Ellipticity of axisymmetric equilibria with flow and pressure anisotropy in single-fluid and Hall magnetohydrodynamics

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The ellipticity criteria for the partial differential equations of axisymmetric single-fluid and Hall magnetohydrodynamic (MHD) equilibria with flow and pressure anisotropy are investigated. The MHD systems are closed with cold ions and electron pressures derived from their parallel heat flux equations, a closure that reproduces the corresponding kinetic dispersion relation. In the single-fluid model, which differs from the double-adiabatic Chew–Goldberger–Low model, it is verified that the elliptic region boundaries occur at poloidal flow velocities equal to wave velocities from the kinetic dispersion relation. For Hall magnetohydrodynamics, a set of anisotropic-pressure equilibrium equations is derived and an ellipticity condition corresponding to a poloidal flow velocity slightly smaller than the ion sound velocity is obtained. © *2007 American Institute of Physics*.
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I. INTRODUCTION

In the ideal magnetohydrodynamics (MHD) description of plasmas, axisymmetric toroidal equilibria with flow are obtained by solving the so-called generalized Grad-Shafranov (GS) equation and the Bernoulli law.[1,2] When the flow is strong, the characteristics of this system of equations are quite different from those for the static case. The generalized GS partial differential equation (PDE) can be either elliptic or hyperbolic depending on the magnitude of the poloidal flow velocity relative to the velocities of MHD waves[1,2] and, in particular, transonic flow profiles become hyperbolic. Transonic poloidal flows are of interest because they may lead to the formation of transport barriers and profile pedestals in tokamaks.[3–5] For these steep plasma profile features, small-scale effects not included in the ideal MHD model should be significant.

Two-fluid effects resolve the Alfvén singularity[6–8] and modify the conditions for ellipticity.[9,10] In order to include further multiscale effects, one has to adopt proper fluid equations since the characteristics of flowing equilibria depend also on the closure models. As a simple example that brings the closure problem to the two-fluid theory, we will consider Hall MHD in the presence of pressure anisotropy. In single-fluid MHD, equilibria with flow and anisotropic pressure have been examined with a kinetic closure model known as Grad's guiding-center-particle (GCP) model[11] or kinetic MHD,[12] and with the Chew-Goldberger-Low (CGL) double-adiabatic model.[13] In the GCP model, the first hyperbolic region is resolved by the Landau damping.[14,15] In the CGL model, the ellipticity criteria are modified quantitatively by the pressure anisotropy.[15,16]

In this paper, we study the conditions for ellipticity of the PDEs for single-fluid and Hall MHD axisymmetric equilibria with flow and pressure anisotropy. We choose to define Hall MHD as two-fluid MHD with cold ions and massless electrons. Hence, this model includes the Hall current and the electron pressure in the generalized Ohm's law. With isotropic electron pressure, it yields a dispersion relation that agrees with kinetic theory without assuming any particular direction for wave propagation.[17] In order to preserve this feature in the presence of pressure anisotropy, we shall adopt here the collisionless magnetized fluid model proposed in Ref. 18, which includes the fluid moment equations for the parallel heat fluxes and closes them by using a shifted, two-temperature Maxwellian distribution function to evaluate the fourth-rank moments. This is different from the CGL double-adiabatic closure which, as is well known,[12] is not consistent with kinetic theory. The assumption of cold ions provides the simplest two-fluid model where we can include consistently anisotropic temperature effects (from the electrons). An analogous cold ion and high-beta anisotropic electron system has been considered, in a kinetic theory formulation, to study the stability of microwave-heated plasmas.[19] In order to include the hot ion effects that are relevant to fusion plasmas, an extension of the model is necessary. However, a consistent treatment of hot ions in a two-fluid framework must include the ion gyroviscosity and other finite ion Larmor radius effects and, with fast poloidal flows, the ion distribution function cannot be taken as close to Maxwellian and a kinetic ion closure is necessary.

The paper is organized as follows. In Sec. II, we introduce our Hall MHD equations in the presence of pressure anisotropy, present the linear dispersion relation for this model, and show its agreement with the kinetic one. In Sec. III, we investigate the equilibrium equations in the single-fluid limit of our anisotropic-pressure model and obtain their ellipticity criterion, which is consistent with the one based on kinetic wave propagation. In Sec. IV, we derive the equilibrium equations for our Hall MHD with flow and anisotropic pressure and examine their ellipticity. A summary is given in Sec. V.

II. BASIC EQUATIONS

The Hall MHD equations in the presence of pressure anisotropy are

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{v}_\nu) = 0 \quad (\nu = i,e), \tag{1}$$

$$m_i n\left(\frac{\partial}{\partial t} + \boldsymbol{v}_i \cdot \nabla\right)\boldsymbol{v}_i = \boldsymbol{j} \times \boldsymbol{B} - \nabla \cdot \boldsymbol{p}_e, \tag{2}$$

$$\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B} = \frac{1}{en}(\boldsymbol{j} \times \boldsymbol{B} - \nabla \cdot \boldsymbol{p}_e), \tag{3}$$

$$\boldsymbol{j} = ne(\boldsymbol{v}_i - \boldsymbol{v}_e), \tag{4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}, \tag{5}$$

$$\mu_0 \boldsymbol{j} = \nabla \times \boldsymbol{B}, \tag{6}$$

$$\boldsymbol{p}_e = p_{e\perp}\boldsymbol{I} + \frac{p_{e\parallel} - p_{e\perp}}{B^2}\boldsymbol{BB}, \tag{7}$$

where n is the density, $\boldsymbol{v}_i$ and $\boldsymbol{v}_e$ are the flow velocities of ions and electrons, m_i is the ion mass, e is the charge, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, $\boldsymbol{j}$ is the current density, and $\boldsymbol{p}_e$ is the electron pressure tensor. The electron mass m_e is neglected in Hall MHD. The right-hand side of the generalized Ohm's law (3) is the Hall term that gives rise to effects characterized by the ion skin depth, $d_i \equiv \sqrt{m_i/\mu_0 n e^2}$. The assumption of cold ions, $p_{i\parallel} = p_{i\perp} = 0$, is consistent with the neglect of finite ion Larmor radius effects such as the ion gyroviscosity, given that the ion Larmor radius ρ_i arises from the ion pressure p_i together with the ion skin depth d_i,

$$\rho_i = d_i\sqrt{\frac{p_i}{B^2/2\mu_0}}. \tag{8}$$

The MHD ordering assumes $|\boldsymbol{v}_i| \sim |\boldsymbol{v}_e| \sim (p_e/nm_i)^{1/2} \ll v_{Te}$, where v_{Te} is the thermal electron velocity. Equations for electron pressures can be obtained from the zero-Larmor-radius fluid moment equations for the parallel heat fluxes,[18]

$$m_\nu B q_{\nu B\parallel}\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right)\ln\left(\frac{B^3 q_{\nu B\parallel}}{n^4}\right) + \frac{3p_{\nu\parallel}}{2}\boldsymbol{B} \cdot \nabla\left(\frac{p_{\nu\parallel}}{n}\right) = 0, \tag{9}$$

$$m_\nu B q_{\nu T\parallel}\left(\frac{\partial}{\partial t} + \boldsymbol{v}_\nu \cdot \nabla\right)\ln\left(\frac{q_{\nu T\parallel}}{n^2}\right) + p_{\nu\parallel}\boldsymbol{B} \cdot \nabla\left(\frac{p_{\nu\perp}}{n}\right) - \frac{(p_{\nu\parallel} - p_{\nu\perp})p_{\nu\perp}}{n}\boldsymbol{B} \cdot \nabla(\ln B) = 0, \tag{10}$$

where $q_{\nu B\parallel}$ and $q_{\nu T\parallel}$ are the parallel and perpendicular components of the parallel heat flux $q_{\nu\parallel} \equiv q_{\nu B\parallel} + q_{\nu T\parallel}$, and are assumed to be of the order of the pressure multiplied by the macroscopic velocity. These equations have been closed by using shifted, two-temperature Maxwellian distribution functions to evaluate the fourth-rank moments. The first terms on the left-hand sides of Eqs. (9) and (10) vanish for massless electrons ($m_e = 0$) and we get the following equations for the electron parallel and perpendicular pressures:

$$\boldsymbol{B} \cdot \nabla\left(\frac{p_{e\parallel}}{n}\right) = 0, \quad \boldsymbol{B} \cdot \nabla\left[\left(\frac{p_{e\parallel}}{p_{e\perp}} - 1\right)B\right] = 0. \tag{11}$$

Based on kinetic theory, it has been reported[15] that the same pressure equations (11) also hold for ions in axisymmetric MHD equilibria with purely toroidal flow $\boldsymbol{v}_i = v_{i\varphi} R \nabla \varphi$. This is understood from Eqs. (9) and (10) by taking $\partial/\partial t = 0$ and $\boldsymbol{v}_i \cdot \nabla = R^{-1} v_{i\varphi} \partial/\partial\varphi = 0$. For massless electrons, the pressure equations (11) do not require either steady-state or purely toroidal flow.

Equations (1)–(7) and (11) form a closed set. The corresponding dispersion relation for waves in a homogeneous plasma, whose derivation is outlined in Appendix A, is

$$(\omega^2 - k_\parallel^2 V_A^2 \tau)[\omega^4 - (k_\perp^2 \alpha + k_\parallel^2 \gamma)V_A^2\omega^2 + k_\parallel^2 C_{s\parallel}^2 (k_\parallel^2 \tau + k_\perp^2 \kappa)V_A^2] = k_\parallel^2 d_i^2 V_A^2 \omega^2 \tau (k_\parallel^2 \tau + k_\perp^2 \sigma)(\omega^2 - k^2 C_{sp}^2), \tag{12}$$

where

$$\alpha \equiv 1 + \frac{2p_{e\parallel} - p_{e\perp}}{B^2/\mu_0}\frac{p_{e\perp}}{p_{e\parallel}}, \tag{13}$$

$$\gamma \equiv 1 + \frac{p_{e\perp}}{B^2/\mu_0}, \tag{14}$$

$$\tau \equiv 1 - \frac{p_{e\parallel} - p_{e\perp}}{B^2/\mu_0}, \tag{15}$$

$$\kappa \equiv 1 + \frac{p_{e\parallel} - p_{e\perp}}{B^2/\mu_0}\frac{2p_{e\perp}}{p_{e\parallel}}, \tag{16}$$

$$\sigma \equiv 1 + \frac{p_{e\parallel} - p_{e\perp}}{B^2/\mu_0}\left(1 + \frac{p_{e\perp}}{p_{e\parallel}}\right), \tag{17}$$

$$C_{sp}^2 \equiv C_{s\parallel}^2 \frac{k_\parallel^2 \tau + k_\perp^2 \kappa}{k_\parallel^2 \tau + k_\perp^2 \sigma}, \tag{18}$$

ω is the frequency of perturbation, $k_\parallel$ and $k_\perp$ are the parallel and perpendicular wave numbers, $V_A = B/\sqrt{\mu_0 n m_i}$ is the Alfvén velocity, and $C_{s\parallel} = \sqrt{p_{e\parallel}/nm_i}$ is the sound velocity based on the electron parallel pressure. If the background has a homogeneous flow velocity $\boldsymbol{v}_i$, the frequency ω in Eq. (12) is replaced with the Doppler-shifted one, $\omega - \boldsymbol{k} \cdot \boldsymbol{v}_i$. When $p_{e\parallel} = p_{e\perp}$, Eq. (12) reduces to the isotropic Hall MHD dispersion relation,[17,20–25] which represents three MHD waves, the shear Alfvén, slow, and fast magnetosonic waves, modified by the Hall term on the right-hand side. In the limit of short wavelength $kd_i \to \infty$, the phase velocities of the waves are 0, C_{sp}, and ∞, with C_{sp} being a modified sound velocity in the presence of pressure anisotropy. In the limit $d_i \to 0$, Eq. (12) is the dispersion relation for single-fluid MHD in the presence of pressure anisotropy and gives the stability condition, i.e., the condition for the real frequency ω, for the firehose instability,

$$\tau = 1 - \frac{p_{e\parallel} - p_{e\perp}}{B^2/\mu_0} > 0, \tag{19}$$

and for the mirror instability,

$$\kappa = 1 + \frac{p_{e\parallel} - p_{e\perp}}{B^2/\mu_0} \frac{2p_{e\perp}}{p_{e\parallel}} > 0, \tag{20}$$

The conditions (19) and (20) coincide with those found in kinetic theory, in contrast with the double-adiabatic CGL result for the mirror instability,[12]

$$\kappa_{CGL} = 1 + \frac{p_{e\parallel} - p_{e\perp}/6}{B^2/\mu_0} \frac{2p_{e\perp}}{p_{e\parallel}} > 0. \tag{21}$$

Furthermore, an identical full dispersion relation can be obtained from kinetic theory for cold ions and massless electrons, $0 = v_{Ti} \ll \omega/|k_\parallel| \ll v_{Te}$, where v_{Ti} is the thermal ion velocities so that the Landau damping can be neglected, as was the case in an isotropic plasma.[17] The derivation is also shown in Appendix A.

III. ANISOTROPIC SINGLE-FLUID EQUILIBRIUM

Omitting the Hall term, i.e., the right-hand side of Eq. (3), Eqs. (1)–(7) and (11) reduce to our anisotropic single-fluid MHD equations. Here we shall consider the corresponding toroidal axisymmetric equilibria, where the magnetic field $\boldsymbol{B}$, the flow velocity $\boldsymbol{v}_i$, the electric field $\boldsymbol{E}$, and the current density $\boldsymbol{j}$ can be written as

$$\boldsymbol{B} = \nabla\psi(R,Z) \times \nabla\varphi + RB_\varphi(R,Z)\nabla\varphi, \tag{22}$$

$$n\boldsymbol{v}_i = \nabla\Psi_i(R,Z) \times \nabla\varphi + nRv_{i\varphi}(R,Z)\nabla\varphi, \tag{23}$$

$$\boldsymbol{E} = -\nabla\Phi(R,Z), \tag{24}$$

$$\mu_0\boldsymbol{j} = \nabla(RB_\varphi) \times \nabla\varphi - \Delta^*\psi\nabla\varphi, \tag{25}$$

where ψ and Ψ_i are the poloidal magnetic flux and ion stream functions, Φ is the electrostatic potential, and $\Delta^* \equiv R^2\nabla\cdot[R^{-2}\nabla]$. Generalizing the formulation in Refs. 2 and 15 to our anisotropic pressures, we obtain the following equations for equilibrium states:

$$p_{e\parallel} = nT_\parallel(\psi), \quad \frac{p_{e\parallel}}{p_{e\perp}} - 1 = \frac{B_0}{B}\Delta(\psi), \tag{26}$$

$$\Psi_i = \Psi_M(\psi), \quad \frac{nV_p}{B_p} = \Psi_M'(\psi), \tag{27}$$

$$\Phi = \Phi_E(\psi), \tag{28}$$

$$\frac{v_{i\varphi}}{R} - \frac{B_\varphi}{nR}\Psi_M'(\psi) = \Phi_E'(\psi), \tag{29}$$

$$\tau_1 RB_\varphi - \mu_0 m_i R^2 \Psi_M'(\psi)\Phi_E'(\psi) = I_M(\psi), \tag{30}$$

the Bernoulli law,

$$m_i\left(\frac{v_i^2}{2} - Rv_{i\varphi}\Phi_E'\right) + T_\parallel \ln\left(\frac{np_{e\parallel}}{p_{e\perp}}\right) = H(\psi),$$

or

$$\frac{m_i}{2}\left[\left(\frac{B\Psi_M'}{n}\right)^2 - (R\Phi_E')^2\right] + T_\parallel \ln\left(\frac{np_{e\parallel}}{p_{e\perp}}\right) = H(\psi), \tag{31}$$

and the generalized GS equation,

$$\mu_0^{-1}\nabla\cdot\left(\frac{\tau_1}{R^2}\nabla\psi\right) + nT_\parallel' - \frac{B_0\Delta' p_{e\parallel}}{B_0\Delta + B} - nT_\parallel' \ln\left(\frac{p_{e\parallel}n}{p_{e\perp}}\right) + nH' + \frac{B_\varphi I_M'}{\mu_0 R} + m_i\Psi_M''\left(RB_\varphi\Phi_E' + \frac{B^2\Psi_M'}{n}\right) + nm_iRv_{i\varphi}\Phi_E'' = 0, \tag{32}$$

where

$$\tau_1 \equiv \tau - M_{Ap}^2 = 1 - \frac{p_{e\parallel} - p_{e\perp}}{B^2/\mu_0} - M_{Ap}^2.$$

$B_p \equiv |\nabla\psi|/R$, $V_p \equiv |\nabla\Psi_i|/nR$ are the poloidal components of the magnetic field and the flow velocity, $B = (B_p^2 + B_\varphi^2)^{1/2}$, $v_i = (V_p^2 + v_{i\varphi}^2)^{1/2}$ are the total magnetic field and flow velocity magnitudes, B_0 is a constant, $M_{Ap} = V_p/V_{Ap}$ is the poloidal Alfvén Mach number, $V_{Ap} \equiv B_p/\sqrt{\mu_0 n m_i}$ is the poloidal Alfvén velocity, and the prime denotes derivative with respect to ψ. Equations (26) result from the pressure equations (11). One can obtain Eqs. (27)–(29) from the projection of Faraday's law along $\nabla\varphi$, $\boldsymbol{B}$ and $\nabla\psi$, and Eqs. (30)–(32) from those of the momentum equation, respectively. There are six arbitrary functions of ψ: $T_\parallel$, Δ, Ψ_M, Φ_E, I_M, and H. In cylindrical coordinates (R,φ,Z), the second-order derivatives of Eq. (32) are

$$[\tau_1 + 2\dot{\tau}_1(\partial_R\psi)^2]\partial_{RR}^2\psi + 4\dot{\tau}_1(\partial_R\psi)(\partial_Z\psi)\partial_{RZ}^2\psi + [\tau_1 + 2\dot{\tau}_1(\partial_Z\psi)^2]\partial_{ZZ}^2\psi,$$

where $\dot{\tau}_1 = \partial\tau_1/\partial|\nabla\psi|^2$. The characteristic determinant D is given by

$$D = \tau_1^2 B^2 \alpha(M_{Ap}^2 - \kappa\beta_\parallel/\alpha)/X_1(M_{Ap}^2),$$

where

$$X_1(M_{Ap}^2) = B_p^2 M_{Ap}^4 - (\alpha B_\varphi^2 + \gamma B_p^2)M_{Ap}^2 + \beta_\parallel(\tau B_p^2 + \kappa B_\varphi^2),$$

$\beta_\parallel = p_{e\parallel}/(B^2/\mu_0)$. The generalized GS equation is elliptic if $D < 0$. In the absence of poloidal flow, the equilibrium is elliptic if $\tau > 0$ and $\kappa > 0$, which are the kinetic stability conditions for the firehose and mirror modes. In the presence of poloidal flow, there are three elliptic regions: $M_{Ap}^2 < \kappa\beta_\parallel/\alpha$, $M_s^2 < M_{Ap}^2 < \tau$, and $\tau < M_{Ap}^2 < M_f^2$, where M_s^2 and M_f^2 ($M_s^2 < M_f^2$) are the roots of $X_1(M_{Ap}^2) = 0$ and correspond to the phase velocities of the slow and fast magnetosonic waves modified by the pressure anisotropy. These critical velocities can be obtained from the dispersion relation (12) in its $d_i \to 0$ limit, i.e., its left-hand side, and are different from those found with the double-adiabatic CGL model.[15,16]

IV. ANISOTROPIC HALL MHD EQUILIBRIUM

The equilibrium equations for our anisotropic Hall MHD ($d_i \neq 0$) are

$$\nabla \cdot (n\boldsymbol{v}_\nu) = 0, \tag{33}$$

$$\boldsymbol{E} = -\boldsymbol{v}_e \times \boldsymbol{B} - \frac{1}{en}\left[\nabla p_{e\perp} + \boldsymbol{B}\cdot\nabla\left(\frac{p_{e\parallel}-p_{e\perp}}{B^2}\boldsymbol{B}\right)\right], \tag{34}$$

$$\boldsymbol{E} = -\boldsymbol{v}_i \times \boldsymbol{B} + \frac{m_i}{e}(\boldsymbol{v}_i \cdot \nabla \boldsymbol{v}_i), \tag{35}$$

$$\nabla \times \boldsymbol{E} = 0, \tag{36}$$

and Eqs. (4), (6), and (11). In addition to (22)–(24), we introduce the following variables:

$$n\boldsymbol{v}_e = \nabla \Psi_e \times \nabla \varphi + nRv_{e\varphi}\nabla \varphi, \tag{37}$$

where Ψ_e is the electron stream function. The poloidal and toroidal components of the current density are given by

$$RB_\varphi = \mu_0 e(\Psi_i - \Psi_e), \tag{38}$$

$$-\Delta^* \psi = en\mu_0 R(v_{i\varphi} - v_{e\varphi}). \tag{39}$$

The axisymmetric toroidal equilibria are described by Eq. (26) and the following equations:

$$\Psi_e + \frac{RB_\varphi}{eB^2}(p_{e\parallel} - p_{e\perp}) = \Psi_{*e}(\psi),$$
$$RB_\varphi = \mu_0 e[\Psi_i - \Psi_{*e}(\psi)]/\tau, \tag{40}$$

$$e\Phi - T_\parallel(\psi)\ln\left(\frac{np_{e\parallel}}{p_{e\perp}}\right) = e\Phi_*(\psi), \tag{41}$$

$$v_{e\varphi} = \frac{1}{en}R\nabla\cdot\left(\frac{p_{e\parallel}-p_{e\perp}}{R^2B^2}\nabla\psi\right) + \frac{B_\varphi}{n}\frac{d\Psi_{*e}}{d\psi} + R\frac{d\Phi_*}{d\psi} + \frac{R}{e}\frac{dT_\parallel}{d\psi}\ln\left(\frac{np_{e\parallel}}{p_{e\perp}}\right) - \frac{R}{e}\frac{dT_\parallel}{d\psi} + \frac{R(p_{e\parallel}-p_{e\perp})}{en}\frac{d\ln\Delta}{d\psi}, \tag{42}$$

$$\psi + \frac{m_i}{e}Rv_{i\varphi} = \chi(\Psi_i), \tag{43}$$

the Bernoulli law for ions,

$$\frac{m_i}{2}v_i^2 + e\Phi = W(\Psi_i), \tag{44}$$

and the coupled GS equations for ψ and Ψ_i,

$$\frac{R^2}{\mu_0}\nabla\cdot\left(\frac{\tau}{R^2}\nabla\psi\right) + enRv_{i\varphi} - eRB_\varphi\frac{d\Psi_{*e}}{d\psi} - R^2(p_{e\parallel}-p_{e\perp})\frac{d\ln\Delta}{d\psi} - neR^2\frac{d\Phi_*}{d\psi} + nR^2\left[1 - \ln\left(\frac{np_{e\parallel}}{p_{e\perp}}\right)\right]\frac{dT_\parallel}{d\psi} = 0, \tag{45}$$

$$m_iR^2\nabla\cdot\left(\frac{1}{nR^2}\nabla\Psi_i\right) - eRB_\varphi + enRv_{i\varphi}\frac{d\chi}{d\Psi_i} - nR^2\frac{dW}{d\Psi_i} = 0. \tag{46}$$

One can obtain Eqs. (40)–(42) from the projection of Eq. (34) along $\nabla\varphi$, $\boldsymbol{B}$ and $\nabla\psi$, and Eqs. (43), (44), and (46) from that of Eq. (35) along $\nabla\varphi$, $\boldsymbol{v}_i$, and $\nabla\Psi_i$, respectively. The GS equation for ψ, Eq. (45), is given by substituting Eqs. (42) and (43) into Eq. (39). There are six arbitrary functions of ψ or Ψ_i: $T_\parallel(\psi)$, $\Delta(\psi)$, $\Phi_*(\psi)$, $\Psi_{*e}(\psi)$, $\chi(\Psi_i)$, and $W(\Psi_i)$. The second-order derivatives of GS equations (45) and (46) are

$$\tau[\Delta^*\psi + \nabla\psi\cdot\nabla\ln\tau] = \tau\left[\Delta^*\psi + \frac{\partial\ln\tau}{\partial|\nabla\psi|^2}\nabla\psi\cdot\nabla(|\nabla\psi|^2) + \frac{\partial\ln\tau}{\partial|\nabla\Psi_i|^2}\nabla\psi\cdot\nabla(|\nabla\Psi_i|^2)\right] + \cdots, \tag{47}$$

$$\Delta^*\Psi_i - \nabla\Psi_i\cdot\nabla\ln n = \Delta^*\Psi_i - \frac{\partial\ln n}{\partial|\nabla\psi|^2}\nabla\Psi_i\cdot\nabla(|\nabla\psi|^2) - \frac{\partial\ln n}{\partial|\nabla\Psi_i|^2}\nabla\Psi_i\cdot\nabla(|\nabla\Psi_i|^2) + \cdots, \tag{48}$$

where

$$\frac{\partial\ln\tau}{\partial|\nabla\psi|^2} = \frac{(1-\tau)m_i}{2R^2B^2X_2}\left[V_p^2\left(2 + \frac{p_{e\perp}}{p_{e\parallel}}\right) - C_{s\parallel}^2\left(1 + \frac{2p_{e\perp}}{p_{e\parallel}}\right)\right], \tag{49}$$

$$\frac{\partial\ln\tau}{\partial|\nabla\Psi_i|^2} = -\frac{(1-\tau)m_i}{2R^2n^2X_2}, \tag{50}$$

$$\frac{\partial\ln n}{\partial|\nabla\psi|^2} = -\frac{(1-\tau)\tau}{2R^2\mu_0 nX_2}, \tag{51}$$

$$\frac{\partial\ln n}{\partial|\nabla\Psi_i|^2} = \frac{m_i(B_p^2\tau + B_\varphi^2\sigma)}{2R^2n^2B^2X_2}, \tag{52}$$

$$X_2 \equiv \frac{m_i(B_p^2\tau + B_\varphi^2\sigma)}{B^2}(V_p^2 - C_{sp}'^2), \tag{53}$$

$$C_{sp}'^2 \equiv C_{s\parallel}^2\frac{B_p^2\tau + B_\varphi^2\kappa}{B_p^2\tau + B_\varphi^2\sigma}. \tag{54}$$

The derivation of Eqs. (49)–(52) are shown in Appendix B. Then, the the coupled GS equations are rewritten as

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} \partial^2_{RR}\psi \\ \partial^2_{RR}\Psi_i \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} \partial^2_{RZ}\psi \\ \partial^2_{RZ}\Psi_i \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\begin{pmatrix} \partial^2_{ZZ}\psi \\ \partial^2_{ZZ}\Psi_i \end{pmatrix} + \cdots = 0, \tag{55}$$

where

$$a_1 \equiv \tau\left[1 + 2\frac{\partial \ln \tau}{\partial|\nabla\psi|^2}(\partial_R\psi)^2\right], \tag{56}$$

$$a_2 \equiv 2\tau\frac{\partial \ln \tau}{\partial|\nabla\Psi_i|^2}(\partial_R\psi)(\partial_R\Psi_i), \tag{57}$$

$$a_3 \equiv -2\frac{\partial \ln n}{\partial|\nabla\psi|^2}(\partial_R\psi)(\partial_R\Psi_i), \tag{58}$$

$$a_4 \equiv 1 - 2\frac{\partial \ln n}{\partial|\nabla\Psi_i|^2}(\partial_R\Psi_i)^2, \tag{59}$$

$$b_1 \equiv 4\tau\frac{\partial \ln \tau}{\partial|\nabla\psi|^2}(\partial_R\psi)(\partial_Z\psi), \tag{60}$$

$$b_2 \equiv 2\tau\frac{\partial \ln \tau}{\partial|\nabla\Psi_i|^2}[(\partial_Z\psi)(\partial_R\Psi_i) + (\partial_R\psi)(\partial_Z\Psi_i)], \tag{61}$$

$$b_3 \equiv -2\frac{\partial \ln n}{\partial|\nabla\psi|^2}[(\partial_R\psi)(\partial_Z\Psi_i) + (\partial_Z\psi)(\partial_R\Psi_i)], \tag{62}$$

$$b_4 \equiv -4\frac{\partial \ln n}{\partial|\nabla\Psi_i|^2}(\partial_R\Psi_i)(\partial_Z\Psi_i), \tag{63}$$

$$c_1 \equiv \tau\left[1 + 2\frac{\partial \ln \tau}{\partial|\nabla\psi|^2}(\partial_Z\psi)^2\right], \tag{64}$$

$$c_2 \equiv 2\tau\frac{\partial \ln \tau}{\partial|\nabla\Psi_i|^2}(\partial_Z\psi)(\partial_Z\Psi_i), \tag{65}$$

$$c_3 \equiv -2\frac{\partial \ln n}{\partial|\nabla\psi|^2}(\partial_Z\psi)(\partial_Z\Psi_i), \tag{66}$$

$$c_4 \equiv 1 - 2\frac{\partial \ln n}{\partial|\nabla\Psi_i|^2}(\partial_Z\Psi_i)^2. \tag{67}$$

In the absence of pressure anisotropy ($p_{e\parallel}=p_{e\perp}=p_e$), the three tensors in (55) are all diagonal since the coefficients $\partial \ln \tau/\partial|\nabla\Psi_i|^2$ and $\partial \ln n/\partial|\nabla\psi|^2$ vanish; thus, the determinants for ellipticity of Eqs. (45) and (46) are decoupled as shown in Ref. 9. In the presence of pressure anisotropy, the determinants are coupled and cannot be examined separately. One can find the conditions for ellipticity of such systems involving higher-order derivatives by examining the existence of wave-type solutions, a method also applicable to second-order differential equations.[26,27] Here we consider a wave propagating in one-dimensional space R and time Z, and having discontinuity across the wave front $\xi(R,Z)=0$. Using curvilinear coordinates of $\xi(R,Z)=\text{const.}$ and its normal surface $\zeta(R,Z)=\text{constant}$, Eq. (55) is transformed to

$$\begin{bmatrix} F_1(u) & G_1(u) \\ G_2(u) & F_2(u) \end{bmatrix}\begin{pmatrix} \partial^2_{\xi\xi}\psi \\ \partial^2_{\xi\xi}\Psi_i \end{pmatrix} + \cdots = 0, \tag{68}$$

where

$$F_1(u) \equiv a_1 - b_1 u + c_1 u^2 = \tau(u^2+1) + 2\tau\frac{\partial \ln \tau}{\partial|\nabla\psi|^2}[(\partial_Z\psi)u - (\partial_R\psi)]^2, \tag{69}$$

$$F_2(u) \equiv a_4 - b_4 u + c_4 u^2 = u^2 + 1 - 2\frac{\partial \ln n}{\partial|\nabla\Psi_i|^2}[(\partial_Z\Psi_i)u - (\partial_R\Psi_i)]^2, \tag{70}$$

$$G_1(u) \equiv a_2 - b_2 u + c_2 u^2 = 2\tau\frac{\partial \ln \tau}{\partial|\nabla\Psi_i|^2}[(\partial_Z\psi)u - (\partial_R\psi)][(\partial_Z\Psi_i)u - (\partial_R\Psi_i)], \tag{71}$$

$$G_2(u) \equiv a_3 - b_3 u + c_3 u^2 = 2\frac{\partial \ln n}{\partial|\nabla\psi|^2}[(\partial_Z\psi)u - (\partial_R\psi)][(\partial_Z\Psi_i)u - (\partial_R\Psi_i)], \tag{72}$$

and $u \equiv dR/dZ$ means the velocity of the wave front. If we assume the wave is smooth, the discontinuity at the wave front $\xi=0$ appears only in the highest-order derivatives with respect to ξ. It implies

$$\begin{vmatrix} F_1(u) & G_1(u) \\ G_2(u) & F_2(u) \end{vmatrix} = 0, \tag{73}$$

which yields an algebraic equation for u,

$$F(u) = G(u), \tag{74}$$

where

$$F(u) \equiv F_1(u)F_2(u), \tag{75}$$

$$G(u) \equiv -4\tau\frac{\partial \ln \tau}{\partial|\nabla\Psi_i|^2}\frac{\partial \ln n}{\partial|\nabla\psi|^2}[(\partial_Z\psi)u - (\partial_R\psi)]^2[(\partial_Z\Psi_i)u - (\partial_R\Psi_i)]^2. \tag{76}$$

If u is real, there exist wave-type, i.e., hyperbolic, solutions for the partial differential equations. Ellipticity of the coupled GS equations requires the nonexistence of real solutions of u, for Eq. (74). The discriminant for the solutions of $F_1(u)=0$ is

$$D_1 = -4\tau^2\left(1 + 2\frac{\partial \ln \tau}{\partial|\nabla\psi|^2}|\nabla\psi|^2\right), \tag{77}$$

$$= -\frac{4B^2\tau^2\sigma[V_p^2 - (\kappa/\sigma)C_{s\parallel}^2]}{(B_p^2\tau + B_\varphi^2\sigma)(V_p^2 - C'^2_{sp})}, \tag{78}$$

and that of $F_2(u)=0$ is

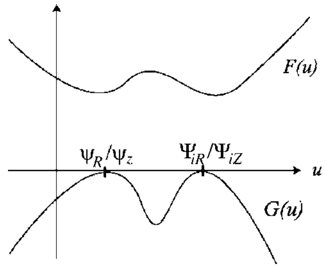

FIG. 1. Sketch of $F(u)$ and $G(u)$.

$$D_2 = -4\left[1 - 2\frac{\partial \ln n}{\partial|\nabla\Psi_i|^2}|\nabla\Psi_i|^2\right], \tag{79}$$

$$= \frac{4C_{sp}'^2}{V_p^2 - C_{sp}'^2}. \tag{80}$$

The right-hand side of Eq. (74) is not zero in the presence of both poloidal flow and pressure anisotropy. We consider the three cases as follows: (i) If the pressure is isotropic, D_1 is always negative and $D_2<0$ yields the condition for ellipticity shown in Ref. 9,

$$V_p^2 < C_s^2 \equiv p_e/nm_i. \tag{81}$$

(ii) If there is no poloidal flow, D_2 is always negative and $D_1<0$ is satisfied by the condition for ellipticity of single-fluid MHD equilibria with purely toroidal flow, $\tau>0$ and $\kappa>0$, due to the relation

$$\sigma - \kappa = (p_{e\parallel} - p_{e\perp})^2/(p_{e\parallel}B^2/\mu_0) > 0. \tag{82}$$

(iii) Provided the condition in (ii) holds, we examine now the dependence of the ellipticity condition on the poloidal flow velocity in the presence of pressure anisotropy. Since

$$C_{sp}'^2 - \frac{\kappa}{\sigma}C_{s\parallel}^2 = \frac{B_p^2\tau(\sigma-\kappa)}{\sigma(B_p^2\tau + B_\varphi^2\sigma)}C_{s\parallel}^2 > 0, \tag{83}$$

both $D_1<0$ and $D_2<0$ are satisfied by

$$V_p^2 < (\kappa/\sigma)C_{s\parallel}^2. \tag{84}$$

The relation (84) gives the relations

$$\frac{\partial \ln \tau}{\partial|\nabla\psi|^2} > -(2B_p^2R^2)^{-1} > -[2(\partial_Z\psi)^2]^{-1} \tag{85}$$

and

$$1 - 2\frac{\partial \ln n}{\partial|\nabla\Psi_i|^2}(\partial_Z\Psi_i)^2 > 0. \tag{86}$$

Thus, $F(u)$ is positive when $\tau>0$ and $\kappa>0$. Since

$$G(u) = -\frac{m_i\tau^2}{\mu_0 n}\left(\frac{1-\tau}{2R^2nX}\right)^2 \leqslant 0, \tag{87}$$

the nonexistence of real roots of $F(u)=0$ is a sufficient condition for the nonexistence of real roots of $F(u)=G(u)$ (see Fig. 1). The condition (84) is therefore a sufficient condition for the ellipticity. We note that the condition for ellipticity (84) is obtained from F_1, i.e., the GS equation for ψ, Eq. (45), not from that for Ψ_i, Eq. (46), as in the absence of pressure anisotropy, and the critical velocity $\sqrt{\kappa/\sigma}C_{s\parallel}$ is slightly smaller than the ion sound velocity C_{sp}'.

V. SUMMARY

In this paper, we have obtained conditions for ellipticity of axisymmetric toroidal equilibria with flow and pressure anisotropy. We have adopted pressure equations that reproduce the kinetic dispersion relation in the limit of cold ions and massless electrons. The ellipticity conditions in our single-fluid model are related to the velocities of waves from the kinetic dispersion relation and are different from those of the double-adiabatic CGL model. For anisotropic Hall MHD, we have obtained a sufficient condition for ellipticity that corresponds to a poloidal flow velocity slightly smaller than the sound velocity.

The fluid moment equations used in this paper include the Hall term and the pressure anisotropy for electrons. To include more small-scale effects such as the gyroviscosity and the Landau damping, a more advanced closure model applicable to finite ion pressures should be adopted.

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APPENDIX A: DERIVATION OF THE DISPERSION RELATION

Here we derive the linear dispersion relation in an infinite homogeneous background with constant density, magnetic field, and anisotropic pressures, and vanishing flow velocity. Linearizing Eqs. (1)–(7) and (11) with the background magnetic field in the z direction, $\boldsymbol{B}_0=B_0\boldsymbol{e}_z$, and the plane-wave perturbations varying proportional to $\exp[i(-\omega t+k_\perp x+k_\parallel z)]$, we obtain

$$\omega n_1 = n_0(\boldsymbol{k}\cdot\boldsymbol{v}_{i1}), \tag{A1}$$

$$-n_0m_i\omega\boldsymbol{v}_{i1} = \frac{B_0}{\mu_0}(k_\parallel\boldsymbol{B}_1 - \boldsymbol{k}B_{1z}) - \boldsymbol{k}\cdot\boldsymbol{p}_{e1}, \tag{A2}$$

$$-\omega\boldsymbol{B}_1 = k_\parallel B_0\boldsymbol{v}_{i1} - \boldsymbol{B}_0(\boldsymbol{k}\cdot\boldsymbol{v}_{i1}) - \frac{ik_\parallel B_0}{\mu_0 n_0 e}(\boldsymbol{k}\times\boldsymbol{B}_1) + \boldsymbol{k} \times\left(\frac{i}{n_0e}\boldsymbol{k}\cdot\boldsymbol{p}_{e1}\right), \tag{A3}$$

$$\frac{ik_\parallel B_0}{n_0}\left(p_{e\parallel 1}-\frac{p_{e\parallel 0}}{n_0}n_1\right)=0,$$
$$\frac{ik_\parallel B_0}{n_0}\left(p_{e\perp 1}-\frac{p_{e\perp 0}}{n_0}n_1\right)-\frac{p_{e\parallel 0}-p_{e\perp 0}}{n_0}\frac{p_{e\perp 0}}{p_{e\parallel 0}}ik_\parallel B_{1z}=0, \quad \text{(A4)}$$

where $\boldsymbol{k}=k_\perp \boldsymbol{e}_x+k_\parallel \boldsymbol{e}_z$ is the wave-number vector. The subscripts 0 and 1 denote equilibrium and perturbation quantities, respectively. The gradient of the electron pressure tensor

$$\nabla\cdot\boldsymbol{p}_e=\nabla p_{e\perp}+\boldsymbol{B}\cdot\nabla\left(\frac{p_{e\parallel}-p_{e\perp}}{B^2}\boldsymbol{B}\right) \quad \text{(A5)}$$

can be linearized as

$$\boldsymbol{k}\cdot\boldsymbol{p}_{e1}=\boldsymbol{k}p_{e\perp 1}+\frac{p_{e\parallel 0}-p_{e\perp 0}}{B_0}k_\parallel \boldsymbol{B}_1 +k_\parallel \boldsymbol{B}_0\left(\frac{p_{e\parallel 1}-p_{e\perp 1}}{B_0}-2\frac{p_{e\parallel 0}-p_{e\perp 0}}{B_0^2}B_{1z}\right), \quad \text{(A6)}$$

and the magnetic field strength B can be approximated as

$$B\simeq\sqrt{B_0^2+2\boldsymbol{B}_0\cdot\boldsymbol{B}_1}\simeq B_0+B_{1z}. \quad \text{(A7)}$$

Eliminating the variables with the procedure similar to that used in Ref. 22 and dropping the subscript 0 for the equilibrium quantities, we obtain the dispersion relation (12).

The dispersion relation (12) is also obtained from kinetic theory. The wave equation for a homogeneous plasma is given by

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{E})+\boldsymbol{\epsilon}\cdot\boldsymbol{E}=0, \quad \text{(A8)}$$

where $\boldsymbol{n}=\boldsymbol{k}c/\omega$, c is the speed of light and $\boldsymbol{\epsilon}$ is the dielectric tensor. Equation (A8) is rewritten in matrix form,

$$\begin{pmatrix}\epsilon_{xx}-n_z^2 & \epsilon_{xy} & \epsilon_{xz}+n_xn_z\\ \epsilon_{yx} & \epsilon_{yy}-n_x^2-n_z^2 & \epsilon_{yz}\\ \epsilon_{zx}+n_zn_x & \epsilon_{zy} & \epsilon_{zz}-n_x^2\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}=0. \quad \text{(A9)}$$

Assuming $0=v_{Ti}\ll\omega/|k_\parallel|\ll v_{Te}$ and $\omega\ll\omega_{ci}$, the dielectric tensor $\boldsymbol{\epsilon}$ given from the linearized Vlasov equation[28] can be expanded as follows:

$$\epsilon_{xx}\simeq(c^2/V_A^2)\left[\alpha_1+\frac{k_\parallel^2V_A^2}{\omega^2}\left(\frac{p_{e\parallel}-p_{e\perp}}{B^2/\mu_0}\right)\right], \quad \text{(A10)}$$

$$\epsilon_{xy}=-\epsilon_{yx}\simeq i(c^2/V_A^2)\alpha_1(\omega/\omega_{ci}), \quad \text{(A11)}$$

$$\epsilon_{xz}=\epsilon_{zx}\simeq-\frac{k_\parallel k_\perp c^2}{\omega^2}\left(\frac{p_{e\parallel}-p_{e\perp}}{B^2/\mu_0}\right), \quad \text{(A12)}$$

$$\epsilon_{yy}\simeq\frac{c^2}{V_A^2}\left[\alpha_1+\frac{k_\parallel^2V_A^2}{\omega^2}\left(1-2\frac{k_\perp^2p_{e\perp}}{k_\parallel^2p_{e\parallel}}\right)\left(\frac{p_{e\parallel}-p_{e\perp}}{B^2/\mu_0}\right)\right], \quad \text{(A13)}$$

$$\epsilon_{yz}=-\epsilon_{zy}\simeq-i\frac{c^2}{V_A^2}\frac{k_\perp p_{e\perp}}{k_\parallel p_{e\parallel}}\frac{\omega_{ci}}{\omega}, \quad \text{(A14)}$$

$$\epsilon_{zz}\simeq\frac{\omega_{pi}^2}{\omega^2k_\parallel^2C_{s\parallel}^2}\left[\omega^2-k_\parallel^2C_{s\parallel}^2\left(1-\frac{k_\perp^2V_A^2}{\omega_{ci}^2}\frac{p_{e\parallel}-p_{e\perp}}{B^2/\mu_0}\right)\right], \quad \text{(A15)}$$

where $\alpha_1\equiv(1-\omega^2/\omega_{ci}^2)^{-1}$, $\omega_{ci}\equiv eB/m_i$ is the ion cyclotron frequency, and $\omega_{pi}\equiv\sqrt{ne^2/\epsilon_0m_i}$ is the ion plasma frequency. Substituting (A10)–(A15) into Eq. (A9), its determinant gives

$$\begin{vmatrix}\omega^2-k_\parallel^2V_A^2\tau & -i\frac{\omega}{\omega_{ci}}V_A^2(k_\parallel^2\tau+k_\perp^2\kappa) & \frac{k_\perp}{k_\parallel}\left(\omega^2-k_\parallel^2V_A^2\tau-\frac{p_{e\parallel}-p_{e\perp}}{p_{e\parallel}}\omega^2\right)\\ i\frac{\omega}{\omega_{ci}}k_\parallel^2V_A^2\tau & \omega^2-V_A^2(k_\parallel^2\tau+k_\perp^2\kappa) & i\frac{k_\perp}{k_\parallel}\omega\omega_{ci}\left(1-\frac{k_\parallel^2V_A^2\tau}{\omega_{ci}^2}-\frac{p_{e\parallel}-p_{e\perp}}{p_{e\parallel}}\right)\\ \frac{k_\parallel k_\perp V_A^2\tau}{\omega_{ci}^2}k_\parallel^2C_{s\parallel}^2 & ik_\parallel k_\perp C_{s\parallel}^2\frac{\omega}{\omega_{ci}}\frac{p_{e\perp}}{p_{e\parallel}} & \omega^2-k_\parallel^2C_{s\parallel}^2\left(1+\frac{k_\perp^2V_A^2\tau}{\omega_{ci}^2}\right)\end{vmatrix}=0, \quad \text{(A16)}$$

which exactly coincides with the dispersion relation (12).

APPENDIX B: DERIVATION OF EQS. (49)–(52)

Equations for B^2, τ, and Eq. (44) are rewritten by using Eqs. (26), (40), (41), and (43) as

$$B^2=\frac{|\nabla\psi|^2}{R^2}+\left(\frac{\mu_0e}{\tau R}\right)^2[\Psi_i-\Psi_{*e}(\psi)]^2, \quad \text{(B1)}$$

$$\tau=1-\frac{nT_\parallel B_0\Delta(\psi)}{[B_0\Delta(\psi)+B]B^2/\mu_0}, \quad \text{(B2)}$$

$$\frac{m_i}{2R^2}\left\{\frac{|\nabla\Psi_i|^2}{n^2}+\frac{e^2}{m_i^2}[\chi(\Psi_i)-\psi]^2\right\}+e\Phi_*(\psi)+T_\parallel(\psi)\ln\left\{n\left[1+\frac{B_0\Delta(\psi)}{B}\right]\right\}=W(\Psi_i). \quad \text{(B3)}$$

The gradients of Eqs. (B1)–(B3) are

$$\nabla B^2 = -\frac{2B^3}{R}\nabla R - 2\left(\frac{\mu_0 e}{\tau R}\right)^2[\Psi_i - \Psi_{*e}(\psi)]^2 \nabla \ln \tau + \frac{\nabla(|\nabla\psi|^2)}{R^2} + 2\left(\frac{\mu_0 e}{\tau R}\right)^2(\Psi_i - \Psi_{*e})\left(\nabla\Psi_i - \frac{d\Psi_{*e}}{d\psi}\nabla\psi\right), \tag{B4}$$

$$\nabla\tau = (\tau - 1)\left[\nabla \ln n - \frac{2B_0\Delta + 3B}{2B^2(B_0\Delta + B)}\nabla B^2 + \left(\frac{d\ln T_\parallel}{d\psi} + \frac{B}{B_0\Delta + B}\frac{d\ln\Delta}{d\psi}\right)\nabla\psi\right], \tag{B5}$$

$$\frac{m_i}{2R^2n^2}\left[-2|\nabla\Psi_i|^2\nabla\ln n + \nabla(|\nabla\Psi_i|^2) + 2\left(\frac{ne}{m_i}\right)^2(\chi-\psi)\left(\frac{d\chi}{d\Psi_i}\nabla\Psi_i - \nabla\psi\right)\right] - \frac{m_i}{R^3n^2}\left[|\nabla\Psi_i|^2 + \left(\frac{ne}{m_i}\right)^2(\chi-\psi)^2\right]\nabla R$$
$$+ e\frac{d\Phi_*}{d\psi}\nabla\psi + \ln\left[n\left(1+\frac{B_0\Delta}{B}\right)\right]\frac{dT_\parallel}{d\psi}\nabla\psi + T_\parallel\nabla\ln n + \frac{T_\parallel B_0}{B_0\Delta + B}\frac{d\Delta}{d\psi}\nabla\psi - \frac{T_\parallel B_0\Delta}{2B^2(B_0\Delta + B)}\nabla B^2 - \frac{dW}{d\Psi_i}\nabla\Psi_i = 0. \tag{B6}$$

Substituting Eq. (B4) to Eqs. (B5) and (B6), one obtains

$$(1-\tau)\nabla\ln n + \left[\tau + (1-\tau)\left(2 + \frac{p_{e\perp}}{p_{e\parallel}}\right)\frac{B_\varphi^2}{B^2}\right]\nabla\ln\tau$$
$$= (1-\tau)\left(1+\frac{p_{e\perp}}{2p_{e\parallel}}\right)\left[\frac{\nabla(|\nabla\psi|^2)}{B^2R^2} - 2\nabla\ln R - \frac{2e\mu_0 B_\varphi}{B^2R\tau}\nabla\Psi_i\right] - (1-\tau)\left[\frac{d\ln T_\parallel}{d\psi} + \frac{p_{e\perp}}{p_{e\parallel}}\frac{d\ln\Delta}{d\psi} + \left(2+\frac{p_{e\perp}}{p_{e\parallel}}\right)\frac{e\mu_0 B_\varphi}{B^2R\tau}\frac{d\Psi_{*e}}{d\psi}\right]\nabla\psi \tag{B7}$$

and

$$-\left(m_iV_p^2 - \frac{p_{e\parallel}}{n}\right)\nabla\ln n + (1-\tau)\frac{B_\varphi^2}{\mu_0 n}\nabla\ln\tau$$
$$= \frac{1-\tau}{2nR^2\mu_0}\nabla(|\nabla\psi|^2) - \frac{m_i\nabla(|\nabla\Psi_i|^2)}{2n^2R^2} + \left[m_iv_i^2 - (1-\tau)\frac{B^2}{\mu_0 n}\right]\nabla\ln R - \left[\ln\left(\frac{np_{e\parallel}}{p_{e\perp}}\right)\frac{dT_\parallel}{d\psi}\right.$$
$$\left.+ (1-\tau)\left(\frac{B^2}{n}\frac{d\ln\Delta}{d\psi} + \frac{eB_\varphi}{\tau nR}\frac{d\Psi_{*e}}{d\psi}\right) + e\frac{d\Phi_*}{d\psi} - \frac{ev_{i\varphi}}{R}\right]\nabla\psi + \left[\frac{(1-\tau)eB_\varphi}{\tau nR} - \frac{ev_{i\varphi}}{R}\frac{d\chi}{d\Psi_i} + \frac{dW}{d\Psi_i}\right]\nabla\Psi_i, \tag{B8}$$

which yield the coefficients (49)–(52) of the expansions

$$\nabla\ln\tau = \frac{\partial\ln\tau}{\partial|\nabla\psi|^2}\nabla(|\nabla\psi|^2) + \frac{\partial\ln\tau}{\partial|\nabla\Psi_i|^2}\nabla(|\nabla\Psi_i|^2) + \cdots, \tag{B9}$$

$$\nabla\ln n = \frac{\partial\ln n}{\partial|\nabla\psi|^2}\nabla(|\nabla\psi|^2) + \frac{\partial\ln n}{\partial|\nabla\Psi_i|^2}\nabla(|\nabla\Psi_i|^2) + \cdots. \tag{B10}$$

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