§2. Integration Formula of the Magnetic Field Produced by the Finite Size Helical Coil with Arbitrary Polygonal Cross Section

Todoroki, J.

In stellarator devices the size of the helical coil is an important parameter, because the magnetic configuration is quite sensitive to the current distribution on the external coils. The integration formulae for the magnetic field produced by the helical coil with rectangular cross section are given in Ref.1-2. The results are extended to the helical coils with arbitrarily polygonal cross section.

In the description of the helical coil, a spatial curve $\boldsymbol{r}_G(l)$ called as the guiding curve is introduced, l being the arc length along the curve.. On the curve, two unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are introduced, so that the three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and the tangent vector $\boldsymbol{t}$, form the orthogonal triad. The curvatures and torsion are defined as

$$\kappa_u \equiv \boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}l},\quad \kappa_v \equiv \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}l},\quad \tau \equiv \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}l}.$$

The point on the helical coil is expressed in the local coordinate system (ξ, η, l) as

$$\boldsymbol{r}_c(\xi,\eta,l) = \boldsymbol{r}_G(l) + \xi\boldsymbol{u}(l) + \eta\boldsymbol{v}(l).$$

The coordinates for the point where the magnetic field is calculated is also represented in terms of the local coordinates

$$\boldsymbol{r}_p(x',y',z',l) = \boldsymbol{r}_G(l) + x'\boldsymbol{u}(l) + y'\boldsymbol{v}(l) + z'\boldsymbol{t}(l).$$

The magnetic field can be expressed by the following integrals

$$\boldsymbol{B} = \oint\{\delta B_u\boldsymbol{u}(l) + \delta B_v\boldsymbol{v}(l) + \delta B_t\boldsymbol{t}(l)\}\mathrm{d}l,$$

where the each component can be represented by the double integral

$$\delta B_k = \frac{\mu_0}{4\pi}\iint_{\Omega_c} W_k(l,\xi,\eta;x',y',z)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

for $k = u, v, t$.

We assume the coil boundary in x'-y' plane is composed of N straight line elements; the corners are numbered from 1 to N anticlockwise, and the coordinates of j-th corner are (X_j, Y_j) with $(X_N, Y_N) \equiv (X_0, Y_0)$. We introduce the directional cosines α_j and β_j, and write equations of the line connecting j-th and $j-1$-th points in the form

$$\lambda_j = \alpha_j(X - x') + \beta_j(Y - y').$$

Finally, the components of the magnetic field can be expressed in the following form

$$\delta B_k = \frac{\mu_0 j_c}{4\pi}\sum_{j=1}^{N} F_k(j) \qquad (\text{for } k = u, v, t),$$

$$\begin{aligned}\hat{F}_u(j) = {} & [-\alpha_j\tau z - \beta_j\gamma - \alpha_j(\alpha_j\kappa_v - \beta_j\kappa_u)\lambda_j]L^j_{j-1} \\ & + \beta_j(\alpha_j\kappa_v - \beta_j\kappa_u)\{R_j - R_{j-1}\} \\ & - (\tau x' + \kappa_v z)\Phi^j_{j-1},\end{aligned}$$

$$\begin{aligned}\hat{F}_v(j) = {} & [\alpha_j\gamma - \beta_j\tau z + \beta_j(\beta_j\kappa_u - \alpha_j\kappa_v)\lambda_j]L^j_{j-1} \\ & + \alpha_j(\beta_j\kappa_u - \alpha_j\kappa_v)\{R_j - R_{j-1}\} \\ & - (\tau y' - \kappa_u z)\Phi^j_{j-1},\end{aligned}$$

$$F_t(j) = \tau[\lambda_j - \alpha_j x' - \beta_j y']L^j_{j-1} + 2\tau z\Phi^j_{j-1},$$

where

$$L^j_{j-1} \equiv \log\left|\frac{R_j + \omega_j}{R_{j-1} + \hat{\omega}_{j-1}}\right| \equiv \log\left|\frac{R_{j-1} - \hat{\omega}_{j-1}}{R_j - \omega_j}\right|,$$

$$\Phi^j_{j-1} \equiv \arctan\left(\frac{\Phi_j - \hat{\Phi}_{j-1}}{1 + \Phi_j\hat{\Phi}_{j-1}}\right),$$

with

$$R_j \equiv \sqrt{x_j^2 + y_j^2 + z^2},\quad x_j \equiv X_j - x',\quad y_j \equiv Y_j - y'$$

$$\Phi_j \equiv \frac{\alpha_j z^2 + \lambda_j x_j}{\beta_j z R_j},\quad \hat{\Phi}_{j-1} \equiv \frac{\alpha_j z^2 + \lambda_j x_{j-1}}{\beta_j z R_{j-1}},$$

$$\omega_j = \alpha_j(Y_j - y') - \beta_j(X_j - x'),$$

$$\hat{\omega}_{j-1} = \alpha_j(Y_{j-1} - y') - \beta_j(X_{j-1} - x'),$$

$$\gamma \equiv 1 - \kappa_u x' - \kappa_v y'.$$

The formulae can be extended easily to the case when the cross section is varying along the guiding curve. The formulae for the vector potentials are also given.

[1] Todoroki, J., Kakuyuugo Kenkyu, **57** (1987), 318.
[2] Todoroki, J., Kakuyuugo Kenkyu, **63** (1990), 217.
[3] Todoroki, J., Japanese J. Applied Phys.,**43** (2004), 1209.