0}(\mathbf{Z}, t) f_{a0}(\mathbf{Z}, t) L_{GYa2}(\mathbf{Z}, t), \tag{70}$$

where f_{a0} is the zeroth-order part of the gyrocenter distribution function and L_{GYa2} is the second-order gyrocenter Lagrangian defined by

$$\begin{aligned} L_{GYa2}(\mathbf{Z}, t) \equiv{}& \frac{e_a}{c} \mathbf{v}_{Ba} \cdot \langle \widehat{\mathbf{A}}(\mathbf{X} + \rho_{a1}, t) \rangle_\xi \\ & - \frac{e_a^2}{2 m_a c^2} \langle |\widehat{\mathbf{A}}(\mathbf{X} + \rho_{a1}, t)|^2 \rangle_\xi + \frac{e_a^2}{2B} \frac{\partial}{\partial \mu} \langle (\widetilde{\psi}_a)^2 \rangle_\xi. \end{aligned} \tag{71}$$

We note here that $L_{GYa1} + L_{GYa2}$ corresponds to the opposite sign of $e_a \Psi_a$ defined by Eq. (A24). The term $(e_a^2/2B)(\partial \langle (\widetilde{\psi}_a)^2 \rangle_\xi / \partial \mu)$ in Eq. (71) is a part of $\frac{1}{2} e_a \langle \{ \widetilde{S}_a, \widetilde{\psi}_a \} \rangle_\xi$ in Eq. (A24), while the remaining part of $\frac{1}{2} e_a \langle \{ \widetilde{S}_a, \widetilde{\psi}_a \} \rangle_\xi$ is removed in L_{GYa2} because, when it is retained, its contribution to L_{GK2} is of higher order in ε than that of the terms included in Eq. (71). As noted after Eq. (A25) in Appendix A, one of the second-order terms, $(e_a/c)\mathbf{v}_{Ba} \cdot \langle \widehat{\mathbf{A}} \rangle_\xi$, is often neglected in conventional studies although this term is kept here to accurately derive the gyrokinetic Ampère's law later.

The Lagrangian L_F is defined by[12]

$$L_F \equiv \frac{1}{8\pi} \int_V d^3 x \left[|\mathbf{E}_L(\mathbf{x}, t)|^2 - |\mathbf{B}(\mathbf{x}, t) + \widehat{\mathbf{B}}(\mathbf{x}, t)|^2 + \frac{2}{c} \lambda(\mathbf{x}, t) \nabla \cdot \widehat{\mathbf{A}}(\mathbf{x}, t) \right], \tag{72}$$

where the longitudinal (or irrotational) part $\mathbf{E}_L$ of the electric field is written in terms of the electrostatic potential ϕ as

$$\mathbf{E}_L \equiv -\nabla \phi \tag{73}$$

and λ plays the role of the Lagrange undetermined multiplier to derive the Coulomb gauge condition

$$\nabla \cdot \widehat{\mathbf{A}} = 0 \tag{74}$$

from $\delta I / \delta \lambda = 0$ (or $\delta L_{GKF} / \delta \lambda = \delta L_F / \delta \lambda = 0$). Equation (72) is used for making the Darwin approximation to remove electromagnetic waves propagating at light speed.

From the condition that $\delta I = 0$ holds for the variation of $\mathbf{Z}_a(t)$ which is fixed at $t = t_1, t_2$, we can derive the gyrocenter motion equations for $\mathbf{Z}_a(t)$ and accordingly the gyrokinetic Vlasov equation for the distribution function f_a, which is constant along the gyrocenter phase-space orbit represented by $\mathbf{Z}_a(t)$. This is a variational derivation of the gyrokinetic Vlasov equation based on the Lagrangian picture of the gyrocenter phase-space motion.[12] The resultant gyrokinetic Vlasov equation is given by removing the collision term from Eq. (1). In the Eulerian picture (or the Euler–Poincaré formulation),[17–19,38–41] we use the expression in the last line of Eq. (67) and consider the variations of f_a and $\dot{\mathbf{Z}}$ as functions of $(\mathbf{Z}, t)$ to derive the gyrokinetic Vlasov equation from $\delta I = 0$. Effects of the collision term, if included, on the local energy and momentum balance equations can be clarified following the same procedure as shown in Refs. 17 and 18.

In the present case, Eq. (70) is used for the second-order Lagrangian to make the linear polarization-magnetization approximation, in which the deviation of f_a from f_{a0} does not enter the polarization and magnetization terms proportional to ϕ and $\widehat{\mathbf{A}}$ in the gyrokinetic Poisson and Ampère equations as shown later.[12] It also should be noted that in the gyrokinetic equation derived in this approximation, quadratic terms with respect to ϕ and $\widehat{\mathbf{A}}$ are removed from the gyrocenter phase-space velocity $d\mathbf{Z}/dt$.

The gyrokinetic Poisson's equation is derived from the variational derivative of the action integral I with respect to the electrostatic potential ϕ. Since the time derivative of ϕ never appears in the Lagrangian density L_{GKF}, the above-mentioned condition can be replaced by $\delta L_{GKF} / \delta \phi = 0$, which leads to

$$\nabla \cdot \mathbf{E}_L = 4\pi \sum_a e_a \int d^6 Z\, \delta^3(\mathbf{X} + \rho_{a1} - \mathbf{x}) \left(D_a f_a + D_{a0} \frac{e_a \widetilde{\psi}_a}{B} \frac{\partial f_{a0}}{\partial \mu} \right). \tag{75}$$

In $\mathscr{O}(e n_0)$ and $\mathscr{O}(\varepsilon e n_0)$, the ensemble-averaged part of Eq. (75) gives the quasineutrality conditions

$$0 = \sum_a e_a n_{a0} \equiv \sum_a e_a \int d^6 Z\, D_{a0} f_{a0} \delta^3(\mathbf{X} - \mathbf{x}) \tag{76}$$

and

$$0 = \sum_a e_a \langle n_{a1}^{(g)} \rangle_{\text{ens}} \equiv \sum_a e_a \int d^6 Z\, D_{a0} \langle f_{a1} \rangle_{\text{ens}} \delta^3(\mathbf{X} - \mathbf{x}), \tag{77}$$

respectively. The fluctuation part of Eq. (74) is written as

$$\nabla\cdot\widehat{\mathbf{E}}_L = 4\pi\sum_a e_a\int d^6Z\, D_{a0}\delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})\times\left(\widehat{f}_a+\frac{e_a\widetilde{\widetilde{\psi}}_a}{B}\frac{\partial f_{a0}}{\partial\mu}\right), \tag{78}$$

which is valid up to the lowest order, $\mathscr{O}(\varepsilon e n_0)$. Here and hereafter, we do not consider the particle species dependence in using the ordering parameter $\varepsilon\sim\rho_a/L$ and $\mathscr{O}(e_a n_{a0})$. Such dependence may occur due to large mass and charge differences although they should be treated using subsidiary parameters other than ε. We can confirm that Eqs. (75)–(78) are consistent with the results derived from using Eqs. (21), (22), and (A11) for Poisson's equation $\nabla\cdot\mathbf{E}_L = 4\pi\sum_a e_a n_a^{(p)}$.

The gyrokinetic Ampère's law is derived from the variational derivative of the action integral I with respect to the fluctuation part $\widehat{\mathbf{A}}$ of the vector potential. Since the time derivative of $\widehat{\mathbf{A}}$ never appears in the Lagrangian density L_{GKF}, we can use $\delta L_{GKF}/\delta\widehat{\mathbf{A}} = 0$ to obtain

$$\nabla\times(\mathbf{B}+\widehat{\mathbf{B}}) = \frac{4\pi}{c}\mathbf{j}-\frac{1}{c}\nabla\lambda, \tag{79}$$

where the electric current density is given by

$$\mathbf{j} = \sum_a e_a\int d^6Z\,\delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})[D_a(\mathbf{Z},t)f_a(\mathbf{Z},t)\mathbf{v}_c + D_{a0}\left\{f_{a0}\left(\mathbf{v}_{Ba}-\frac{e_a}{m_a c}\widehat{\mathbf{A}}\right)+\frac{e_a\widetilde{\psi}_a}{B}\frac{\partial f_{a0}}{\partial\mu}\mathbf{v}_c\right\}\Bigg]. \tag{80}$$

We see that Eq. (80) agrees with the result shown in Eq. (55). The longitudinal (or irrotational) part of Eq. (79) gives

$$\nabla\lambda = 4\pi\mathbf{j}_L. \tag{81}$$

From the transverse (or solenoidal) part of Eq. (79), the gyrokinetic Ampère's law is written as

$$\nabla\times(\mathbf{B}+\widehat{\mathbf{B}}) = \frac{4\pi}{c}\mathbf{j}_T. \tag{82}$$

In Eqs. (81) and (82), $\mathbf{j}_L$ and $\mathbf{j}_T$ represent the longitudinal and transverse parts of $\mathbf{j}$, respectively. It is noted here that an arbitrary vector field $\mathbf{a}$ is written as $\mathbf{a}=\mathbf{a}_L+\mathbf{a}_T$ where the longitudinal and transverse parts of $\mathbf{a}$ are given by $\mathbf{a}_L(\mathbf{x}) = -(4\pi)^{-1}\nabla\int d^3x'(\nabla'\cdot\mathbf{a}(\mathbf{x}'))/|\mathbf{x}-\mathbf{x}'|$ and $\mathbf{a}_T(\mathbf{x}) = (4\pi)^{-1}\nabla\times(\nabla\times\int d^3x'\,\mathbf{a}(\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|)$, respectively.[23]

The ensemble-averaged part and the fluctuation part of Eq. (82) are written as

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\langle\mathbf{j}\rangle_{\mathrm{ens}T}, \tag{83}$$

and

$$\nabla\times\widehat{\mathbf{B}} = \frac{4\pi}{c}\widehat{\mathbf{j}}_T, \tag{84}$$

respectively, where the ensemble-averaged part and fluctuation part of the current density are given by

$$\begin{aligned}\langle\mathbf{j}\rangle_{\mathrm{ens}} &= \sum_a e_a\int d^6Z\,\delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})\big[D_{a0}\langle f_{a1}\rangle_{\mathrm{ens}}\mathbf{v}_c + (D_{a0}+D_{a1})f_{a0}\mathbf{v}_c + D_{a0}f_{a0}\mathbf{v}_{da}\big]\\ &= \left\{\sum_a e_a\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}\langle f_{a1}\rangle_{\mathrm{ens}}U + \frac{c}{B}\left((P_\parallel)_0-(P_\perp)_0\right)(\mathbf{b}\cdot\nabla\times\mathbf{b})\right\}\mathbf{b}\\ &\quad+\frac{c}{B}\mathbf{b}\times\nabla\cdot\left\{(P_\parallel)_0\mathbf{bb}+(P_\perp)_0(\mathbf{I}-\mathbf{bb})\right\}\end{aligned} \tag{85}$$

and

$$\widehat{\mathbf{j}} = \sum_a e_a\int d^6Z\,\delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})D_{a0}\cdot\left(\widehat{f}_a\mathbf{v}_c - f_{a0}\frac{e_a}{m_a c}\widehat{\mathbf{A}}+\frac{e_a\widetilde{\widetilde{\psi}}_a}{B}\frac{\partial f_{a0}}{\partial\mu}\mathbf{v}_c\right), \tag{86}$$

respectively. On the right-hand side of Eq. (85), $(P_\perp)_0\equiv\sum_a(P_{a\perp})_0$ and $(P_\parallel)_0\equiv\sum_a(P_{a\parallel})_0$ are used and the definitions of $(P_{a\perp})_0$ and $(P_{a\parallel})_0$ are found in Eqs. (47) and (49), respectively. When f_{a0} takes the form of the local Maxwellian distribution with no mean flow, we have the isotropic equilibrium pressure $(P_\perp)_0 = (P_\parallel)_0 = P_0$. It should be noted that Eqs. (80), (85), and (86) are valid up to the lowest in ε. In Appendix D, using the WKB representation, the turbulent parts of Poisson and Ampère equations in Eqs. (78) and (84) are shown to agree with the results derived in earlier works.[25,26]

VI. CONCLUSIONS

In this paper, effects of both equilibrium and gyroradius scale electromagnetic turbulence are included to derive expressions of polarization and magnetization in terms of the distribution function in the gyrocenter phase-space coordinates. These expressions presented in Eqs. (22) and (28) include infinite series expansion with respect to the gyroradius vector, which is defined in the gyrocenter coordinates by Eqs. (A11)–(A17), where effects of the turbulent fields are taken into account.

To the leading (or first) order in the normalized gyroradius parameter ε, the polarization flow vanishes and the ensemble-averaged (or non-turbulent) part of the particle flow consists of the gyrocenter and magnetization flows, which agrees with the result called the magnetization law in the drift kinetics.[24] On the other hand, the leading-order turbulent part of the particle flow is given by the sum of the turbulent parts of the polarization, magnetization, and gyrocenter flows. Thus, a practical extension of the drift kinetic magnetization law is made to gyrokinetic systems with electromagnetic fluctuations and collisions. The compact expression of the particle flow, including both mean and turbulent parts, is given in Eq. (55), which is valid to the leading order and useful for evaluating the total current density to self-consistently determine the magnetic field in full-f global gyrokinetic simulations.[1–9]

The effect of collisions appears as the classical transport in the second-order mean particle flow. In toroidally confined plasmas, the first-order mean (or ensemble-averaged) particle flow is tangential to the magnetic surface, so that the mean particle transport flux across the magnetic surface is of the second-order and it is verified to contain classical, neoclassical, and turbulent transport processes, which determine plasma particle confinement in a transport timescale.

The Lagrangian is presented for variational derivation of the gyrokinetic Poisson and Ampère equations, which properly include mean and turbulent parts. It is shown that the diamagnetic current can be correctly included in the mean part of Ampère's law derived from the variational principle using the Lagrangian, which retains the second-order term given by the inner product of the turbulent vector potential and the drift velocity consisting of the curvature drift and the ∇B drift. The resultant expressions of Ampère's law [Eq. (82)] and the current density [Eq. (80)] are useful especially for the full-f global electromagnetic gyrokinetic simulations to accurately treat high-beta plasmas. Properly taking account of the difference between the phase space coordinates in the classical gyrokinetic formulation and the modern formulation employed in the present work, the equivalence between descriptions of electromagnetic gyrokinetic turbulent fluctuations in the two formulations is clarified as shown in Appendixes C and D. The turbulent parts of the gyrokinetic Poisson and Ampère equations in Eqs. (75) and (82) are confirmed to agree with the results derived from the classical gyrokinetic formulation using the WKB representation in earlier works. Thus, these equations present a basic model for global full-f gyrokinetic simulations which is also consistent with the local turbulence model used in flux-tube gyrokinetic simulations.[42–46] Based on the presented Lagrangian, local energy and momentum balance equations for the gyrokinetic system with electromagnetic turbulence and collisions can be derived following the same formulation as given by our previous work in the case of electrostatic turbulence.[18] Details of the derivation will be reported in a future work.

ACKNOWLEDGMENTS

This work is supported in part by the JSPS Grants-in-Aid for Scientific Research Grant No. 19H01879 and in part by the NIFS Collaborative Research Program No. NIFS20KNTT055.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: GYROCENTER COORDINATES AND EQUATIONS OF MOTION

We consider motion of a charged particle in a strong magnetic field. The particle mass and charge are denoted by m_a and e_a, respectively, where the subscript a represents the particle species. The magnetic field is assumed to consist of the background part $\mathbf{B} \equiv \nabla \times \mathbf{A}$ and the small fluctuation part $\widehat{\mathbf{B}} \equiv \nabla \times \widehat{\mathbf{A}}$. The particle's position and velocity are denoted by $\mathbf{x}$ and $\mathbf{v}$, respectively. The velocity $\mathbf{v}$ is written by the sum of the parallel and perpendicular components as

$$\mathbf{v} \equiv v_\parallel \mathbf{b} + \mathbf{v}_\perp, \tag{A1}$$

where the unit vector $\mathbf{b} \equiv \mathbf{B}/B$ in the direction parallel to the magnetic field is evaluated at the particle's position $\mathbf{x}$. Using a right-handed orthogonal triad of unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{b})$ which are regarded as functions of $(\mathbf{x}, t)$, we represent the perpendicular velocity as

$$\mathbf{v}_\perp \equiv -v_\perp(\sin\xi_0\, \mathbf{e}_1 + \cos\xi_0\, \mathbf{e}_2), \tag{A2}$$

where $v_\perp \equiv |\mathbf{v}_\perp|$. We now define the particle phase-space coordinates $\mathbf{z}$ by

$$\mathbf{z} \equiv (\mathbf{x}, v_\parallel, \mu_0, \xi_0), \tag{A3}$$

where

$$\mu_0 \equiv \frac{m_a v_\perp^2}{2B(\mathbf{x}, t)}. \tag{A4}$$

Using the Lie transformation technique, the gyrocenter phase-space coordinates

$$\mathbf{Z} \equiv (\mathbf{X}, U, \mu, \xi) \tag{A5}$$

are obtained, such that the Lagrangian for the particle motion is transformed into a function which is independent of the gyrophase angle variable ξ, as shown later in Eq. (A20). The relations of the gyrocenter coordinates $\mathbf{Z} \equiv (\mathbf{X}, U, \mu, \xi)$ to the particle coordinates $\mathbf{z} \equiv (\mathbf{x}, v_\parallel, \mu_0, \xi_0)$ are given by

$$\begin{aligned}\mathbf{X} = \mathbf{x} - \frac{v_\perp}{\Omega_a}\mathbf{a} + \frac{v_\perp}{\Omega_a^2}\Bigg[\left\{v_\parallel(\mathbf{b}\cdot\nabla\times\mathbf{b}) - \frac{v_\perp}{2B}(\mathbf{a}\cdot\nabla B)\right\}\mathbf{a} \\ + \left\{2v_\parallel(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{c}) + \frac{v_\perp}{8}(\mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{c} - 5\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{a})\right\}\mathbf{b}\Bigg] \\ + \frac{1}{B}\left[\left(\widehat{\mathbf{A}} + \frac{c}{e_a}\nabla\widetilde{S}_a\right)\times\mathbf{b} + \mathbf{b}\int d\xi_0\, \widetilde{\widehat{A}_\parallel}\right],\end{aligned} \tag{A6}$$

$$U = v_\parallel - \frac{v_\perp}{\Omega_a}\left[v_\parallel(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{a}) + \frac{v_\perp}{4}(3\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{c} - \mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{a})\right] + \frac{e_a}{m_a c}\widehat{A}_\parallel, \tag{A7}$$

$$\begin{aligned}\mu = \frac{m_a v_\perp^2}{2B} + \frac{m_a v_\perp^2}{B\Omega_a}\Bigg[\frac{v_\parallel^2}{v_\perp}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{a}) \\ + \frac{v_\parallel}{4}(3\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{c} - \mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{a}) + \frac{v_\perp}{2B}(\mathbf{a}\cdot\nabla B)\Bigg] \\ + \frac{e_a}{B}\left(\frac{1}{c}\mathbf{v}_\perp\cdot\widehat{\mathbf{A}} + \widetilde{\psi}_a\right),\end{aligned} \tag{A8}$$

and

$$\begin{aligned}\xi = \xi_0 + \frac{1}{\Omega_a}\Bigg[\frac{v_\parallel^2}{v_\perp}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{c}) + \frac{v_\parallel}{4}(\mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{c} - \mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{a}) \\ + v_\perp\left(\mathbf{c}\cdot\frac{\nabla B}{B} - \mathbf{a}\cdot\nabla\mathbf{c}\cdot\mathbf{a}\right)\Bigg] - \frac{e_a}{m_a c}\frac{\partial\widetilde{S}_a}{\partial\mu_0},\end{aligned} \tag{A9}$$

where $\Omega_a \equiv e_a B(\mathbf{x}, t)/(m_a c)$, $v_\perp \equiv (2\mu_0 B(\mathbf{x}, t)/m_a)^{1/2}$, $\mathbf{c} \equiv \mathbf{v}_\perp/v_\perp$, $\mathbf{a} \equiv \mathbf{b}\times\mathbf{c}$, $\widehat{A}_\parallel \equiv \widehat{\mathbf{A}}\cdot\mathbf{b}$, and the definitions of $\widetilde{\psi}_a$ and $\widetilde{S}_a$ are shown later in Eqs. (A18) and (A19), respectively. Equation (A6) for the gyrocenter position $\mathbf{X}$ is valid up to the second order in the normalized gyroradius parameter ε, while Eqs. (A7)–(A9) are up to the first order. When there are no fluctuation fields, the formulas in Eqs. (A6)–(A9) agree with those given by Littlejohn,[47] except that Eq. (A7) is given here in a slightly different way, in order to remove the $\mathscr{O}(\varepsilon)$ term of the Hamiltonian in Ref. 47. The same procedure as in Ref. 12 is used to include the effects of the fluctuation fields in Eqs. (A6)–(A9).

We can inversely solve Eqs. (A6)–(A9) to represent the particle position vector $\mathbf{x}$ by the function of the gyrocenter coordinates $\mathbf{Z}$ as

$$\mathbf{x} = \mathbf{X} + \rho_a(\mathbf{Z}, t), \tag{A10}$$

where the gyroradius vector $\rho_a(\mathbf{Z}, t)$ is expanded in ε as

$$\rho_a(\mathbf{Z}, t) = \rho_{a1}(\mathbf{Z}, t) + \rho_{a2}(\mathbf{Z}, t) + \cdots. \tag{A11}$$

The lowest-order part of ρ_a is given by

$$\rho_{a1}(\mathbf{Z}, t) \equiv \frac{\mathbf{b}(\mathbf{X}, t) \times \mathbf{v}_c(\mathbf{Z}, t)}{\Omega_a(\mathbf{X}, t)}, \tag{A12}$$

where $\mathbf{v}_c$ is defined by

$$\mathbf{v}_c \equiv U\mathbf{b}(\mathbf{X}, t) - W[\sin\xi\, \mathbf{e}_1(\mathbf{X}, t) + \cos\xi\, \mathbf{e}_2(\mathbf{X}, t)] \tag{A13}$$

and

$$W \equiv \left(\frac{2\mu B(\mathbf{X}, t)}{m_a}\right)^{1/2}. \tag{A14}$$

To the lowest order in ε, the particle velocity $\mathbf{v}$ and the gyroradius vector $\rho \equiv \mathbf{x} - \mathbf{X}$ are represented by $\mathbf{v}_c$ and ρ_{a1}, respectively. The second-order part of ρ_a is written as

$$\rho_{a2}(\mathbf{Z}, t) \equiv \langle\rho_{a2}\rangle_{\mathrm{ens}} + \widehat{\rho}_{a2}, \tag{A15}$$

where the ensemble-average and fluctuation parts of ρ_{a2} are given by

$$\begin{aligned}
\langle\rho_{a2}\rangle_{\mathrm{ens}} \equiv\, & \mathbf{b}\left[-\frac{W^2}{8\Omega_a^2}(3\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{a} + \mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{c}) - \frac{2UW}{\Omega_a^2}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{c})\right] \\
& + \mathbf{a}\left[\frac{c}{\Omega_a B}(\mathbf{a}\cdot\langle\mathbf{E}_1\rangle_{\mathrm{ens}}) - \frac{W^2}{2\Omega_a^2}(\mathbf{a}\cdot\nabla\ln B)\right. \\
& \left. + \frac{UW}{4\Omega_a^2}(\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{c} - 3\mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{a}) - \frac{U^2}{\Omega_a^2}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{a})\right] \\
& + \mathbf{c}\left[\frac{c}{\Omega_a B}(\mathbf{c}\cdot\langle\mathbf{E}_1\rangle_{\mathrm{ens}}) - \frac{W^2}{\Omega_a^2}(\mathbf{c}\cdot\nabla\ln B)\right. \\
& \left. + \frac{UW}{4\Omega_a^2}(\mathbf{a}\cdot\nabla\mathbf{b}\cdot\mathbf{a} - \mathbf{c}\cdot\nabla\mathbf{b}\cdot\mathbf{c}) - \frac{U^2}{\Omega_a^2}(\mathbf{b}\cdot\nabla\mathbf{b}\cdot\mathbf{c})\right]
\end{aligned} \tag{A16}$$

and

$$\begin{aligned}
\widehat{\rho}_{a2} &= \frac{\mathbf{b}}{B}\times\widehat{\mathbf{A}} + \left\{\mathbf{X} + \boldsymbol{\rho}_{a1}, \widetilde{\widehat{S}}_a\right\} \\
&= -\frac{c}{BW}\left(\widehat{\phi} - \frac{U}{c}\widehat{A}_\parallel - \langle\widehat{\psi}_a\rangle_\xi\right)\mathbf{a} + \frac{m_a c W}{B^2}\frac{\partial}{\partial\mu}\left(\int\widetilde{\widehat{\psi}}_a\, d\xi\right)\mathbf{c} \\
&\quad - \frac{1}{B}\left[\widehat{\mathbf{A}} + \frac{c}{\Omega_a}\nabla\left(\int\widetilde{\widehat{\psi}}_a\, d\xi\right)\right]\times\mathbf{b} - \frac{1}{B}\left(\int\widetilde{\widehat{A}}_\parallel\, d\xi\right)\mathbf{b},
\end{aligned} \tag{A17}$$

respectively. The definitions of $\langle\cdots\rangle_\xi$ and $\widetilde{\ }$ are given in Eqs. (3) and (4), respectively, and $\widehat{\psi}_a \equiv \psi_a - \langle\psi_a\rangle_{\mathrm{ens}}$ is the fluctuation part of ψ_a, which is defined in terms of the electrostatic potential ϕ and the fluctuation part $\widehat{\mathbf{A}}$ of the vector potential as

$$\psi_a \equiv \phi(\mathbf{X} + \rho_{a1}, t) - \frac{\mathbf{v}_c}{c}\cdot\widehat{\mathbf{A}}(\mathbf{X} + \rho_{a1}, t). \tag{A18}$$

Here, we also define

$$\widetilde{S}_a \equiv \frac{m_a c}{B}\int\widetilde{\psi}_a d\xi, \tag{A19}$$

where the integral constant is determined from the condition $\langle\widetilde{S}_a\rangle_\xi = 0$. We now note that $\widetilde{\widehat{\mathbf{A}}}$, $\widetilde{\psi}_a$, and $\widetilde{S}_a$ are defined above as functions of $\mathbf{Z} \equiv (\mathbf{X}, U, \mu, \xi)$ and t, although when they are substituted into the formulas for the coordinate transformation from $\mathbf{z}$ to $\mathbf{Z}$ [see Eqs. (A6)–(A9)], the independent variables $(\mathbf{X}, U, \mu, \xi)$ for the functions $\widetilde{\widehat{\mathbf{A}}}$, $\widetilde{\psi}_a$, and $\widetilde{S}_a$ should be replaced with $(\mathbf{x} - \rho_{a1}(\mathbf{z}, t), v_\parallel, \mu_0, \xi_0)$ to keep the validity of the formulas up to the orders described after Eq. (A9). Here, the finite gyroradius ρ_{a1} cannot be neglected because fluctuations are considered to have $\mathscr{O}(\rho_a)$ wavelengths in directions perpendicular to $\mathbf{B}$.

In the gyrocenter coordinates, the Lagrangian for the charged particle of motion is given by

$$L_{GYa}(\mathbf{Z}, \dot{\mathbf{Z}}, t) \equiv \frac{e_a}{c}\mathbf{A}_a^*\cdot\dot{\mathbf{X}} + \frac{m_a c}{e_a}\mu\,\dot{\xi} - H_{GYa}(\mathbf{Z}, t), \tag{A20}$$

where the modified vector potential $\mathbf{A}_a^*$ is defined by

$$\mathbf{A}_a^* \equiv \mathbf{A}(\mathbf{X}, t) + \frac{m_a c}{e_a}U\mathbf{b}(\mathbf{X}, t) - \frac{m_a c^2}{e_a^2}\mu\mathbf{W}(\mathbf{X}, t) \tag{A21}$$

and

$$\mathbf{W} \equiv \nabla\mathbf{e}_1\cdot\mathbf{e}_2 + \frac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b})\mathbf{b}. \tag{A22}$$

Here, the gyrocenter Hamiltonian H_{GYa} is defined by

$$H_{GYa} \equiv \frac{1}{2}m_a U^2 + \mu B + e_a\Psi_a. \tag{A23}$$

The fluctuations are included in the Hamiltonian $H_{GY\,a}$ through the term $e_a\Psi_a$ defined by

$$e_a\Psi_a \equiv e_a\langle\psi_a\rangle_\xi - \frac{e_a}{c}\mathbf{v}_{Ba}\cdot\langle\widehat{\mathbf{A}}\rangle_\xi + \frac{e_a^2}{2m_a c^2}\langle|\widehat{\mathbf{A}}|^2\rangle_\xi - \frac{e_a}{2}\left\langle\left\{\widetilde{S}_a, \widetilde{\psi}_a\right\}\right\rangle_\xi, \tag{A24}$$

where $\{\cdot,\cdot\}$ represents the Poisson bracket, defined by Eqs. (29)–(33) in Ref. 12, and

$$\mathbf{v}_{Ba} \equiv \frac{c}{e_a B}\mathbf{b}\times\left(m_a U^2\mathbf{b}\cdot\nabla\mathbf{b} + \mu\nabla B\right) \tag{A25}$$

is the first-order drift velocity consisting of the curvature drift and the ∇B drift. On the right-hand side of Eq. (A24), the first term is of $\mathscr{O}(\varepsilon)$ and the others are of $\mathscr{O}(\varepsilon^2)$. There the third and fourth terms are quadratic in the fluctuations, while the second term $-(e_a/c)\mathbf{v}_{Ba}\cdot\langle\widehat{\mathbf{A}}\rangle_\xi$ is given by the product of the average drift velocity and the fluctuation vector potential. The latter term $-(e_a/c)\mathbf{v}_{Ba}\cdot\langle\widehat{\mathbf{A}}\rangle_\xi$ is often neglected in conventional studies, although it is retained here for accuracy up to $\mathscr{O}(\varepsilon^2)$.

The gyrocenter equations of motion are derived from the Euler–Lagrange equations using the gyrocenter Lagrangian in Eq. (A20). Using the Hamiltonian in Eq. (A23), they are given in the form

$$\frac{d\mathbf{Z}}{dt} = \{\mathbf{Z}, H_{GYa}\} + \{\mathbf{Z}, \mathbf{X}\} \cdot \frac{e_a}{c}\frac{\partial \mathbf{A}_a^*}{\partial t}, \tag{A26}$$

which are rewritten as[12]

$$\frac{d\mathbf{X}}{dt} = \frac{1}{B_{a\parallel}^*}\left[\left(U + \frac{e_a}{m_a}\frac{\partial \Psi_a}{\partial U}\right)\mathbf{B}_a^* + c\mathbf{b} \times \left(\frac{\mu}{e_a}\nabla B + \nabla\Psi_a + \frac{1}{c}\frac{\partial \mathbf{A}_a^*}{\partial t}\right)\right], \tag{A27}$$

$$\frac{dU}{dt} = -\frac{\mathbf{B}_a^*}{m_a B_{a\parallel}^*} \cdot \left(\mu\nabla B + e_a\nabla\Psi_a + \frac{e_a}{c}\frac{\partial \mathbf{A}_a^*}{\partial t}\right), \tag{A28}$$

$$\frac{d\mu}{dt} = 0, \tag{A29}$$

and

$$\frac{d\xi}{dt} = \Omega_a + \mathbf{W} \cdot \frac{d\mathbf{X}}{dt} + \frac{e_a^2}{m_a c}\frac{\partial \Psi_a}{\partial \mu}, \tag{A30}$$

where $\mathbf{B}_a^*$ and $B_{a\parallel}^*$ are defined in terms of $\mathbf{A}_a^*$ in Eq. (A21) as

$$\mathbf{B}_a^* \equiv \nabla \times \mathbf{A}_a^*, \quad \text{and} \quad B_{a\parallel}^* \equiv \mathbf{B}_a^* \cdot \mathbf{b}, \tag{A31}$$

respectively. Since the gyrocenter Lagrangian L_{GY} is independent of the gyrophase variable ξ, the time derivatives of the gyrocenter variables do not depend on ξ and the magnetic moment $\mu = (e_a/m_a c)(\partial L_{GY}/\partial\dot{\xi})$ is conserved, as seen in Eqs. (A27)–(A30). The gyrocenter motion given by Eqs. (A27)–(A30) satisfies Liouville's theorem, which is expressed as

$$\frac{\partial D_a(\mathbf{Z}, t)}{\partial t} + \frac{\partial}{\partial \mathbf{Z}} \cdot \left(D_a(\mathbf{Z}, t)\frac{d\mathbf{Z}}{dt}\right) = 0, \tag{A32}$$

where the Jacobian $D_a(\mathbf{Z}, t)$ is given by

$$D_a(\mathbf{Z}, t) = \frac{B_{a\parallel}^*}{m_a}. \tag{A33}$$

APPENDIX B: EXPANSION OF $d\mathbf{X}/dt$ AND $d\rho_a/dt$ IN ε

In this appendix, $d\mathbf{X}/dt$ and $d\rho_a/dt$ are expanded in the normalized gyroradius parameter ε. To begin with, the zeroth-order gyrocenter velocity is parallel to the background magnetic field and given by

$$\left(\frac{d\mathbf{X}}{dt}\right)_0 = U\mathbf{b}(\mathbf{X}, t), \tag{B1}$$

which contains no fluctuation part. The first-order gyrocenter velocity is written as

$$\left(\frac{d\mathbf{X}}{dt}\right)_1 = \left\langle\left(\frac{d\mathbf{X}}{dt}\right)_1\right\rangle_{\text{ens}} + \widehat{\left(\frac{d\mathbf{X}}{dt}\right)}_1, \tag{B2}$$

where the ensemble-averaged part and the fluctuation part are given by

$$\left\langle\left(\frac{d\mathbf{X}}{dt}\right)_1\right\rangle_{\text{ens}} = \frac{c}{e_a B}\mathbf{b} \times \left(m_a U^2\mathbf{b}\cdot\nabla\mathbf{b} + \mu\nabla B + e_a\nabla\langle\phi_1\rangle_{\text{ens}}\right) \equiv \mathbf{v}_{da}, \tag{B3}$$

and

$$\widehat{\left(\frac{d\mathbf{X}}{dt}\right)}_1 = -\frac{e_a}{m_a c}\langle\widehat{A}_\parallel\rangle_\xi\mathbf{b} + \frac{c}{B}\mathbf{b} \times \nabla\langle\widehat{\psi}_a\rangle_\xi \equiv \widehat{\mathbf{v}}_{ga}, \tag{B4}$$

respectively. Regarding the second-order gyrocenter velocity, only its ensemble-averaged part is given here as

$$\left\langle\left(\frac{d\mathbf{X}}{dt}\right)_2\right\rangle_{\text{ens}} = -\frac{U}{\Omega_a}\left[(\mathbf{b}\cdot\nabla\times\mathbf{b})\,\mathbf{v}_{da} + \frac{\mu B}{m_a\Omega_a}(\nabla\times\mathbf{W})_\perp\right] + \frac{c}{B}\left(-\nabla\langle\phi_2\rangle_{\text{ens}} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) \times \mathbf{b} \equiv \mathbf{v}_{da2}. \tag{B5}$$

The zeroth-order part of $d\rho_a/dt$ is given by the perpendicular component of the particle velocity as

$$\left(\frac{d\rho_a}{dt}\right)_0 = \Omega_a\frac{\partial \rho_{a1}}{\partial \xi} = (\mathbf{v}_c)_\perp \equiv -\left(\frac{2\mu B}{m_a}\right)^{1/2}[\sin\xi\,\mathbf{e}_1 + \cos\xi\,\mathbf{e}_2]. \tag{B6}$$

The first-order part of $d\rho_a/dt$ is written as

$$\left(\frac{d\rho_a}{dt}\right)_1 = \left\langle\left(\frac{d\rho_a}{dt}\right)_1\right\rangle_{\text{ens}} + \widehat{\left(\frac{d\rho_a}{dt}\right)}_1, \tag{B7}$$

where

$$\left\langle\left(\frac{d\rho_a}{dt}\right)_1\right\rangle_{\text{ens}} = U\mathbf{b}\cdot\left(\nabla\rho_{a1} + \mathbf{W}\frac{\partial \rho_{a1}}{\partial \xi}\right) + \Omega_a\frac{\partial\langle\rho_{a2}\rangle_{\text{ens}}}{\partial \xi} \tag{B8}$$

and

$$\widehat{\left(\frac{d\rho_a}{dt}\right)}_1 = -\frac{e_a}{m_a c}(\widetilde{\widehat{A}_\parallel}\mathbf{b} + \widehat{\mathbf{A}}_\perp) - \frac{c}{B}\mathbf{b}\times\nabla\langle\widehat{\psi}_a\rangle_\xi + \left\{(\mathbf{v}_c)_\perp, \widetilde{\widehat{S}}_a\right\}. \tag{B9}$$

The second-order ensemble-averaged part of $(c/e_a)\mathbf{M}_a$ is derived from Eqs. (28), (B1), (B3), (B6), and (B8) as

$$\begin{aligned}\left\langle\left(\frac{c}{e_a}\mathbf{M}_a\right)_2\right\rangle_{\text{ens}} = &\frac{c}{e_a}\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})(D_{a0}f_{a1} + D_{a1}f_{a0})(-\mu\mathbf{b}) \\ &+\frac{1}{\Omega_a}\left[\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}f_{a0}U\mathbf{v}_{da}\right. \\ &-2\frac{c}{e_a}\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}f_{a0}U\mu\mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b} \\ &\left.+\frac{c}{2e_a}\mathbf{b}\times\nabla\left(\int d^6Z\,\delta^3(\mathbf{X}-\mathbf{x})D_{a0}f_{a0}U\mu\right)\right],\end{aligned} \tag{B10}$$

where D_{a0} and D_{a1} are given by

$$D_{a0} = \frac{B}{m_a}, \quad D_{a1} = \frac{c}{e_a} U \mathbf{b} \cdot (\nabla \times \mathbf{b}). \tag{B11}$$

APPENDIX C: ZEROTH AND FIRST-ORDER DISTRIBUTION FUNCTIONS

We here consider the zeroth and first-order distribution functions in the normalized gyroradius parameter ε, and present the kinetic equations satisfied by these distribution functions. As for the zeroth-order distribution function, Maxwellian and non-Maxwellian cases are treated.

1. Case of Maxwellian zeroth-order distribution

To the zeroth order in ε, Eq. (1) is written as

$$\dot{\mathbf{Z}}_0 \cdot \frac{\partial f_{a0}}{\partial \mathbf{Z}} = \sum_b C_{ab}^{(p)}[f_{a0}, f_{b0}], \tag{C1}$$

where $\dot{\mathbf{Z}}_0$ represents the zeroth-order part of $\dot{\mathbf{Z}} \equiv d\mathbf{Z}/dt$. The collision terms appear on the right-hand side of Eq. (C1) because the collision frequency is regarded here as of the same order as the transit frequency ω_{Ta}.

In Ref. 29, it is shown using Eq. (C1) and the property of the collision operator regarding the entropy production that, in the magnetic confinement system with nested toroidal magnetic surfaces, the collision term vanishes and f_{a0} is the Maxwellian equilibrium f_{aM} distribution function with no means flow, and satisfies

$$\mathbf{b} \cdot \nabla f_{aM}(\mathbf{X}, \mathscr{E}_c, t) = 0, \tag{C2}$$

where $\mathscr{E}_c$ represents the zeroth-order particle energy given by

$$\mathscr{E}_c = \frac{1}{2} m_a U^2 + \mu B + e_a \langle \phi_1 \rangle_{\text{ens}}. \tag{C3}$$

It should be noted that, in Eq. (C2), $\nabla \equiv \partial/\partial \mathbf{X}$ acts on f_{aM} with $\mathscr{E}_c$ fixed. Then we can write

$$\begin{aligned} f_{a0} &= f_{aM}(\mathbf{X}, \mathscr{E}_c) \\ &= n_{a0} \left(\frac{m_a}{2\pi T_{a0}} \right)^{3/2} \exp\left(-\frac{\mathscr{E}_c - e_a \langle \phi_1 \rangle_{\text{ens}}}{T_{a0}} \right), \end{aligned} \tag{C4}$$

where n_{a0}, T_{a0} and $\langle \phi_1 \rangle_{\text{ens}}$ need to be flux surface functions because of Eq. (C2).

Next we find from Eq. (1) that the first-order ensemble-averaged gyrocenter distribution function $\langle f_{a1} \rangle_{\text{ens}}$ satisfies

$$\begin{aligned} &\dot{\mathbf{Z}}_0 \cdot \frac{\partial \langle f_{a1} \rangle_{\text{ens}}}{\partial \mathbf{Z}} + \langle \dot{\mathbf{Z}}_1 \rangle_{\text{ens}} \cdot \frac{\partial f_{aM}}{\partial \mathbf{Z}} \\ &= \sum_b (C_{ab}^{(p)})^L [\langle f_{a1} \rangle_{\text{ens}}, \langle f_{b1} \rangle_{\text{ens}}] \\ &\equiv \sum_b \langle C_{ab}^{(p)} [\langle f_{a1} \rangle_{\text{ens}}, f_{b0}] + C_{ab}^{(p)} [f_{a0}, \langle f_{b1} \rangle_{\text{ens}}] \rangle_\xi, \end{aligned} \tag{C5}$$

where $(C_{ab}^{(p)})^L$ represents the linearized collision operator. Equation (C5) is the so-called linearized drift kinetic equation, which is used as a basic equation for the neoclassical transport theory.[29–31]

From the fluctuation part of Eq. (C10), the governing equation for the first-order fluctuation part of the gyrocenter distribution function is obtained as

$$\frac{\partial}{\partial t} \widehat{f}_{a1} + \left\{ \widehat{f}_{a1}, \mathscr{E}_c \right\} + \left\{ f_{aM} + \widehat{f}_{a1}, e \langle \widehat{\psi}_a \rangle_\xi \right\} = \left\langle \left(C_{ab}^{(g)} \right)^L \left[\widehat{f}_{a1}, \widehat{f}_{b1} \right] \right\rangle_\xi, \tag{C6}$$

where effects of gyroradius scale perpendicular wavelengths of $\widehat{f}_{a1}$ are taken into account in defining the collision operator $(C_{ab}^{(g)})^L$ by

$$(C_{ab}^{(g)})^L \left[\widehat{f}_{a1}, \widehat{f}_{b1} \right] \equiv e^{\boldsymbol{\rho}_{a1} \cdot \nabla} (C_{ab}^{(p)})^L \left[e^{-\boldsymbol{\rho}_{a1} \cdot \nabla} \widehat{f}_{a1}, e^{-\boldsymbol{\rho}_{b1} \cdot \nabla} \widehat{f}_{b1} \right]. \tag{C7}$$

Here, $\widehat{f}_{a1}$ is given by the sum of adiabatic and nonadiabatic parts as

$$\widehat{f}_{a1} = -\frac{e_a \langle \widehat{\psi}_a \rangle_\xi}{T_{a0}} f_{aM} + \widehat{h}_a, \tag{C8}$$

which is substituted into Eq. (C6) to derive the equation for $\widehat{h}_a$

$$\begin{aligned} &\frac{\partial}{\partial t} \widehat{h}_a + \left\{ \widehat{h}_a, \mathscr{E}_c + e \langle \widehat{\psi}_a \rangle_\xi \right\} - \sum_b \left\langle (C_{ab}^{(g)})^L \left[\widehat{f}_{a1}, \widehat{f}_{b1} \right] \right\rangle_\xi \\ &= e_a \frac{\partial \langle \widehat{\psi}_a \rangle_\xi}{\partial t} \frac{f_{aM}}{T_{a0}} - \left\{ \mathbf{X}, e_a \langle \widehat{\psi}_a \rangle_\xi \right\} \cdot \frac{f_{aM}(\mathbf{X}, \mathscr{E}_c)}{\partial \mathbf{X}}. \end{aligned} \tag{C9}$$

2. Case of non-Maxwellian zeroth-order distribution

In the zeroth-order in ε, Eq. (1) gives

$$\dot{\mathbf{Z}}_0 \cdot \frac{\partial f_{a0}}{\partial \mathbf{Z}} = 0, \tag{C10}$$

where the collision term is neglected by assuming the collision frequency to be sufficiently small. It is seen from Eq. (C10) that the zeroth-order distribution function $f_0 = f_0(\mathbf{X}, \mathscr{E}_c, \mu)$ satisfies

$$\mathbf{b} \cdot \nabla f_{a0}(\mathbf{X}, \mathscr{E}_c, \mu) = 0, \tag{C11}$$

where $\mathscr{E}_c$ is defined in Eq. (C3) and $\nabla \equiv \partial/\partial \mathbf{X}$ acts on f_{aM} with $\mathscr{E}_c$ fixed in the same way as in Eq. (C2).

From the fluctuation part of Eq. (1), the governing equation for the first-order fluctuation part of the gyrocenter distribution function is obtained as

$$\begin{aligned} &\frac{\partial}{\partial t} \widehat{f}_{a1} + \left\{ \widehat{f}_{a1}, \mathscr{E}_c \right\} + \left\{ f_{a0} + \widehat{f}_{a1}, e_a \langle \widehat{\psi}_a \rangle_\xi \right\} \\ &= \sum_b \left\langle (C_{ab}^{(g)})^L \left[\widehat{f}_{a1}, \widehat{f}_{b1} \right] \right\rangle_\xi, \end{aligned} \tag{C12}$$

where the collision term is retained for including collisional effects on gyrokinetic turbulence. Here, $\widehat{f}_{a1}$ is given by the sum of adiabatic and nonadiabatic parts as

$$\widehat{f}_{a1} = e_a \langle \widehat{\psi}_a \rangle_\xi \frac{\partial f_{a0}(\mathbf{X}, \mathscr{E}_c, \mu)}{\partial \mathscr{E}_c} + \widehat{h}_a, \tag{C13}$$

which is substituted into Eq. (C12) to derive the equation for h_a

$$\begin{aligned} &\frac{\partial}{\partial t} \widehat{h}_a + \left\{ \widehat{h}_a, \mathscr{E}_c + e_a \langle \widehat{\psi}_a \rangle_\xi \right\} - \sum_b \left\langle (C_{ab}^{(g)})^L \left[\widehat{f}_{a1}, \widehat{f}_{b1} \right] \right\rangle_\xi \\ &= -e_a \frac{\partial \langle \widehat{\psi}_a \rangle_\xi}{\partial t} \frac{\partial f_{a0}(\mathbf{X}, \mathscr{E}_c, \mu)}{\partial \mathscr{E}_c} - \left\{ \mathbf{X}, e_a \langle \widehat{\psi}_a \rangle_\xi \right\} \cdot \frac{\partial f_{a0}(\mathbf{X}, \mathscr{E}_c, \mu)}{\partial \mathbf{X}}. \end{aligned} \tag{C14}$$

It is found that the nonlinear gyrokinetic equation in Ref. 27 can be reproduced from Eq. (C14) while neglecting the collision term and using the WKB representation described in Appendix D.

Substituting Eq. (C13) into Eqs. (78) and (86), the gyrokinetic Poisson and Ampère equations are written as

$$-\nabla^2\widehat{\phi} = 4\pi\sum_a e_a\int d^3X\, d\mathscr{E}_c\, d\mu\, d\xi \sum_{\sigma=\pm1}\frac{B}{m_a^2|U|} \times \delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})\left[e_a\widehat{\phi}\frac{\partial f_{a0}}{\partial\mathscr{E}_c} + e_a\left(\widehat{\phi}-\frac{U}{c}\widehat{A}_\parallel - \langle\widehat{\psi}_a\rangle_\xi\right)\times\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}+\widehat{h}_a\right] \tag{C15}$$

and

$$-\nabla^2\widehat{\mathbf{A}} = \frac{4\pi}{c}\sum_a e_a\int d^3X\, d\mathscr{E}_c\, d\mu\, d\xi \sum_{\sigma=\pm1}\frac{B}{m_a^2|U|} \times \delta^3(\mathbf{X}+\rho_{a1}-\mathbf{x})\left[U\mathbf{b}\left\{e_a\left(\widehat{\phi}-\frac{U}{c}\widehat{A}_\parallel - \langle\widehat{\psi}_a\rangle_\xi\right)\times\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}+\widehat{h}_a\right\} + (\mathbf{v}_c)_\perp\left\{-\langle\widehat{\psi}_a\rangle_\xi\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}+\widehat{h}_a\right\}\right], \tag{C16}$$

respectively, where $\sigma \equiv U/|U|$ and $|U| \equiv [(2/m_a)(\mathscr{E}_c - \mu B - e_a\langle\phi_1\rangle_{\rm ens})]^{1/2}$ are used and the integration in $\mathscr{E}_c$ and μ are done over the region defined by $0 \le \mu B \le \mathscr{E}_c - e_a\langle\phi_1\rangle_{\rm ens}$.

It is useful to consider a case in which the distribution function $f_a^{(p)}$ in the particle coordinates is used instead of the distribution function f_a in the gyrocenter coordinates. These functions are related to each other by

$$f_a^{(p)}(\mathbf{x},\mathscr{E},\mu_0,\xi_0,t) = f_a(\mathbf{X},\mathscr{E}_c,\mu,\xi,t), \tag{C17}$$

where $\mathscr{E}$ and $\mathscr{E}_c$ are used as independent variables instead of $v_\parallel$ and U, respectively. Here, following Ref. 25, $\mathscr{E}$ is defined by:

$$\mathscr{E} \equiv \frac{1}{2}m_a v^2 + e_a\Phi, \tag{C18}$$

where Φ is the equilibrium electrostatic potential and corresponds to $\langle\phi\rangle_{\rm ens}$ in our notation. The relation between $\mathscr{E}_c$ and $\mathscr{E}$ is written as

$$\mathscr{E}_c = \mathscr{E} + \Delta\mathscr{E}. \tag{C19}$$

Then, using Eqs. (A7), (A8), (C3), and (C18), the fluctuation part $\Delta\widehat{\mathscr{E}}$ of $\Delta\mathscr{E}$ is obtained up to the leading order in ε as

$$\Delta\widehat{\mathscr{E}} = e_a\left(\frac{1}{c}\mathbf{v}\cdot\widehat{\mathbf{A}} + \widetilde{\widehat{\psi}}_a\right) = e_a\left(\widehat{\phi} - \langle\widehat{\psi}_a\rangle_\xi\right). \tag{C20}$$

Equation (A8) is rewritten as

$$\mu = \mu_0 + \Delta\mu \tag{C21}$$

and the fluctuation part $\Delta\widehat{\mu}$ of $\Delta\mu$ is given up to the leading order in ε as

$$\Delta\widehat{\mu} = \frac{e_a}{B}\left(\frac{1}{c}\mathbf{v}_\perp\cdot\widehat{\mathbf{A}} + \widetilde{\widehat{\psi}}_a\right) = \frac{e_a}{B}\left(\widehat{\phi} - \frac{1}{c}v_\parallel\widehat{A}_\parallel - \langle\widehat{\psi}_a\rangle_\xi\right). \tag{C22}$$

Noting that the zeroth-order parts of f_a and $f_a^{(p)}$ are both given by the same function f_{a0}, and using Eqs. (C13), (C20), and (C22), the first-order fluctuation part $\widehat{f}_{a1}^{(p)}$ of $f_a^{(p)}$ is written as

$$\begin{aligned}\widehat{f}_{a1}^{(p)}&(\mathbf{x},\mathscr{E},\mu_0,t)\\ &= \widehat{f}_{a1}(\mathbf{x}-\boldsymbol{\rho}_{a1},\mathscr{E},\mu_0,t) + \left(\Delta\widehat{\mathscr{E}}\frac{\partial}{\partial\mathscr{E}} + \Delta\widehat{\mu}\frac{\partial}{\partial\mu_0}\right)f_{a0}(\mathbf{x},\mathscr{E},\mu_0,t)\\ &= e_a\widehat{\phi}\frac{\partial f_{a0}}{\partial\mathscr{E}} + e_a\left(\widehat{\phi} - \frac{v_\parallel}{c}\widehat{A}_\parallel - \langle\widehat{\psi}_a\rangle_\xi\right)\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}\\ &\quad + \widehat{h}_a(\mathbf{x}-\boldsymbol{\rho}_{a1},\mathscr{E},\mu_0,t).\end{aligned} \tag{C23}$$

We find from using Eq. (C23) that Eqs. (C15) and (C16) are rewritten in the well-known forms as $-\nabla^2\widehat{\phi} = \sum_a e_a\int d^6z'\,\delta^3(\mathbf{x}'-\mathbf{x})\,\widehat{f}^{(p)}(\mathbf{z}')$ and $\nabla\times\widehat{\mathbf{B}} = (4\pi/c)\sum_a e_a\int d^6z'\,\delta^3(\mathbf{x}'-\mathbf{x})\widehat{f}^{(p)}(\mathbf{z}')\mathbf{v}'$, respectively.

APPENDIX D: WKB REPRESENTATION

Here, we consider any variable Q, the fluctuation part $\widehat{Q}$ of which has small wavelengths of the order of the gyroradius ρ in directions perpendicular to the background magnetic field. Then, we use the WKB (or ballooning) representation[25–27] for $\widehat{Q}$

$$\widehat{Q}(\mathbf{x},t) = \sum_{\mathbf{k}_\perp}\widehat{Q}_{\mathbf{k}_\perp}(\mathbf{x},t)\exp\left[iS_{\mathbf{k}_\perp}(\mathbf{x},t)\right], \tag{D1}$$

where $\widehat{Q}_{\mathbf{k}_\perp}(\mathbf{x},t)$ has the same gradient scale length L as that of the equilibrium field, while the eikonal $S_{\mathbf{k}_\perp}(\mathbf{x},t)$ represents the rapid variation with the wave number vector $\mathbf{k}_\perp \equiv \nabla S_{\mathbf{k}_\perp} (\sim \rho^{-1})$ which satisfies $\mathbf{k}_\perp\cdot\mathbf{b} = 0$.

The first-order fluctuation part $\widehat{f}_{a1}^{(p)}(\mathbf{z},t)$ of the distribution function in the particle coordinates is given by the WKB representation as

$$\widehat{f}_{a1}^{(p)}(\mathbf{z},t) = \sum_{\mathbf{k}_\perp}\widehat{f}_{a1\mathbf{k}_\perp}^{(p)}(\mathbf{z},t)\exp\left[iS_{\mathbf{k}_\perp}(\mathbf{x},t)\right]. \tag{D2}$$

The first-order fluctuation part $\widehat{f}_{a1}(\mathbf{Z},t)$ of the gyrocenter distribution function and its nonadiabatic part $\widehat{h}_a(\mathbf{Z},t)$ are given by the WKB representation as

$$\begin{bmatrix}\widehat{f}_{a1}(\mathbf{Z},t)\\ \widehat{h}_a(\mathbf{Z},t)\end{bmatrix} = \sum_{\mathbf{k}_\perp}\begin{bmatrix}\widehat{f}_{a1\mathbf{k}_\perp}(\mathbf{Z},t)\\ \widehat{h}_{a\mathbf{k}_\perp}(\mathbf{Z},t)\end{bmatrix}\exp\left[iS_{\mathbf{k}_\perp}(\mathbf{X},t)\right], \tag{D3}$$

where the gyrocenter position vector $\mathbf{X}$ is used in the eikonal $S_{\mathbf{k}_\perp}(\mathbf{X},t)$ instead of the particle position vector $\mathbf{x}$. From Eqs. (C13) and (C23), we have

$$\widehat{f}_{a1\mathbf{k}_\perp} = e_a\langle\widehat{\psi}_a\rangle_{\xi\mathbf{k}_\perp}\frac{\partial f_{a0}(\mathbf{X},\mathscr{E}_c,\mu)}{\partial\mathscr{E}_c} + \widehat{h}_{a\mathbf{k}_\perp} \tag{D4}$$

and

$$\widehat{f}_{a1\mathbf{k}_\perp}^{(p)} = e_a\widehat{\phi}_{\mathbf{k}_\perp}\frac{\partial f_{a0}}{\partial\mathscr{E}_c} + e_a\left(\widehat{\phi}_{\mathbf{k}_\perp} - \frac{U}{c}\widehat{A}_{\parallel\mathbf{k}_\perp} - \langle\widehat{\psi}_a\rangle_{\xi\mathbf{k}_\perp}e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_{a1}}\right)\times\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu} + \widehat{h}_{a\mathbf{k}_\perp}e^{-i\mathbf{k}_\perp\cdot\boldsymbol{\rho}_{a1}}, \tag{D5}$$

respectively, and Eq. (C14) is rewritten as

$$\left(\frac{\partial}{\partial t}+U\mathbf{b}\cdot\nabla+\mathbf{k}_\perp\cdot\mathbf{v}_{da}\right)\hat{h}_{a\mathbf{k}_\perp} - \sum_b e^{i\mathbf{k}_\perp\cdot\rho_{a1}}\left\langle (C_{ab}^{(p)})^L\left[\hat{h}_a e^{-i\mathbf{k}_\perp\cdot\rho_{a1}},\hat{h}_b e^{-i\mathbf{k}_\perp\cdot\rho_{b1}}\right]\right\rangle_\xi = -e_a\left(\frac{\partial f_{a0}}{\partial \mathscr{E}_c}\frac{\partial}{\partial t}+i\frac{c}{B}(\mathbf{b}\times\mathbf{k}_\perp)\cdot\nabla f_{a0}\right)\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}_\perp} + \frac{c}{B}\sum_{\mathbf{k}'_\perp+\mathbf{k}''_\perp=\mathbf{k}_\perp}\left[\mathbf{b}\cdot(\mathbf{k}'_\perp\times\mathbf{k}''_\perp)\right]\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}'_\perp}\hat{h}_{\mathbf{k}''_\perp}, \tag{D6}$$

where

$$\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}_\perp} = J_0\left(\frac{k_\perp W}{\Omega_a}\right)\left(\hat{\phi}_{\mathbf{k}_\perp}-\frac{U}{c}\hat{A}_{\|\mathbf{k}_\perp}\right)+J_1\left(\frac{k_\perp W}{\Omega_a}\right)\frac{W\hat{B}_{\|\mathbf{k}_\perp}}{c\,k_\perp}. \tag{D7}$$

Here, J_0 and J_1 are the first and second-order Bessel functions, respectively.

In the WKB representation, the fluctuation part of the gyrokinetic Poisson's equation in Eq. (C15) and that of the gyrokinetic Ampère's law in Eq. (C16) are given by

$$k_\perp^2\hat{\phi}_{\mathbf{k}_\perp} = 4\pi\sum_a e_a\int d\mathscr{E}_c\,d\mu\sum_{\sigma=\pm1}\frac{2\pi B}{m_a^2|U|}\left[e_a\hat{\phi}_{\mathbf{k}_\perp}\frac{\partial f_{a0}}{\partial\mathscr{E}_c} + e_a\left(\hat{\phi}_{\mathbf{k}_\perp}-\frac{U}{c}\hat{A}_{\|\mathbf{k}_\perp}-J_0(k_\perp W/\Omega_a)\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}_\perp}\right)\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu} + J_0(k_\perp W/\Omega_a)\hat{h}_{a\mathbf{k}_\perp}\right] \tag{D8}$$

and

$$k_\perp^2\hat{\mathbf{A}}_{\mathbf{k}_\perp} = \frac{4\pi}{c}\sum_a e_a\int d\mathscr{E}_c\,d\mu\sum_{\sigma=\pm1}\frac{2\pi B}{m_a^2|U|}\times\left[U\mathbf{b}\left\{e_a\left(\hat{\phi}_{\mathbf{k}_\perp}-\frac{U}{c}\hat{A}_{\|\mathbf{k}_\perp}-J_0(k_\perp W/\Omega_a)\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}_\perp}\right)\times\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}+J_0(k_\perp W/\Omega_a)\hat{h}_{a\mathbf{k}_\perp}\right\}+i\frac{\mathbf{b}\times\mathbf{k}_\perp}{k_\perp}\times WJ_1(k_\perp W/\Omega_a)\left\{-e_a\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}_\perp}\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}+\hat{h}_{a\mathbf{k}_\perp}\right\}\right], \tag{D9}$$

respectively. The component of Eq. (D9) in the direction parallel to the background magnetic field is written as

$$k_\perp^2\hat{A}_{\|\mathbf{k}_\perp} = \frac{4\pi}{c}\sum_a e_a\int d\mathscr{E}_c\,d\mu\sum_{\sigma=\pm1}\frac{2\pi B}{m_a^2|U|}U\times\left[e_a\left(\hat{\phi}_{\mathbf{k}_\perp}-\frac{U}{c}\hat{A}_{\|\mathbf{k}_\perp}-J_0(k_\perp W/\Omega_a)\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}_\perp}\right)\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}+J_0(k_\perp W/\Omega_a)\hat{h}_{\mathbf{k}_\perp}\right], \tag{D10}$$

where $\hat{A}_{\|\mathbf{k}_\perp}\equiv\hat{\mathbf{A}}_{\mathbf{k}_\perp}\cdot\mathbf{b}$. Taking the inner product of Eq. (D9) and $-i\mathbf{b}\times\mathbf{k}_\perp/k_\perp$ gives

$$-k_\perp\hat{B}_{\|\mathbf{k}_\perp} = \frac{4\pi}{c}\sum_a e_a\int d\mathscr{E}_c\,d\mu\sum_{\sigma=\pm1}\frac{2\pi B}{m_a^2|U|}\times WJ_1(k_\perp W/\Omega_a)\left(-e_a\langle\hat{\psi}_a\rangle_{\xi\mathbf{k}_\perp}\frac{1}{B}\frac{\partial f_{a0}}{\partial\mu}+\hat{h}_{a\mathbf{k}_\perp}\right), \tag{D11}$$

where $\hat{B}_{\|\mathbf{k}_\perp}\equiv i(\mathbf{k}_\perp\times\hat{\mathbf{A}}_{\mathbf{k}_\perp})\cdot\mathbf{b}$. It is found from the inner product of Eq. (D9) and $\mathbf{k}_\perp$ that the Coulomb gauge condition, $\mathbf{k}_\perp\cdot\hat{\mathbf{A}}_{\mathbf{k}_\perp}=0$, holds. Equations (D8), (D10), and (D11) agree with the gyrokinetic Poisson and Ampère equations derived in earlier works[25,26] using the WKB representation.

REFERENCES

1. Z. Lin, T. S. Hahm, W. W. Lee, W. M. Tang, and R. B. White, Science **281**, 1835 (1998).
2. Y. Idomura, Phys. Plasmas **24**, 080701 (2017).
3. S. Ku, C. S. Chang, and P. H. Diamond, Nucl. Fusion **49**, 115021 (2009).
4. W. X. Wang, T. S. Hahm, S. Ethier, G. Rewoldt, W. W. Lee, W. M. Tang, S. M. Kaye, and P. H. Diamond, Phys. Rev. Lett. **102**, 035005 (2009).
5. V. Grandgirard, J. Abiteboul, J. Bigot, T. Cartier-Michaud, N. Crouseilles, G. Dif-Pradalier, Ch. Ehrlacher, D. Esteve, X. Garbet, Ph. Ghendrih, G. Latu, M. Mehrenberger, C. Norscini, Ch. Passeron, F. Rozar, Y. Sarazin, E. Sonnendrücker, A. Strugarek, and D. Zarzoso, Comput. Phys. Commun. **207**, 35 (2016).
6. A. Bottino and E. Sonnendrücker, J. Plasma Phys. **81**, 435810501 (2015).
7. J. A. Heikkinen, S. J. Janhunen, T. P. Kiviniemi, and F. Ogando, J. Comput. Phys. **227**, 5582 (2008).
8. N. R. Mandell, A. Hakim, G. W. Hammett, and M. Francisquez, J. Plasma Phys. **86**, 905860109 (2020).
9. S. Matsuoka, H. Sugama, and Y. Idomura, Phys. Plasmas **28**, 064501 (2021).
10. R. G. Littlejohn, J. Math. Phys. **23**, 742 (1982).
11. A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).
12. H. Sugama, Phys. Plasmas **7**, 466 (2000).
13. B. Scott and J. Smirnov, Phys. Plasmas **17**, 112302 (2010).
14. A. J. Brizard and N. Tronko, Phys. Plasmas **18**, 082307 (2011).
15. F. I. Parra and I. Calvo, Plasma Phys. Controlled Fusion **53**, 045001 (2011).
16. H. Sugama, M. Nunami, M. Nakata, and T.-H. Watanabe, Phys. Plasmas **24**, 020701 (2017); H. Sugama, Rev. Mod. Plasma Phys. **1**, 9 (2017).
17. H. Sugama, M. Nunami, S. Satake, and T.-H. Watanabe, Phys. Plasmas **25**, 102506 (2018).
18. H. Sugama, S. Matsuoka, M. Nunami, and S. Satake, Phys. Plasmas **28**, 022312 (2021).
19. E. Hirvijoki, J. W. Burby, D. Pfefferlé, and A. J. Brizard, J. Phys. A: Math. Theor. **53**, 235204 (2020).
20. H. Qin, J. W. Burby, and R. C. Davidson, Phys. Rev. E **90**, 043102 (2014).
21. P. Fan, H. Qin, and J. Xiao, "Discovering exact local energy-momentum conservation laws for electromagnetic gyrokinetic system by high-order field theory on heterogeneous manifolds," arXiv:2006.11039 (2020).
22. A. J. Brizard, J. Plasma Phys. **87**, 905870307 (2021).
23. J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998), Chap. 6.
24. R. D. Hazeltine and J. D. Meiss, *Plasma Confinement* (Addison-Wesley, Redwood City, California, 1992), Chap. 4.5.
25. T. M. Antonsen, Jr. and B. Lane, Phys. Fluids **23**, 1205 (1980).
26. P. J. Catto, W. M. Tang, and D. E. Baldwin, Plasma Phys. **23**, 639 (1981).
27. E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982).
28. J. W. Burby and A. J. Brizard, Phys. Lett. A **383**, 2172 (2019).
29. F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **48**, 239 (1976).
30. S. P. Hirshman and D. J. Sigmar, Nucl. Fusion **21**, 1079 (1981).
31. P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge, 2002), Chap. 8.
32. W. Horton, *Turbulent Transport in Magnetized Plasmas*, 2nd ed. (World Scientific, Singapore, 2018), Chap. 12.
33. A. J. Brizard, Phys. Plasmas **11**, 4429 (2011).
34. H. Sugama, T.-H. Watanabe, and M. Nunami, Phys. Plasmas **22**, 082306 (2015).

[35]I. G. Abel, G. G. Plunk, E. Wang, M. Barnes, S. C. Cowley, W. Dorland, and A. A. Schekochihin, Rep. Prog. Phys. **76**, 116201 (2013).
[36]H. Sugama and W. Horton, Phys. Plasmas **5**, 2560 (1998).
[37]H. Sugama, M. Okamoto, W. Horton, and M. Wakatani, Phys. Plasmas **3**, 2379 (1996).
[38]J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, Texts in Applied Mathematics (Springer, 1999), Chap. 13.5.
[39]H. Cendra, D. D. Holm, M. J. W. Hoyle, and J. E. Marsden, J. Math. Phys. **39**, 3138 (1998).
[40]W. A. Newcomb, Nucl. Fusion Suppl. Part **2**, 451 (1962).
[41]J. Squire, H. Qin, W. M. Tang, and C. Chandre, Phys. Plasmas **20**, 022501 (2013).
[42]A. M. Dimits, G. Bateman, M. A. Beer, B. I. Cohen, W. Dorland, G. W. Hammett, C. Kim, J. E. Kinsey, M. Kotschenreuther, A. H. Kritz, L. L. Lao, J. Mandrekas, W. M. Nevins, S. E. Parker, A. J. Redd, D. E. Shumaker, R. Sydora, and J. Weiland, Phys. Plasmas **7**, 969 (2000).
[43]F. Jenko, W. Dorland, M. Kotschenreuther, and B. N. Rogers, Phys. Plasmas **7**, 1904 (2000).
[44]J. Candy and R. E. Waltz, J. Comput. Phys. **186**, 545 (2003).
[45]T.-H. Watanabe and H. Sugama, Nucl. Fusion **46**, 24 (2006).
[46]A. G. Peeters, Y. Camenen, F. J. Casson, W. A. Hornsby, A. P. Snodin, D. Strintzi, and G. Szepesi, Comput. Phys. Commun. **180**, 2650 (2009).
[47]R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).