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Parallel momentum input by tangential neutral beam injections in stellarator and heliotron plasmas

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The configuration dependence of parallel momentum inputs to target plasma particle species by tangentially injected neutral beams is investigated in non-axisymmetric stellarator/heliotron model magnetic fields by assuming the existence of magnetic flux-surfaces. In parallel friction integrals of the full Rosenbluth-MacDonald-Judd collision operator in thermal particles' kinetic equations, numerically obtained eigenfunctions are used for excluding trapped fast ions that cannot contribute to the friction integrals. It is found that the momentum inputs to thermal ions strongly depend on magnetic field strength modulations on the flux-surfaces, while the input to electrons is insensitive to the modulation. In future plasma flow studies requiring flow calculations of all particle species in more general non-symmetric toroidal configurations, the eigenfunction method investigated here will be useful. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4929789]

I. INTRODUCTION

Recently, impurity flow velocities of NBI (neutral beam injection) heated plasmas in Heliotron-J were successfully explained by the neoclassical transport theory.[1,2] That study applied a recently developed moment equation approach for general non-symmetric toroidal plasmas, including the external momentum input.[2] In the moment method, problems, including the field particle portion $C_{ab}(f_{aM},f_b)$ of the linearized collision operator with the local Maxwellian distribution f_{aM}, are converted to generalized parallel force balance expressed in an algebraic form. The recent study handled the external parallel momentum input by including the parallel friction collision moments $\int v_\| L_j^{(3/2)}(x_a^2)C_{a\mathrm{f}}(f_{aM},f_\mathrm{f})d^3\mathbf{v}$ of each target plasma species (denoted by the subscript "a") with the fast ions ("f") in this simultaneous algebraic equation. Here, $L_j^{(\alpha)}(K)\equiv(e^K K^{-\alpha}/j!)d^j(e^{-K}K^{j+\alpha})/dK^j$ is the Laguerre (Sonine) polynomial corresponding to the algebraic expression of the energy space structure and $x_a^2\equiv m_a v^2/(2\langle T_a\rangle)$. The fast ion birth profile was obtained by using the HFREYA and MCNBI, which are parts of a widely used NBI analysis code FIT3D.[3] Although the prompt orbit effect in non-symmetric toroidal configurations just after the beam ionization is taken into account in this method, a simple analytical formula of the fast ions' slowing down velocity distribution $f_\mathrm{f}(\mathbf{x},\mathbf{v})$ for uniform magnetic field strength $\mathbf{B}\cdot\nabla B=0$ is used for these collision integrals. It means that the fast ion trapping effect, which will be important for lower energy regions of $f_\mathrm{f}(\mathbf{x},\mathbf{v})$ broadened to full pitch angle range, is neglected. Therefore, a more systematic method for the friction collision moments in general non-symmetric toroidal configurations is required for more quantitative understandings of physical processes determining plasma flows.

5D-simulation methods[4,5] also may be thought to be applicable for investigating this kind of fast ion drift orbit effect in the slowing down process especially in cases of perpendicular injection of the beams.[5] In this type of injection, generating the fast ions in the trapped pitch-angle range, methods for handling the complicated bounce-center motion of the trapped ions will be required. However, for the tangential NBI used in the studies to investigate its parallel flow driving effect,[1] the beam ionization occurs at the circulating pitch-angle range. The fast ion trapping discussed here is that when these circulating fast ions enter into toroidally trapped pitch-angle range as a result of the pitch-angle-scattering (PAS) collision in the slowing down process, they do not contribute to the parallel friction moments. This reduction of the friction between the fast ions and target plasma species is analogous to the neoclassical parallel viscosity of the thermalized particles, as discussed below. This type of trapped fast ions should be excluded in these integrals. For the studies of physics of target plasma species, this exclusion is an important requirement and the behaviors of deeply trapped fast ions are not the purpose. It corresponds also to a basic idea of the moment method[6] that the field particle portion $C_{ab}(f_{aM},f_b)$ is an integral operator,[2] in which the higher Legendre orders in $f_b(\mathbf{x},\mathbf{v})$ expressing its detailed pitch-angle space structure are reduced. A more important requirement for the studies of multi-ion-species target plasmas is to know momentum and energy transfer ratios to each target plasma species and energy space structure of $C_{a\mathrm{f}}(f_{aM},f_\mathrm{f}^{(l=1)})$ as the specific Legendre order $l=1$ in the collision with the fast ion, which are governed by the slowing down and the pitch-angle-scattering collision of the circulating fast ions. In addition to the computational cost for handling the deeply trapped fast ions, there is another problem. If the drift approximation, including the perpendicular guiding center motion, is applied to the unbalanced tangential injections, the parallel force moment of the fast ions' drift kinetic equation (DKE) cannot reproduce the force moment of the Landau equation (Vlasov-Fokker-Planck equation) without the gyro-phase-averaging. Therefore, here we shall apply the

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eigenfunction method, which is originally proposed for the α-particle diffusion in axisymmetric tokamaks,[7] for plasma flow studies based on the parallel force balance, including the neoclassical parallel viscosity of both of the fast ions and target plasma species in non-symmetric stellarator/heliotron configurations.

The rest of this work is organized as follows. In Sec. II, the concept of flux-surface coordinates systems for NBI heated plasmas is explained. In neoclassical theories for stellarator/heliotron plasmas,[8–10] various integral theorems had been used for handling the 2-D real space of poloidal and toroidal angles in the coordinates systems,[11] such as Boozer and Hamada coordinates. However, it has been clarified in many experimental studies on the Shafranov shifts of the NBI heated plasmas[12,13] including resultant changes of the **B**-field strength modulation on the surfaces[14,15] that a modification of the Pfirsch-Schlüter current due to the large radial gradient of the fast ions' parallel pressure $\partial p_{\|\mathrm{f}}/\partial s$ is not negligible. The validity of the previously used theorems in these situations is explained in this section. The charge conservation $\nabla\cdot\mathbf{J}=0$ in plasmas with the anisotropic pressure is another important issue. The relation of these problems with the recent analyses is discussed there. The drift kinetic equation for the fast ions is introduced in Sec. III. Since our present study is focused on the friction collision integrals for the target plasma species, specific approximations are used there. The application of the eigenfunction method to non-symmetric stellarator/heliotron configurations is explained in Secs. IV and V with numerical examples. A summary is given in Sec. VI. Since these issues are related to (1) radial transport of general particle species, (2) analytical expressions of the fast ion velocity distribution and its energy integrals, and (3) $\int d^3\mathbf{v}$ integral formulas of the test particle portion of the linearized collision operator $C_{ab}(f_a,f_{b\mathrm{M}})$, they are described in Appendix. Formulas shown there hold also for fast ions in NBI-heated or burning plasmas, and for the anisotropic pressure equilibriums.

II. FLUX-SURFACE COORDINATES SYSTEM FOR NBI HEATED PLASMAS

When including the unbalanced tangential NBI in the MHD equilibrium and transport calculation based on the equilibrium, the following definition of perpendicular and parallel pressures is useful:

$$2p_{\perp a}\equiv m_a\int|\mathbf{v}_\perp-\mathbf{u}_{\perp a}|^2 f_a d^3\mathbf{v}=m_a\int v_\perp^2 f_a d^3\mathbf{v}-n_a m_a u_{\perp a}^2, \tag{1}$$

$$p_{\|a}\equiv m_a\int v_\|^2 f_a d^3\mathbf{v}. \tag{2}$$

Here, m_a, $n_a\equiv\int f_a d^3\mathbf{v}$, and $n_a\mathbf{u}_a\equiv\int\mathbf{v}f_a d^3\mathbf{v}$ are the mass, density, and particle flux of the species number a, respectively. Notations $\mathbf{F}_\|\equiv\mathbf{b}(\mathbf{b}\cdot\mathbf{F})\equiv\mathbf{b}F_\|$ and $\mathbf{F}_\perp\equiv\mathbf{F}-\mathbf{F}_\|$ [$\mathbf{b}\equiv\mathbf{B}/B$: the unit vector tangential to the magnetic field] for the parallel and perpendicular components of arbitrary vectors $\mathbf{F}(\mathbf{x})$ are used hereafter. The viscosity tensor also is defined by

$$\boldsymbol{\pi}_a\equiv m_a\int\{(\mathbf{v}-\mathbf{u}_{\perp a})(\mathbf{v}-\mathbf{u}_{\perp a})-|\mathbf{v}-\mathbf{u}_{\perp a}|^2\mathbf{I}/3\}f_a d^3\mathbf{v}, \tag{3}$$

with the unit tensor $\mathbf{I}$, and it is assumed that this tensor has the symmetric CGL (Chew-Goldberger-Low) form $\boldsymbol{\pi}_a=(p_{\|a}-p_{\perp a})(\mathbf{bb}-\mathbf{I}/3)$. Then parallel and perpendicular components of the force balance

$$\nabla\cdot(p_a\mathbf{I}+\boldsymbol{\pi}_a)+m_a\nabla\cdot\{n_a(\mathbf{u}_a\mathbf{u}_a-\mathbf{u}_{\|a}\mathbf{u}_{\|a})\} - e_a n_a\left(\mathbf{E}+\frac{\mathbf{u}_a\times\mathbf{B}}{c}\right)=\mathbf{F}_{a1}+m_a\int\mathbf{v}_\| S_a(\mathbf{x},v,\xi)d^3\mathbf{v} \tag{4}$$

as the $\int\mathbf{v}d^3\mathbf{v}$ integral of the Landau equation using $\partial(n_a\mathbf{u}_a)/\partial t=0$ and $p_a\equiv(2p_{\perp a}+p_{\|a})/3$ can be written more explicitly by following formulas for the CGL tensor:[16]

$$\begin{aligned}\mathbf{b}\cdot\nabla\cdot\boldsymbol{\pi}_a&=\frac{2}{3}\mathbf{b}\cdot\nabla(p_{\|a}-p_{\perp a})-(p_{\|a}-p_{\perp a})\mathbf{b}\cdot\nabla\ln B\\&=\frac{2}{3}B^{3/2}\mathbf{b}\cdot\nabla\frac{p_{\|a}-p_{\perp a}}{B^{3/2}},\\\mathbf{b}\cdot\nabla\cdot(p_a\mathbf{I}+\boldsymbol{\pi}_a)&=\frac{1}{2}\mathbf{b}\cdot\left(\nabla(p_{\|a}+p_{\perp a})+B^2\nabla\frac{p_{\|a}-p_{\perp a}}{B^2}\right),\end{aligned} \tag{5}$$

$$\begin{aligned}(\nabla\cdot\boldsymbol{\pi}_a)_\perp&=\frac{1}{3}\nabla_\perp(p_{\perp a}-p_{\|a})-(p_{\perp a}-p_{\|a})\mathbf{b}\cdot\nabla\mathbf{b}\\&\cong\frac{B^3}{3}\nabla_\perp\frac{p_{\perp a}-p_{\|a}}{B^3},\\\{\nabla\cdot(p_a\mathbf{I}+\boldsymbol{\pi}_a)\}_\perp&=\frac{1}{2}\left(\nabla_\perp(p_{\perp a}+p_{\|a})+B^2\nabla_\perp\frac{p_{\perp a}-p_{\|a}}{B^2}\right).\end{aligned} \tag{6}$$

Here, $\nabla_\perp\equiv\nabla-\mathbf{b}(\mathbf{b}\cdot\nabla)$, the steady-state Ampere's law $c\nabla\times\mathbf{B}=4\pi\mathbf{J}$ for $\mathbf{J}\equiv\sum_a e_a n_a\mathbf{u}_a$ and a low-perpendicular-beta approximation $8\pi\sum_a p_{\perp a}/B^2\ll 1$ for the **B**-field curvature $\mathbf{b}\cdot\nabla\mathbf{b}=\nabla_\perp\ln B+\frac{4\pi}{c}\mathbf{J}\times\mathbf{B}/B^2\cong\nabla_\perp\ln B$ are used in $(\nabla\cdot\boldsymbol{\pi}_a)_\perp$. The inertia force can be rewritten as $\nabla\cdot\{n_a(\mathbf{u}_a\mathbf{u}_a-\mathbf{u}_{\|a}\mathbf{u}_{\|a})\}=n_a(\mathbf{u}_a\cdot\nabla\mathbf{u}_a-\mathbf{u}_{\|a}\cdot\nabla\mathbf{u}_{\|a})+\mathbf{u}_{\|a}\nabla\cdot(n_a\mathbf{u}_{\perp a})$ by using the particle conservation $\nabla\cdot(n_a\mathbf{u}_a)=0$. It is neglected here since $m_a n_a u_{\perp a}^2, m_a n_a u_{\perp a}u_{\|a}\ll p_a$ for general particle species. An assumption of $m_a n_a u_{\|a}^2\ll p_a$, which does not hold for the fast ions $a=\mathrm{f}$ in unbalanced tangential NBI operations, is not required in this approximation.

The local parallel force balance and the local perpendicular current in the MHD equilibrium equation, which are given by summation of the force balance equation of all particle species with using the charge neutrality $\sum_a e_a n_a=0$ and the momentum conservation $\sum_a\mathbf{F}_{a1}=0$ of the friction integral $\mathbf{F}_{a1}\equiv m_a\int\mathbf{v}\sum_b C_{ab}(f_a,f_b)d^3\mathbf{v}$ of the Coulomb collision operator, are

$$\mathbf{b}\cdot\left(\nabla\sum_a(p_{\|a}+p_{\perp a})+B^2\nabla\sum_a\frac{p_{\|a}-p_{\perp a}}{B^2}\right)=0, \tag{7}$$

and

$$\mathbf{J}_\perp=-\frac{c}{2B^2}\left(\nabla\sum_a(p_{\perp a}+p_{\|a})+B^2\nabla\sum_a\frac{p_{\perp a}-p_{\|a}}{B^2}\right)\times\mathbf{B}, \tag{8}$$

respectively. The parallel momentum input $\int \mathbf{v}_{\|} S_a d^3\mathbf{v}$ due to the source term $S_a(\mathbf{x}, v, \xi)$ [$\xi \equiv v_{\|}/v$: cosine of pitch-angle in the spherical velocity coordinates], which exists only in the fast ions' kinetic equation $a = \mathrm{f}$ in unbalanced tangential NBI operations giving surface-averaged force $\sum_a \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle = m_\mathrm{f}\langle B\int v_{\|} S_\mathrm{f} d^3\mathbf{v}\rangle$, is neglected in this local parallel force balance since it is a 1st order of $(v_\mathrm{b}\tau_\mathrm{S})^{-1}$, as explained in Sec. III. We shall consider only the 0th order in constructing the flux-surface coordinates. When

$$\mathbf{b}\cdot\nabla\sum_a (p_{\|a} + p_{\perp a}) = 0 = \mathbf{b}\cdot\nabla\sum_a \frac{p_{\|a} - p_{\perp a}}{B^2} \tag{9}$$

can be assumed, $\sum_a (p_{\|a} + p_{\perp a}) = \mathrm{const}$ contour surfaces satisfy also $\mathbf{J}\cdot\nabla\sum_a (p_{\|a} + p_{\perp a}) = 0$. These kinds of surfaces are usually called flux-surface,[11] and we shall use s as an arbitrary label for them. As in Refs. 8–10, this s can be arbitrary surface-quantities in the following discussion, such as ψ, V, and the minor radius r. However, it also should be noted that Eq. (9) is not generally guaranteed, and thus effects of the deviation $\mathbf{b}\cdot\nabla\sum_a \frac{p_{\|a} - p_{\perp a}}{B^2} \neq 0$ also should be investigated after the explanation of the "ideal" condition $\mathbf{B}\cdot\nabla s = \mathbf{J}\cdot\nabla s = 0$. In this ideal condition, $\mathbf{J}\cdot\nabla\sum_a (p_{\|a} + p_{\perp a}) = 0 = \mathbf{J}\cdot\nabla\sum_a (p_{\|a} - p_{\perp a})/B^2$ and the formula

$$\begin{aligned}\nabla\cdot(H\nabla F\times\mathbf{B}) &= \nabla F\times\mathbf{B}\cdot\nabla H - H(\nabla F)\cdot\nabla\times\mathbf{B}\\ &= \nabla F\times\mathbf{B}\cdot\nabla H - H\frac{4\pi}{c}\mathbf{J}\cdot\nabla F\end{aligned} \tag{10}$$

for arbitrary scalar functions $F(\mathbf{x})$, $H(\mathbf{x})$, which is valid when $c\nabla\times\mathbf{B} = 4\pi\mathbf{J}$ holds, give

$$\nabla\cdot\mathbf{J}_{\|} = -\nabla\cdot\mathbf{J}_{\perp} = \frac{c}{2}\left(\frac{\partial}{\partial s}\sum_a (p_{\perp a} + p_{\|a})\right)\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2} \tag{11}$$

for the parallel current $\mathbf{J}_{\|}$. Therefore, a basic characteristic of the current in cases with the anisotropic heating can be understood as a sum of that in isotropic pressure equilibrium with the pressure $\sum_a (p_{\perp a} + p_{\|a})/2$ and the second term in Eq. (8) as a divergence free perpendicular component.

The straight field line (SFL) coordinates[11] (s, θ, ζ) [θ, ζ: the poloidal and toroidal angles, respectively] giving the contravariant and covariant expressions

$$\mathbf{B} = \psi'\nabla s\times\nabla\theta + \chi'\nabla\zeta\times\nabla s = B_s\nabla s + B_\theta\nabla\theta + B_\zeta\nabla\zeta, \tag{12}$$

and the Jacobian $\sqrt{g} \equiv [(\nabla s\times\nabla\theta)\cdot(\nabla\zeta)]^{-1} = (\psi' B_\zeta + \chi' B_\theta)/B^2$ can be constructed when only this $\nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{J} = \mathbf{B}\cdot\nabla s = \mathbf{J}\cdot\nabla s = 0$ is satisfied. Here, volume integrals $\chi(s) \equiv \frac{1}{4\pi^2}\int_V (\mathbf{B}\cdot\nabla\theta) d^3\mathbf{x}$ and $\psi(s) \equiv \frac{1}{4\pi^2}\int_V (\mathbf{B}\cdot\nabla\zeta) d^3\mathbf{x}$ for the volume V enclosed by the surface $s = \mathrm{const}$ correspond to poloidal and toroidal magnetic fluxes, respectively, and $' \equiv d/ds$ indicates radial gradients of them. A relation between the volume and the Jacobian is $V' = \oint\oint\sqrt{g}\, d\theta d\zeta$. For the flux-surface-average operation $\langle\cdot\rangle \equiv \oint\oint\cdot\sqrt{g}\, d\theta d\zeta/\oint\oint\sqrt{g}\, d\theta d\zeta$, there are two important theorems for the parallel and perpendicular gradients $\mathbf{B}\cdot\nabla F$, $\nabla s\times\mathbf{B}\cdot\nabla F$ of arbitrary scalar quantity $F(\mathbf{x})$. First, $\mathbf{B}\cdot\nabla = (1/\sqrt{g})(\chi'\partial/\partial\theta + \psi'\partial/\partial\zeta)$ of this contravariant expression of $\mathbf{B}$ satisfies

$$\langle H\mathbf{B}\cdot\nabla F\rangle = -\langle F\mathbf{B}\cdot\nabla H\rangle \tag{13}$$

for arbitrary $F(\mathbf{x})$ and $H(\mathbf{x})$. A frequently appearing formula of the surface-averaged parallel force $\langle\mathbf{B}\cdot\nabla\cdot(p_a\mathbf{I} + \boldsymbol{\pi}_a)\rangle = \langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle = -\langle(p_{\|a} - p_{\perp a})\mathbf{B}\cdot\nabla\ln B\rangle$ for Eq. (5), in which the scalar pressure moment p_a is eliminated, it is an important example of consequences of Eq. (13). Next, when the covariant expression $\mathbf{B} = B_s\nabla s + B_\theta\nabla\theta + B_\zeta\nabla\zeta$ is determined to satisfy $c\nabla\times\mathbf{B} = 4\pi\mathbf{J}$, (θ, ζ) can be chosen in a manner in which $B_\zeta = \mathrm{const}$, $B_\theta = \mathrm{const}$, and consequently $\sqrt{g} = (\psi' B_\zeta + \chi' B_\theta)/B^2 = \frac{V'}{4\pi^2}\langle B^2\rangle/B^2$ on the surfaces. This selection is called Boozer coordinates, and its $\nabla s\times\mathbf{B}\cdot\nabla = -(1/\sqrt{g_\mathrm{B}})(B_\zeta^{(\mathrm{Boozer})}\partial/\partial\theta_\mathrm{B} - B_\theta^{(\mathrm{Boozer})}\partial/\partial\zeta_\mathrm{B})$ indicates that

$$\langle H\nabla s\times\mathbf{B}\cdot\nabla F\rangle = -\langle F\nabla s\times\mathbf{B}\cdot\nabla H\rangle. \tag{14}$$

For Eq. (11), the set of Eqs. (13) and (14) gives $-\langle\nabla\cdot\mathbf{J}_\perp\rangle = \langle\nabla\cdot\mathbf{J}_{\|}\rangle = \langle\mathbf{B}\cdot\nabla(J_{\|}/B)\rangle = 0$ as the solubility condition of the charge conservation $\nabla\cdot\mathbf{J} = 0$. This set of theorems does not require the complete isotropic pressure $\sum_a p_{\|a} = \sum_a p_{\perp a}$, and thus we can use it for cases with the external anisotropic heating.

In the "ideal" situations, (θ, ζ) can be chosen in another manner in which not only the $\mathbf{B}$-vector but also the $\mathbf{J}$-vectors are straight lines. Although this selection is known as Hamada coordinates for the isotropic pressure equilibriums, here we call it straight current line (SCL) coordinates. ("Hamada" is used below for coordinates giving the Jacobian $\sqrt{g_\mathrm{H}} = \frac{V'}{4\pi^2}$, which are constants on the surfaces.) The contravariant expression of current $\mathbf{J} = \sqrt{g_\mathrm{SCL}}(J_\mathrm{SCL}^\zeta\nabla s\times\nabla\theta_\mathrm{SCL} + J_\mathrm{SCL}^\theta\nabla\zeta_\mathrm{SCL}\times\nabla s)$ in this selection satisfies $\sqrt{g_\mathrm{SCL}}J_\mathrm{SCL}^\zeta = \mathrm{const}$ and $\sqrt{g_\mathrm{SCL}}J_\mathrm{SCL}^\theta = \mathrm{const}$ on the surfaces, and thus we know by $\mathbf{J}\cdot\nabla = J_\mathrm{SCL}^\theta\partial/\partial\theta_\mathrm{SCL} + J_\mathrm{SCL}^\zeta\partial/\partial\zeta_\mathrm{SCL}$ that

$$\langle H\mathbf{J}\cdot\nabla F\rangle = -\langle F\mathbf{J}\cdot\nabla H\rangle. \tag{15}$$

However, actual situations are not "ideal" for guaranteeing Eq. (15) as noted on Eq. (9). When reading Eq. (7) with an approximation of $B^2 \simeq \langle B^2\rangle$, the equation indicates a characteristic of the $\mathbf{B}$-field lines that they are constrained to follow the $\sum_a p_{\|a} = \mathrm{const}$ contour surfaces. When these surfaces are closed ones surrounding the magnetic axis, it is reasonable to assume $s = \mathrm{const}$ surfaces satisfying $\nabla\cdot\mathbf{B} = 0 = \mathbf{B}\cdot\nabla s$. In contrast to these $\mathbf{B}$-field lines, $\mathbf{J}_\perp$ vectors determined by Eq. (8) are not constrained to the surfaces when Eq. (9) is not satisfied. Even when the $\mathbf{J}_\perp$ vectors deviate from the $\mathbf{B}$-surfaces ($\mathbf{J}\cdot\nabla s \neq 0$) due to the violation of Eq. (9), these vectors are connected by the differential operation $c\nabla\times\mathbf{B} = 4\pi\mathbf{J}$. Therefore, Eqs. (13) and (14) as basic characteristics of Eq. (12) are not easily broken. A violation of Eq. (15) is more easily caused since the contravariant expression of $\mathbf{J}$ does not exist when $\mathbf{J}\cdot\nabla s \neq 0$. Although most of the problems in heating and transport analyses require only Eqs. (13) and (14) and do not require Eq. (15), we shall consider how the charge conservation $\nabla\cdot\mathbf{J}=0$ in the volume V is retained and how the theorem $\langle\mathbf{J}\cdot\nabla F\rangle=0$ is modified in cases with the parallel force component $\mathbf{b}\cdot\nabla\sum_a(p_{\|a}-p_{\perp a})/B^2\neq 0$.

By using Eqs. (8) and (14), the surface-averaged radial current is given by

$$\langle\mathbf{J}\cdot\nabla s\rangle = c\left\langle\sum_a(p_{\perp a}+p_{\|a})\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\nabla\ln B\right\rangle. \tag{16}$$

Because of the Gauss' theorem[11] $\int_0^V\langle\nabla\cdot\mathbf{F}\rangle dV=\int_0^V\langle\nabla\cdot\mathbf{F}_\perp\rangle dV=\langle\mathbf{F}_\perp\cdot\nabla V\rangle=\langle\mathbf{F}\cdot\nabla V\rangle$ ($\because\langle\nabla\cdot\mathbf{F}_\|\rangle=\langle\mathbf{B}\cdot\nabla(F_\|/B)\rangle=0$ due to Eq. (13)) for arbitrary vector field $\mathbf{F}(\mathbf{x})$, the so-called ambipolar condition $\langle\mathbf{J}\cdot\nabla s\rangle=0$ at all radial positions is $\langle\nabla\cdot\mathbf{J}_\perp\rangle=0$ as the solubility condition of $\nabla\cdot\mathbf{J}=0$. In configurations with the stellarator symmetry $B(s,-\theta,-\zeta)=B(s,\theta,\zeta)$, for example, $\sum_a(p_{\perp a}+p_{\|a})$ also should basically be a symmetric phase function $F(s,-\theta,-\zeta)=F(s,\theta,\zeta)$ for retaining the local charge conservation. Although there are no contradictions in this determination of the geometrical shapes of the **B**-surfaces by Eq. (7) and the charge conservation with Eq. (16) when the external parallel force is $\int\mathbf{v}_\| S_f d^3\mathbf{v}=0$, it should be considered that there is one constraint on the real space structure of this force term when its finite values are added. The allowed force that we noted previously as "1st order of $(v_b\tau_S)^{-1}$" is a divergence-free vector $\int\mathbf{v}_\| S_f d^3\mathbf{v}=\langle B\int v_\| S_f d^3\mathbf{v}\rangle\mathbf{B}/\langle B^2\rangle$. Deviations from this form that can be written as parallel gradients of scalar quantities will be problematic in the simultaneous retaining of the geometrical shape and the charge conservation. When we define the scalar $P_S(s,\theta,\zeta)$ by

$$\mathbf{b}\cdot\nabla P_S\equiv m_a\int v_\| S_f d^3\mathbf{v}-m_a\left\langle B\int v_\| S_f d^3\mathbf{v}\right\rangle\frac{B}{\langle B^2\rangle},\quad\langle P_S\rangle=0, \tag{17}$$

the local parallel force balance Eq. (7) is modified to be

$$\mathbf{b}\cdot\left(\frac{1}{2}\nabla\sum_a(p_{\|a}+p_{\perp a})+\frac{B^2}{2}\nabla\sum_a\frac{p_{\|a}-p_{\perp a}}{B^2}-\nabla P_S\right)=0.$$

Even in this case, we shall assume that the structure of the **B**-field has the stellarator symmetry $B(s,-\theta,-\zeta)=B(s,\theta,\zeta)$ as in Eq. (16). Therefore, instead of the aforementioned $\sum_a p_{\|a}=$ const contour surfaces, $\sum_a p_{\|a}-P_S=$ const contour surfaces should be adjusted to this geometrical shape of the **B**-field. However, this adding of $\mathbf{b}\cdot\nabla P_S$ can alter only $\mathbf{b}\cdot\nabla\sum_a(p_{\|a}+p_{\perp a})$ without changing $\mathbf{b}\cdot\nabla\sum_a(p_{\|a}-p_{\perp a})/B^2$ since the anisotropy should be determined to satisfy the aforementioned surface-averaged force balance $-\langle\sum_a(p_{\|a}-p_{\perp a})\mathbf{B}\cdot\nabla\ln B\rangle=m_f\langle B\int v_\| S_f d^3\mathbf{v}\rangle$ and cannot be balanced with $\mathbf{b}\cdot\nabla P_S$. Therefore, when the external parallel force term has forms giving anti-symmetric phase functions $P_S(s,-\theta,-\zeta)=-P_S(s,\theta,\zeta)\neq 0$, there is a serious contradiction that the geometrical shape of the **B**-field requires the anti-symmetric phase component in $\sum_a(p_{\|a}+p_{\perp a})$, while the charge conservation with Eq. (16) forbids that component in the pressure. For retaining geometrical shapes of the **B**-surfaces following Eq. (7) and the charge conservation $\nabla\cdot\mathbf{J}=0$ with Eq. (16) in those configurations simultaneously, it is concluded that Eq. (17) should vanish, and, in particular, the anti-symmetric phase component $P_S(s,-\theta,-\zeta)=-P_S(s,\theta,\zeta)\neq 0$ is forbidden when the **B**-field has the stellarator symmetry $B(s,-\theta,-\zeta)=B(s,\theta,\zeta)$.

Next, we shall show some practically usable formulas for cases, including the parallel force component $\mathbf{b}\cdot\nabla\sum_a(p_{\|a}-p_{\perp a})/B^2\neq 0$ and resulting local radial current $\mathbf{J}\cdot\nabla s\neq 0$. The parallel force balance for

$$\sum_a(p_{\|a}+p_{\perp a})=\sum_a\langle p_{\|a}+p_{\perp a}\rangle+\sum_a\delta(p_{\|a}+p_{\perp a})$$
$$\sum_a\frac{p_{\|a}-p_{\perp a}}{B^2}=\sum_a\left\langle\frac{p_{\|a}-p_{\perp a}}{B^2}\right\rangle+\sum_a\delta\frac{p_{\|a}-p_{\perp a}}{B^2}$$

is

$$\mathbf{b}\cdot\left(\nabla\sum_a\delta(p_{\|a}+p_{\perp a})+B^2\nabla\sum_a\delta\frac{p_{\|a}-p_{\perp a}}{B^2}\right)=0$$
$$\sum_a\delta(p_{\|a}+p_{\perp a})\cong -B^2\sum_a\delta\frac{p_{\|a}-p_{\perp a}}{B^2}. \tag{18}$$

Substituting it into the current formula, Eq. (8), gives

$$\mathbf{J}_\perp=-\frac{c}{2B^2}\nabla\sum_a\langle p_{\perp a}+p_{\|a}\rangle\times\mathbf{B}-\frac{c}{2}\nabla\left(\sum_a\left\langle\frac{p_{\perp a}-p_{\|a}}{B^2}\right\rangle+2\sum_a\delta\frac{p_{\perp a}-p_{\|a}}{B^2}\right)\times\mathbf{B}. \tag{19}$$

This current results in

$$\nabla\cdot\mathbf{J}_\|=-\nabla\mathbf{J}_\perp=\frac{c}{2}\left(\frac{\partial}{\partial s}\sum_a\langle p_{\perp a}+p_{\|a}\rangle\right)\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2} \tag{20}$$

by a low-beta approximation $\beta\equiv 8\pi\sum_a p_a/B^2\ll 1$ for neglecting $\mathbf{J}\cdot\nabla$ in Eq. (10). This is also due to a relation between radial gradient scale lengths $|\frac{\partial}{\partial s}\ln\sum_a\langle p_{\perp a}+p_{\|a}\rangle|\gg|\frac{\partial}{\partial s}\ln\langle B^{-2}\rangle|$ in the approximation

$$\nabla\cdot\left(\nabla\sum_a(p_{\perp a}+p_{\|a})\times\mathbf{B}/B^2\right)=\left(\frac{\partial}{\partial s}\sum_a\langle p_{\perp a}+p_{\|a}\rangle\right)\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2}-\left(\frac{\partial}{\partial s}\left\langle\frac{1}{B^2}\right\rangle\right)\nabla s\times\mathbf{B}\cdot\nabla\sum_a(p_{\perp a}+p_{\|a})$$

for retaining the solubility condition $\langle\nabla\cdot\mathbf{J}_\perp\rangle=0$. As long as the ambipolar condition $\langle\mathbf{J}\cdot\nabla s\rangle=0$ is satisfied, the divergence $\nabla\cdot\mathbf{J}_\|=-\nabla\cdot\mathbf{J}_\perp$ is insensitive to the parallel force perturbation. We shall define a function $\tilde{U}$ by[8]

$$\mathbf{B}\cdot\nabla\frac{\tilde{U}}{B}=(\mathbf{B}\times\nabla s)\cdot\nabla\frac{1}{B^2},\quad\langle B\tilde{U}\rangle=0 \tag{21}$$

for handling this type of parallel flow divergences. Explicit expressions of this function for the **B**-field with the stellarator symmetry $B(s,-\theta,-\zeta)=B(s,\theta,\zeta)$ are given in following Fourier series forms with the symmetric phase:

$$\begin{aligned}\tilde{U} &= \frac{B}{\langle B^2\rangle}\sum_{(m,n)\neq(0,0)}\frac{B_\zeta^{(\mathrm{Boozer})}m+B_\theta^{(\mathrm{Boozer})}n}{\chi' m-\psi' n}\varepsilon_{mn}^{(\mathrm{Boozer})}\cos(m\theta_\mathrm{B}-n\zeta_\mathrm{B})\\ &= \frac{1}{B}\sum_{(m,n)\neq(0,0)}\frac{B_\zeta^{(\mathrm{Boozer})}m+B_\theta^{(\mathrm{Boozer})}n}{\chi' m-\psi' n}\varepsilon_{mn}^{(\mathrm{Hamada})}\cos(m\theta_\mathrm{H}-n\zeta_\mathrm{H}),\\ \varepsilon_{mn}^{(\mathrm{Boozer})} &\equiv \frac{1}{2\pi^2}\int_0^{2\pi}d\theta_\mathrm{B}\int_0^{2\pi}d\zeta_\mathrm{B}\left(\frac{\langle B^2\rangle}{B^2}-1\right)\cos(m\theta_\mathrm{B}-n\zeta_\mathrm{B}),\\ \varepsilon_{mn}^{(\mathrm{Hamada})} &\equiv \frac{1}{2\pi^2}\int_0^{2\pi}d\theta_\mathrm{H}\int_0^{2\pi}d\zeta_\mathrm{H}\left(1-\frac{B^2}{\langle B^2\rangle}\right)\cos(m\theta_\mathrm{H}-n\zeta_\mathrm{H}).\end{aligned}\tag{22}$$

As noted previously, the Boozer coordinates $(s,\theta_\mathrm{B},\zeta_\mathrm{B})$ and the Hamada coordinates $(s,\theta_\mathrm{H},\zeta_\mathrm{H})$ are defined here as systems that have $\sqrt{g_\mathrm{B}}=\frac{V'}{4\pi^2}\langle B^2\rangle/B^2$ and $\sqrt{g_\mathrm{H}}=\frac{V'}{4\pi^2}$, respectively. A relation between these coordinates is shown in Refs. 8 and 17. Equation (21) combined with Eqs. (13) and (14) implies

$$\begin{aligned}\left\langle\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\nabla F\right\rangle &= \left\langle F(\mathbf{B}\times\nabla s)\cdot\nabla\frac{1}{B^2}\right\rangle\\ &= \left\langle F\mathbf{B}\cdot\nabla\frac{\tilde{U}}{B}\right\rangle = -\langle\tilde{U}\mathbf{b}\cdot\nabla F\rangle.\end{aligned}\tag{23}$$

By using it and $\langle\tilde{U}\mathbf{b}\cdot\nabla\sum_a(p_{\|a}+p_{\perp a})\rangle=-\langle\tilde{U}B^2\mathbf{b}\cdot\nabla\sum_a(p_{\|a}-p_{\perp a})/B^2\rangle$, Eq. (16) can be rewritten as another expression that agrees with $\sum_a e_a\langle n_a\mathbf{u}_a\cdot\nabla s\rangle$ given in Appendix A. In addition to this,

$$\begin{aligned}&\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\nabla\\ &= -\frac{1}{\sqrt{g_\mathrm{H}}\langle B^2\rangle}\left(B_\zeta^{(\mathrm{Boozer})}\frac{\partial}{\partial\theta_\mathrm{H}}-B_\theta^{(\mathrm{Boozer})}\frac{\partial}{\partial\zeta_\mathrm{H}}\right)\end{aligned}$$

given by a procedure in Refs. 8 and 17 indicates also that

$$\left\langle F\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\nabla H\right\rangle = -\left\langle H\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\nabla F\right\rangle\tag{24}$$

for arbitrary $F(\mathbf{x})$ and $H(\mathbf{x})$. Equation (24) corresponds to $\langle H\mathbf{J}\cdot\nabla F\rangle=-\langle F\mathbf{J}\cdot\nabla H\rangle$ in isotropic pressure equilibriums with the pressure of $\sum_a\langle p_{\perp a}+p_{\|a}\rangle/2$. The Gauss' theorem gives another important formula

$$\begin{aligned}\langle\nabla\cdot(H\nabla F\times\mathbf{B})\rangle &= \frac{\partial}{\partial V}\langle H\nabla F\times\mathbf{B}\cdot\nabla V\rangle\\ &= -\frac{\partial}{\partial V}\langle H\nabla V\times\mathbf{B}\cdot\nabla F\rangle\\ &= \frac{\partial}{\partial V}\langle F\nabla V\times\mathbf{B}\cdot\nabla H\rangle.\end{aligned}\tag{25}$$

In addition to Eqs. (13) and (14), Eqs.(23)–(25) also had played important roles in the moment equation approach (Refs. 8–10 and references cited therein).

Then, by assuming this form of the parallel current

$$J_\| = \frac{\langle\mathbf{J}\cdot\mathbf{B}\rangle B}{\langle B^2\rangle}-\frac{c}{2}\tilde{U}\frac{\partial}{\partial s}\sum_a\langle p_{\perp a}+p_{\|a}\rangle,\tag{26}$$

we shall derive $\langle\mathbf{J}\cdot\nabla F\rangle$ for arbitrary scalar $F(\mathbf{x})$ except surface-quantities, such as s,χ,ψ, and V. (When F is a surface-quantity, $\langle\mathbf{J}\cdot\nabla F\rangle$ should be calculated by $\langle\mathbf{J}\cdot\nabla s\rangle\partial F/\partial s$ with Eq. (16) or the formula in Appendix A without the $|\varepsilon_{mn}|\ll 1$ approximation in Eqs. (18) and (19) for $\mathbf{J}_\perp$ and without following $\beta\ll 1$ approximation in Eq. (10). The result will vanish by the ambipolar constraint.) By using $\langle\nabla F\times\mathbf{B}\cdot\nabla H\rangle=\langle\nabla\cdot(H\nabla F\times\mathbf{B})\rangle$ as a $\beta\ll 1$ approximation, neglecting $\mathbf{J}\cdot\nabla$ in Eq. (10) for Eq. (25),

$$\begin{aligned}\langle\nabla F\times\mathbf{B}\cdot\nabla H\rangle &= -\frac{\partial}{\partial V}\langle H\nabla V\times\mathbf{B}\cdot\nabla F\rangle\\ &= \frac{\partial}{\partial V}\langle F\nabla V\times\mathbf{B}\cdot\nabla H\rangle\ (\beta\ll 1)\end{aligned}\tag{27}$$

for arbitrary scalars $F(\mathbf{x})$ and $H(\mathbf{x})$ is given. By combining Eqs. (13), (14), (19), (23), (27),

$$\begin{aligned}\langle\mathbf{J}\cdot\nabla F\rangle &= c\frac{\partial}{\partial V}\left\langle F\nabla V\times\mathbf{B}\cdot\nabla\sum_a\frac{p_{\perp a}-p_{\|a}}{B^2}\right\rangle\\ &= -c\frac{\partial}{\partial V}\left\langle\sum_a\frac{p_{\perp a}-p_{\|a}}{B^2}\nabla V\times\mathbf{B}\cdot\nabla F\right\rangle\end{aligned}\tag{28}$$

is obtained. In configurations with the stellarator symmetry, the ambipolar condition with Eq. (16) does not have any limitations on $\sum_a(p_{\perp a}-p_{\|a})/B^2-\sum_a\langle(p_{\perp a}-p_{\|a})/B^2\rangle$ with the symmetric phase $F(s,-\theta,-\zeta)=F(s,\theta,\zeta)$, which is caused by various mechanisms, such as the collisionless detrapping ν regime ripple diffusions of light low-Z species,[18] the resonant viscosity of heavy impurity ions,[10] and a characteristic of fast ions velocity distribution discussed in Sec. III. When $\mathbf{J}\cdot\nabla s\neq 0$ by these reasons, Eq. (28) is a main deviation from the usual SCL coordinates system giving Eq. (15).

One conclusion of this section on a consistency of the $\mathbf{B}$,$\mathbf{J}$ vector fields is that when the $\mathbf{B}$-field has the stellarator symmetry $B(s,-\theta,-\zeta)=B(s,\theta,\zeta)$, the local parallel and radial currents $J_\parallel$ and $\mathbf{J}\cdot\nabla s$ should be functions with a symmetric phase $F(s,-\theta,-\zeta)=F(s,\theta,\zeta)$ and an anti-symmetric phase $F(s,-\theta,-\zeta)=-F(s,\theta,\zeta)$, respectively. Although it is a rigorous constraint for the symmetric configurations where $c_1\partial B/\partial\theta+c_2\partial B/\partial\zeta=0$ holds,[8] non-axisymmetric Fourier components of $\sin(m\theta-n\zeta)$ in $\sum_a(p_{\parallel a}+p_{\perp a})$ and $\sum_a(p_{\parallel a}-p_{\perp a})$ can actually exist in non-symmetric stellarator/heliotron configurations. This is an essential difference between these configurations and is clarified especially when investigating the ambipolar condition $\langle\mathbf{J}\cdot\nabla s\rangle=0$ in cases with the external momentum input $\langle B\int v_\parallel S_\mathrm{f}\mathrm{d}^3\mathbf{v}\rangle\neq 0$ as analyzed in Appendix A. For this reason, the NBI heated stellarator/heliotron plasmas[1] could be analyzed by the pure neoclassical procedure[8–10] without any phenomenological momentum dissipation terms, such as that in Sec. 8 in Ref. 6.

III. DRIFT KINETIC EQUATION FOR UNBALANCED TANGENTIAL NBI

Hereafter, a set of $\sigma\equiv v_\parallel/|v_\parallel|=\pm1$ and $\lambda\equiv\mu B_\mathrm{M}/w\equiv(B_\mathrm{M}/B)v_\perp^2/v^2$ with the maximum magnetic field strength B_M on each flux-surfaces is used mainly as the pitch-angle space parameter, rather than $\xi\equiv v_\parallel/v=\sigma(1-\lambda B/B_\mathrm{M})^{1/2}$ in Eq. (4) and some references, such as Refs. 8 and 19. A range $0\leq\lambda\leq1$ corresponds to the circulating pitch-angle in the full range $0\leq\lambda\leq B_\mathrm{M}/B$. Various pitch-angle integrals for moment equations in the moment method should be obtained by $\int_{-1}^{1}\mathrm{d}\xi=-\sum_\sigma\int_0^{B_\mathrm{M}/B}\{\partial(1-\lambda B/B_\mathrm{M})^{1/2}/\partial\lambda\}\mathrm{d}\lambda$. Fast ions' gyro-phase-averaged velocity distribution $\bar{f}_\mathrm{f}(\mathbf{x},v,\sigma,\lambda)$ discussed here is defined as a part of velocity distribution of a specific ion species, such as proton and deuterium in NBI heated plasmas and helium in burning plasmas. This part does not include the exponential factor $\exp(-m_av^2/2T_a)$ with the temperature $T_a\equiv p_a/n_a$ as shown in following discussions, and is categorized to be one particle species $a=\mathrm{f}$. The remaining component, including the exponential factor, corresponds to the thermalized ions that are categorized to be another particle species $a\neq\mathrm{e,f}$ [e:electron]. It should be handled by usual neoclassical procedures, in which the self-adjoint property of the Coulomb collision is fully utilized,[8–10] and energy scattering/exchange collision effects for the lower Legendre orders $l=0$, 1 are included in Braginskii's matrix elements[6] $\int v^lP_l(\xi)L_j^{(l+1/2)}(x_a^2)C_{ab}(v^lP_l(\xi)L_k^{(l+1/2)}(x_a^2)f_{a\mathrm{M}},f_{b\mathrm{M}})\mathrm{d}^3\mathbf{v}$, $\int v^lP_l(\xi)L_j^{(l+1/2)}(x_a^2)C_{ab}(f_{a\mathrm{M}},v^lP_l(\xi)L_k^{(l+1/2)}(x_b^2)f_{b\mathrm{M}})\mathrm{d}^3\mathbf{v}$. The fast ions' pressure $3p_\mathrm{f}\equiv m_\mathrm{f}\int v^2f_\mathrm{f}\mathrm{d}^3\mathbf{v}-n_\mathrm{f}m_\mathrm{f}u_{\perp\mathrm{f}}^2$ and particle flux $n_\mathrm{f}\mathbf{u}_\mathrm{f}\equiv\int\mathbf{v}f_\mathrm{f}\mathrm{d}^3\mathbf{v}$ as a component of $\mathbf{J}$ in the MHD equilibrium are not negligible as observed in experiments and as discussed in Sec. II. For thermalized particle species $a\neq\mathrm{f}$, $\int v\xi C_{a\mathrm{f}}(f_a,f_\mathrm{f})\mathrm{d}^3\mathbf{v}$ and $\int v^2C_{a\mathrm{f}}(f_a,f_\mathrm{f})\mathrm{d}^3\mathbf{v}$ are important input of parallel momentum and energy, respectively. Since these integrals are non-negligible only due to the fast ions' large initial velocity of $m_\mathrm{f}v_\mathrm{b}^2/2\gg T_\mathrm{e},T_\mathrm{i}$ $(T_\mathrm{i}\equiv\sum_{a\neq\mathrm{e,f}}p_a/\sum_{a\neq\mathrm{e,f}}n_a)$ and the heavy mass $m_\mathrm{f}\gg m_\mathrm{e}$, the density moment $n_\mathrm{f}\equiv\int f_\mathrm{f}\mathrm{d}^3\mathbf{v}$ can be assumed to be

$$Z_\mathrm{f}^2n_\mathrm{f}\ll n_\mathrm{e},Z_\mathrm{eff}n_\mathrm{e}. \tag{29}$$

Hereafter, following previous works related to fast ions,[7,19,20] charge number $Z_a\equiv e_a/e$ also is used to express various collision parameters and $Z_\mathrm{eff}\equiv\sum_{a\neq\mathrm{e,f}}Z_a^2n_a/n_\mathrm{e}$. One reason of Eq. (29) is given in Appendix B.

In this section, we shall consider a determination procedure of this $\bar{f}_\mathrm{f}(\mathbf{x},v,\sigma,\lambda)$ by the drift approximation. Firstly, we should note that the fast ions DKE for the parallel momentum input in the unbalanced tangential NBI operations[1,20] should exclude the perpendicular guiding center drift velocity $\mathbf{v}_\mathrm{df}=(c/e_\mathrm{f})(m_\mathrm{f}v_\parallel^2/B+\mu)\mathbf{b}\times\nabla\ln B$. The equation for the steady-states is given by[20]

$$v_\parallel\mathbf{b}\cdot\nabla\bar{f}_\mathrm{f}=\sum_b C_{\mathrm{f}b}(\bar{f}_\mathrm{f},f_b)+S_\mathrm{f}(\mathbf{x},v,\sigma,\lambda). \tag{30}$$

Here, $\mathbf{b}\cdot\nabla$ is a differential keeping constant (v,σ,λ), i.e., $v_\parallel\mathbf{b}\cdot\nabla=v\xi\mathbf{b}\cdot\nabla_{(v,\xi)=\mathrm{const}}-\frac{v}{2}(1-\xi^2)(\mathbf{b}\cdot\nabla\ln B)\partial/\partial\xi$. By excluding the perpendicular drift term $\mathbf{v}_\mathrm{df}\cdot\nabla$ from Eq. (30), the $\int v\xi\mathrm{d}^3\mathbf{v}$ moment of this equation agrees with the parallel component of Eq. (4) with $a=\mathrm{f}$ and with neglecting $m_\mathrm{f}n_\mathrm{f}u_{\perp\mathrm{f}}^2,m_\mathrm{f}n_\mathrm{f}u_{\perp\mathrm{f}}u_{\parallel\mathrm{f}}\ll p_\mathrm{f}$ in that $\nabla B^{-2}\times\mathbf{B}\cdot\nabla u_{\parallel\mathrm{f}}$ of the parallel velocity moments of the $\bar{f}_\mathrm{f}(\mathbf{x},v,\sigma,\lambda)$ does not appear there. Here, this $\int v\xi\mathrm{d}^3\mathbf{v}$ integral will be obtained by using the formula

$$\begin{aligned}v_\parallel\mathbf{b}\cdot\nabla\{P_l(\xi)F(\mathbf{x},v)\}&=v\frac{l}{2l+1}P_{l-1}(\xi)B^{(l+1)/2}\mathbf{b}\cdot\nabla\left(\frac{F}{B^{(l+1)/2}}\right)\\&+v\frac{l+1}{2l+1}P_{l+1}(\xi)\frac{1}{B^{l/2}}\mathbf{b}\cdot\nabla\left(FB^{l/2}\right)\end{aligned}$$

that is applicable for pitch-angle integrals $\int_{-1}^{1}P_l(\xi)(v_\parallel\mathbf{b}\cdot\nabla\bar{f}_a)\mathrm{d}\xi$ with general Legendre polynomials $P_l(\xi)$. The perpendicular drift term will be important when calculating $\bar{f}_\mathrm{f}^{(\mathrm{even})}\equiv[\bar{f}_\mathrm{f}(\mathbf{x},v,\sigma,\lambda)+\bar{f}_\mathrm{f}(\mathbf{x},v,-\sigma,\lambda)]/2$ as the even component of $v_\parallel$ in $\bar{f}_\mathrm{f}(\mathbf{x},v,\sigma,\lambda)$, since $\mathbf{v}_\mathrm{df}\cdot\nabla\bar{f}_\mathrm{f}^{(\mathrm{even})}$ corresponds to (1) generation of the Pfirsch-Schlüter current in Eq. (26), (2) the bounce-center motion of trapped particles[5] especially in the ripple-trapped pitch-angle range $0\leq\kappa^2\leq1$ of $\kappa^2\equiv\{w-\mu B_0(1+\varepsilon_\mathrm{T}-\varepsilon_\mathrm{H})\}/(2\mu B_0\varepsilon_\mathrm{H})$ in stellarator/heliotron magnetic fields $B/B_0=1+\varepsilon_\mathrm{T}(s,\theta)+\varepsilon_\mathrm{H}(s,\theta)\cos[L\theta-N\zeta+\gamma(s,\theta)]$, where B_0 is the volume averaged field strength, and (3) radial particle and energy transport.[7] However, $\int v\xi L_j^{(3/2)}(x_a^2)C_{a\mathrm{f}}(f_{a\mathrm{M}},\bar{f}_\mathrm{f})\mathrm{d}^3\mathbf{v}$ required for studying physics of target plasma species[2] and/or $\langle Bn_\mathrm{f}u_{\parallel\mathrm{f}}\rangle\equiv\langle B\int v\xi\bar{f}_\mathrm{f}\mathrm{d}^3\mathbf{v}\rangle$ in parallel current[20] $\langle\mathbf{J}\cdot\mathbf{B}\rangle$ are contributions of $\bar{f}_\mathrm{f}^{(\mathrm{odd})}\equiv[\bar{f}_\mathrm{f}(\mathbf{x},v,\sigma,\lambda)-\bar{f}_\mathrm{f}(\mathbf{x},v,-\sigma,\lambda)]/2$. This $\bar{f}_\mathrm{f}^{(\mathrm{odd})}$ should be handled by Eq. (30) excluding $\mathbf{v}_\mathrm{df}\cdot\nabla\bar{f}_\mathrm{f}^{(\mathrm{odd})}$ since this drift term generates a deviation from Eq. (4) especially in the unbalanced NB injections, which is a main purpose of the present work. It also should be noted that $\mathbf{E}\cdot\partial f_{\rm f}/\partial \mathbf{v}$ in the Landau equation as well as the resultant $\mathbf{E}\times\mathbf{B}$ motion is not taken into account in Eq. (30) since

(1) Because of the low density equation (29), this exclusion (corresponding to $e_{\rm f}n_{\rm f}\mathbf{E}=0$ in Eq. (4)) is not a serious inconsistency from the view of the MHD equilibrium.
(2) At present, stellarator/heliotron experiments are conducted without inductive electric fields $\mathbf{E}^{(\mathbf{A})}\equiv -c^{-1}\partial\mathbf{A}/\partial t\approx 0$. Only one purpose for retaining $\langle\mathbf{B}\cdot\mathbf{E}^{(\mathbf{A})}\rangle$ in stellarator/heliotron theories is to confirm the Onsagar symmetric relation between bootstrap currents and Ware pinches in the full neoclassical transport matrix.[10] The electric field is substantially that due to the ambipolar electrostatic potential $\mathbf{E}=-\nabla\Phi$. Since this ambipolar potential is order of $|\nabla\Phi|\sim|(\nabla p_a)/(e_a n_a)|$ $(a\neq {\rm f})$, it is negligible for the drift motions of fast ions with $m_{\rm f}v_{\rm b}^2/2\gg T_{\rm e},T_{\rm i}$, while it is non-negligible for thermalized particles with $m_a v^2/2\sim T_{\rm e},T_{\rm i}$.

Secondly, the collision operator $\sum_b C_{{\rm f}b}(f_{\rm f},f_b)$ in Eq. (30) is simplified in contrast with that in thermalized particles' kinetic equations.[6,8–10] The Coulomb collision operator for the colliding particle species pair a-b is basically used in the linearized form $C_{ab}(f_a,f_b)\cong C_{ab}(f_{a1},f_{b\rm M})+C_{ab}(f_{a\rm M},f_{b1})$ for $f_a=f_{a\rm M}+f_{a1}$. When we linearize the kinetic equations for these thermal particles velocity distributions, $f_{a\rm M}$ is the Maxwellian velocity distribution defined by the surface-averaged density and pressure moments that vanish in $\mathbf{b}\cdot\nabla f_a$, $\nabla s\times\mathbf{b}\cdot\nabla f_a$, and without the velocity moment $\int \mathbf{v}f_{a\rm M}{\rm d}^3\mathbf{v}=0$, while f_{a1} is the poloidally and toroidally varying deviation from $f_{a\rm M}$ (i.e., $\mathbf{b}\cdot\nabla f_{a1}\neq 0$ or $\nabla s\times\mathbf{b}\cdot\nabla f_{a1}\neq 0$). In theories for thermal particles,[6,8–10] this $C_{ab}(f_{a\rm M},f_{b1})$ is retained to include field particles' flows. In other words, $C_{ab}(f_a,f_b)\cong C_{ab}(f_a,f_{b\rm M}(\mathbf{v}-\mathbf{u}_b))$, where $f_{b\rm M}(\mathbf{v}-\mathbf{u}_b)$ is the shifted Maxwellian velocity distribution of the thermal particle species b. These flow velocities of thermal particle species are often comparable $|u_{\|a}|\sim|u_{\|b}|\sim|u_{\|c}|\sim ....$, and therefore $f_{a1}^{(l=1)}\equiv\frac{3}{2}\xi\int_{-1}^{1}\xi f_{a1}{\rm d}\xi$ of all species are regarded to be comparable there because of the Galilean invariant property of the Coulomb collision. In Eq. (30) for fast ions, however, these flow velocities of target thermal particles being $|u_a|\sim|u_b|\sim|u_c|\sim...\ll v_{\rm b}\,(a,b,c,..\neq{\rm f})$ can be neglected and thus $C_{{\rm f}b}(f_{\rm f},f_b)\cong C_{{\rm f}b}(f_{\rm f},f_{b\rm M})$ for $b\neq{\rm f}$ (test particle portion only). Because of an extreme difference between the velocity moments in the tangential NBI, we do not need to retain the Galilean invariant property so rigorously in this $C_{{\rm f}b}(f_{\rm f},f_b)$. In this approximation, for retaining the conservation of momentum and energy, collisions of thermalized particles (a) with the fast ions (f) should be calculated by $C_{a\rm f}(f_a,f_{\rm f})\cong C_{a\rm f}(f_{a\rm M},f_{\rm f})$ (field particle portion only).[2] When Eq. (29) is satisfied, $C_{a\rm f}(f_{a1},f_{\rm f})$ is negligibly smaller than $\sum_{b\neq\rm f}C_{ab}(f_{a1},f_{b\rm M})$. Furthermore, in Eq. (30), because of this low density of the fast ions themselves and the momentum/energy conservation of like-particle collisions $\int\mathbf{v}C_{aa}(f_a,f_a){\rm d}^3\mathbf{v}=0=\int v^2C_{aa}(f_a,f_a){\rm d}^3\mathbf{v}$, the non-linear collision term $C_{\rm ff}(f_{\rm f},f_{\rm f})$ can be omitted. An explicit expression of the exact test particle portion in the spherical velocity coordinates is given by[21]

$$C_{ab}(f_a,f_{b\rm M})=4\pi n_b\left(\frac{e_ae_b}{m_a}\right)^2\ln\Lambda_{ab}\left[\frac{\Phi(x_b)-G(x_b)}{v^3}\mathcal{L}f_a+v^{-2}\frac{\partial}{\partial v}\left\{G(x_b)v\left(\frac{m_av}{T_b}+\frac{\partial}{\partial v}\right)f_a\right\}\right]$$

$$\mathcal{L}\equiv\frac{1}{2}\left(\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi}+\frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right)=\frac{B_{\rm M}}{B}\left(2\frac{v_\|}{v}\frac{\partial}{\partial\lambda}\lambda\frac{v_\|}{v}\frac{\partial}{\partial\lambda}+\frac{1}{2\lambda}\frac{\partial^2}{\partial\phi^2}\right). \quad (31)$$

In Eq. (30) determining the gyro-phase-averaged distribution, the gyro-angle differential $\partial^2/\partial\phi^2$ is not used. The error and the Chandrasekhar functions are defined by

$$\Phi(x)\equiv\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}{\rm d}y=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}$$

and

$$G(x)\equiv\frac{\Phi(x)-x\Phi'(x)}{2x^2}=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+3)n!}x^{2n+1},$$

respectively, and $x_b\equiv\sqrt{m_bv^2/(2T_b)}$. Connection formulas of their $x\ll 1$ and $x\gg 1$ asymptotic limits

$$\Phi(x)-G(x)\cong\left\{\left(\frac{3\sqrt{\pi}}{4x}\right)^{5/2}+1\right\}^{-2/5},$$

$$G(x)\cong\left(\frac{3\sqrt{\pi}}{2x}+2x^2\right)^{-1}$$

also will be useful. The Coulomb logarithm $\ln\Lambda_{ab}=\ln\Lambda_{ba}$ for the colliding species pair a-b is a constant being independent of $(\theta,\zeta,\mathbf{v})$, on each flux-surfaces. Not only the omission of $\partial^2/\partial\phi^2$, a straightforward use of Eq. (31) for $\sum_{b\neq\rm f}C_{{\rm f}b}(f_{\rm f},f_{b\rm M})$ in Eq. (30) is inadequate and thus other minor modifications are required because of the following reason. The straightforward use in this equation with the source term will result in a time evolution of a velocity distribution component, including $\exp(-m_{\rm f}v^2/2T_{\rm i})$ at $m_{\rm f}v^2\sim 2T_{\rm i}$ following the H-theorem. A strongly peaking structure at $m_{\rm f}v^2\sim T_{\rm i}$ shown in Fig. 2 in Ref. 5 is an example. Note that this structure indicates only a qualitative characteristic of the velocity distribution since it corresponds to Maxwellian of protons for which a prior existence in the source and collision terms is assumed. It also should be noted that this energy region $m_{\rm f}v^2\sim T_{\rm i}$ in Ref. 5 could not contribute to the substantial heating power[22] due to a well-known relation $C_{ab}(f_{a\rm M},f_{b\rm M})\propto(T_a/T_b-1)$. When we choose a method preventing this formation of the exponential structure, the approximation should be optimized for $m_{\rm f}v^2\gg 2T_{\rm i}$ and simultaneously the particle conservation $\int\sum_b C_{{\rm f}b}(f_{\rm f},f_{b\rm M}){\rm d}^3\mathbf{v}=0$ should be artificially broken at $m_{\rm f}v^2\sim 2T_{\rm i}$ to make the collision operator sink low energy particles. Although our present approximation method for this purpose is basically identical to that in previous tokamak studies related to fast ions,[7,19,20] we shall summarize the approximation here. Although the collisions with electrons $C_{\rm fe}(f_{\rm f},f_{\rm eM})$ and those with thermal ions $C_{{\rm f}b}(f_{\rm f},f_{b{\rm M}})$ ($b\neq$ e,f) use different approximations, a common modification to optimize them for $m_{\rm f}v^2 \gg 2T_{\rm i}$ is $(m_{\rm f}v/T_b+\partial/\partial v)f_{\rm f} \cong m_{\rm f}vf_{\rm f}/T_b$. The energy transfer rates of the standard RMJ (Rosenbluth-MacDonald-Judd) operator are retained within accuracies neglecting $3T_{\rm e}/m_{\rm f} \ll v_{\rm b}^2$ for f-e collisions and $2T_{\rm i}/m_{\rm i} \ll v_{\rm b}^2$ for f-i collisions. (Comparisons of the momentum/energy transfer rates with those of the standard RMJ operator are shown in Appendix C.) In $C_{\rm fe}(f_{\rm f},f_{\rm eM})$, the pitch- and gyro-angle scattering function $\mathcal{L}f_{\rm f}$ should be simultaneously omitted for retaining the f-e, e-f momentum transfer rate of the standard RMJ within an accuracy neglecting only $m_{\rm e} \ll m_{\rm f}$. This approximation corresponds to a neglect of the second term in the Cartesian coordinates expression $C_{\rm fe}(f_{\rm f},f_{\rm eM}) \cong \tau_{\rm S}^{-1}\sum_\alpha \partial/\partial v_\alpha\{v_\alpha + (T_{\rm e}/m_{\rm f})\partial/\partial v_\alpha\}f_{\rm f}$ for a velocity range $v<(3\sqrt{\pi}/4)^{1/3}\sqrt{2T_{\rm e}/m_{\rm e}}$ to optimize it for $m_{\rm f}v^2 \gg 2T_{\rm e}$. The $\mathcal{L}f_{\rm f}$ operator in $C_{\rm fe}(f_{\rm f},f_{\rm eM})$ is only a minor component of that in $\sum_{b\neq{\rm f}} C_{{\rm f}b}(f_{\rm f},f_{b{\rm M}})$ with $Z_{\rm eff}>1$ in general v-space regions, especially in $v<(3\sqrt{\pi}/4)\sqrt{2T_{\rm e}/m_{\rm e}}$. In the collision with thermal ions $C_{{\rm f}b}(f_{\rm f},f_{b{\rm M}})$ ($b\neq$ e,f), the standard form Eq. (31) is used in its $T_{\rm i}/m_{\rm f}\to 0$ limit and consequently $G(x_b)v(m_{\rm f}v/T_b+\partial/\partial v)f_{\rm f} \cong (m_{\rm f}/m_b)f_{\rm f}$. This replacement in the energy scattering term is not only for the optimization for $m_{\rm f}v^2 \gg 2T_{\rm i}$ but also the artificial break of the particle conservation $\int C_{{\rm f}b}(f_{\rm f},f_{b{\rm M}})d^3\mathbf{v}=0$ for obtaining the steady-state solution of Eq. (30). In this use in the $T_{\rm i}/m_{\rm f}\to 0$ limit, the momentum transfer rate of the standard RMJ is retained within an accuracy neglecting $2T_{\rm i}/m_{\rm i} \ll v_{\rm b}^2$. Now it is concluded for Eq. (30) that

$$
\begin{aligned}
\sum_b C_{{\rm f}b}(f_{\rm f},f_b) &\cong \sum_{b\neq{\rm f}} C_{{\rm f}b}(f_{\rm f},f_{b{\rm M}}) \cong C_{\rm f}^{\rm PAS}f_{\rm f} + C_{\rm f}^{\rm ES}f_{\rm f}\\
C_{\rm f}^{\rm PAS}f_{\rm f} &\equiv 4\pi\frac{e^4Z_{\rm f}^2}{m_{\rm f}^2}\left(\sum_{b\neq{\rm e,f}}\langle n_b\rangle Z_b^2\ln\Lambda_{{\rm f}b}\right)v^{-3}\mathcal{L}f_{\rm f} \equiv \frac{Z_2}{\tau_{\rm S}}\frac{v_{\rm c}^3}{v^3}\mathcal{L}f_{\rm f}\\
C_{\rm f}^{\rm ES}f_{\rm f} &\equiv 4\pi\frac{e^4Z_{\rm f}^2}{m_{\rm f}}v^{-2}\frac{\partial}{\partial v}\left\{\left(\langle n_{\rm e}\rangle\ln\Lambda_{\rm fe}\frac{v^2}{\langle T_{\rm e}\rangle}G(x_{\rm e}) + \sum_{b\neq{\rm e,f}}\frac{\langle n_b\rangle Z_b^2\ln\Lambda_{{\rm f}b}}{m_b}\right)f_{\rm f}\right\}\\
&\equiv \frac{1}{\tau_{\rm S}}v^{-2}\frac{\partial}{\partial v}\left\{\left(v^2v_{\rm Te}\frac{3\sqrt{\pi}}{2}G(x_{\rm e})+v_{\rm c}^3\right)f_{\rm f}\right\}.
\end{aligned}
\tag{32}
$$

Here, the following parameters and/or variables are used:

$$
v_{\rm Te}\equiv\sqrt{2\langle T_{\rm e}\rangle/m_{\rm e}},\quad x_{\rm e}\equiv v/v_{\rm Te},\quad \frac{3\sqrt{\pi}}{4\tau_{\rm S}}\equiv\frac{4\pi e^4Z_{\rm f}^2\langle n_{\rm e}\rangle\ln\Lambda_{\rm fe}}{m_{\rm f}m_{\rm e}v_{\rm Te}^3},\quad Z_2\equiv\frac{1}{m_{\rm f}}\frac{\sum_{a\neq{\rm e,f}}\langle n_a\rangle Z_a^2\ln\Lambda_{{\rm f}a}}{\sum_{a\neq{\rm e,f}}\langle n_a\rangle Z_a^2\ln\Lambda_{{\rm f}a}/m_a},
$$

$$
v_{\rm c}^3\equiv\frac{3\sqrt{\pi}}{4}v_{\rm Te}^3\frac{m_{\rm e}}{\langle n_{\rm e}\rangle\ln\Lambda_{\rm fe}}\sum_{a\neq{\rm e,f}}\frac{\langle n_a\rangle Z_a^2\ln\Lambda_{{\rm f}a}}{m_a}.
$$

A possibility of $x_{\rm e}^3\sim 3\sqrt{\pi}/4$ in $C_{\rm f}^{\rm ES}$ at low-$T_{\rm e}$ regions is allowed in following calculations. The approximation of $C_{\rm f}^{\rm PAS}$ is justified later also by the resulting $\mathbf{v}$-space structure of $\bar{f}_{\rm f}(\mathbf{x},v,\sigma,\lambda)$. Since this velocity distribution in NBI-heated or burning plasmas is generated as a response to the source term being the delta function in the energy space $S_{\rm f}(\mathbf{x},v,\sigma,\lambda)\propto\delta(v-v_{\rm b})/v^2$, it includes the unit step function in the energy space[7,19,20] $U(v_{\rm b}-v)$. Actually, it is not a rigorous step function but is a continuous function having an exponential decay structure[19]

$$
U_{\rm c}(v_{\rm b}-v)\equiv\begin{cases}1 \text{ for } v\le v_{\rm b}\\ \exp\left[-C_{\rm D}(v-v_{\rm b})/v_{\rm b}\right] \text{ for } v\ge v_{\rm b}\end{cases}
$$

$$
C_{\rm D}\equiv\frac{m_{\rm f}\sum_{a\neq{\rm f,e}}\langle n_a\rangle Z_a^2\ln\Lambda_{{\rm f}a}/m_a + m_{\rm f}\left(v_{\rm b}^2/\langle T_{\rm e}\rangle\right)\langle n_{\rm e}\rangle\ln\Lambda_{\rm fe}G(v_{\rm b}/v_{\rm Te})}{\left(\langle T_{\rm i}\rangle/v_{\rm b}^2\right)\sum_{a\neq{\rm f,e}}\langle n_a\rangle Z_a^2\ln\Lambda_{{\rm f}a}/m_a + \langle n_{\rm e}\rangle\ln\Lambda_{\rm fe}G(v_{\rm b}/v_{\rm Te})}
$$

that satisfies $(\partial/\partial v)\{\sum_{b\neq{\rm f}}\langle n_b\rangle e_b^2\ln\Lambda_{{\rm f}b}G(x_b)v(m_{\rm f}v/\langle T_b\rangle+\partial/\partial v)\bar{f}_{\rm f}\}=0$ for the full part of the energy scattering term in Eq. (31) at an energy space region of $0<(v-v_{\rm b})/v_{\rm b}\sim T_{\rm e}/(m_{\rm f}v_{\rm b}^2), T_{\rm i}/(m_{\rm f}v_{\rm b}^2)\ll 1$. In spite of this, the solution in $v<v_{\rm b}$ should be obtained by using Eq. (32) when investigating the steady-states. Both substituting $U(v_{\rm b}-v)$ into $C_{\rm f}^{\rm ES}$ and substituting $U_{\rm c}(v_{\rm b}-v)$ into the full energy scattering operator result in the same delta function at the initial velocity that is balanced with the source by

$$C_{\rm f}^{\rm ES}U(v_{\rm b}-v)=4\pi\left(\frac{e_{\rm f}}{m_{\rm f}}\right)^2 v^{-2}\frac{\partial}{\partial v}\left\{\sum_{b\neq{\rm f}}\langle n_b\rangle e_b^2\ln\Lambda_{{\rm f}b}G(x_b)v\left(\frac{m_{\rm f}v}{\langle T_b\rangle}+\frac{\partial}{\partial v}\right)U_{\rm c}(v_{\rm b}-v)\right\}$$
$$=-\frac{1}{\tau_{\rm S}}\left\{v_{\rm b}^2 v_{\rm Te}\frac{3\sqrt{\pi}}{2}G\left(\frac{v_{\rm b}}{v_{\rm Te}}\right)+v_{\rm c}^3\right\}\frac{\delta(v_{\rm b}-v)}{v^2}\ \text{at}\ |v-v_{\rm b}|/v_{\rm b}\ll 1.$$

Since this high-energy tail region $0<(v-v_{\rm b})/v_{\rm b}\ll 1$ does not have any essential roles in practically important integral formulas, such as $\int L_j^{(1/2)}(x_a^2)C_{a{\rm f}}(f_{a{\rm M}},f_{\rm f}){\rm d}^3\mathbf{v}$, $\int \mathbf{v}L_j^{(3/2)}(x_a^2)C_{a{\rm f}}(f_{a{\rm M}},f_{\rm f}){\rm d}^3\mathbf{v}$ of the field particle portion,[2] $\int v^2 C_{{\rm f}b}(f_{\rm f},f_{b{\rm M}}){\rm d}^3\mathbf{v}$, $\int \mathbf{v}v^n C_{{\rm f}b}(f_{\rm f},f_{b{\rm M}}){\rm d}^3\mathbf{v}$ of the test particle portion (Appendix C), and/or $n_{\rm f}\mathbf{u}_{\rm f}\equiv\int\mathbf{v}f_{\rm f}{\rm d}^3\mathbf{v}$ of the velocity distribution,[20] we assume $f_{\rm f}\propto U(v_{\rm b}-v)$ at $|v-v_{\rm b}|/v_{\rm b}\ll 1$ for the 0th order of $\rho_{\rm p}/L_r$ [$\rho_{\rm p}$: typical poloidal gyro-radius, L_r: typical radial gradient scale length] in discussions below.

Thirdly, there is a constraint on the real space structure $S_{\mathbf{x}\lambda}(\mathbf{x},\sigma,\lambda)$ in the source term $S_{\rm f}(\mathbf{x},v,\sigma,\lambda)=S_{\mathbf{x}\lambda}(\mathbf{x},\sigma,\lambda)\delta(v-v_{\rm b})/v^2$. The constraint is due to fast ions initial parallel drift motions just after the beam ionization (or the nuclear reaction generating the α-particles) which conserve the magnetic moment $\mu=m_{\rm f}v_\perp^2/2B$. The resultant real space structure should be $S_{\mathbf{x}\lambda}(\mathbf{x},\sigma,\lambda)=S_{\mathbf{x}\lambda}(s,\sigma,\lambda)$ especially in the circulating pitch-angle range $0\le\lambda\le 1$, and consequently $\int v_\parallel v^n S_{\rm f}{\rm d}^3\mathbf{v}\propto B(\theta,\zeta)$ on each flux-surfaces for arbitrary integer $n\sim 1$. We already discussed in Sec. II this characteristic of the source term from the viewpoint of the **B**,**J** vector fields determination in the MHD equilibrium. Here, we explain it in another viewpoint of the fast ions drift motion and their collision equation (32). The solution method for Eq. (30) explains this reason. A typical collision time $\tau_{\rm S}$ in Eq. (32) corresponds to the longest time scale in various Braginskii's collision times, which express the time scale of collisions between thermalized particles. Because of this collision time scale and the fast initial velocity $m_{\rm f}v_{\rm b}^2/2\gg T_{\rm e},T_{\rm i}$, the method is an asymptotic expansion that uses the inverse mean free path ν/v as the expansion parameter.[7,20] This method is analogous to the banana regime expansion for thermalized particles' energy regions of $\nu/v\ll(\delta B/B)^{3/2}/L_{\rm c}$, where $1/L_{\rm c}\sim\mathbf{b}\cdot\nabla\ln B$ is the characteristic length along the **B**-field line.[23–25] In Eq. (30), the 0th order of ν/v should satisfy $\mathbf{b}\cdot\nabla\bar{f}_{\rm f}^0=0$ and thus $\bar{f}_{\rm f}^{0\,({\rm odd})}$ exists only in $0\le\lambda\le 1$, i.e.,

$$\begin{aligned}\bar{f}_{\rm f}^{0\,({\rm odd})}&\equiv[\bar{f}_{\rm f}^0(\mathbf{x},v,\sigma,\lambda)-\bar{f}_{\rm f}^0(\mathbf{x},v,-\sigma,\lambda)]/2\\&=[\bar{f}_{\rm f}^0(\mathbf{x},v,\sigma,\lambda)-\bar{f}_{\rm f}^0(\mathbf{x},v,-\sigma,\lambda)]U(1-\lambda)/2\\&=\bar{f}_{\rm f}^{0\,({\rm odd})}(s,v,\sigma,\lambda)\end{aligned}\tag{33}$$

and $\bar{f}_{\rm f}^{0\,({\rm odd})}(\lambda=1)=0$. As mentioned in the introduction, this is the fast ion trapping effect, in which the trapped pitch-angle range cannot contribute to integrals in a form of $\langle B\int\xi F(v)f_{\rm f}{\rm d}^3\mathbf{v}\rangle$, such as $\langle B\int v\xi L_j^{(3/2)}(x_a^2)C_{a{\rm f}}(f_{a{\rm M}},f_{\rm f}){\rm d}^3\mathbf{v}\rangle$. Then the 1st order of ν/v is governed by $v_\parallel\mathbf{b}\cdot\nabla\bar{f}_{\rm f}^1=(C_{\rm f}^{\rm PAS}+C_{\rm f}^{\rm ES})\bar{f}_{\rm f}^0+S_{\rm f}(\mathbf{x},v,\sigma,\lambda)$. The solubility condition of this 1st order equation in $0\le\lambda\le 1$ and $v<v_{\rm b}$ is

$$\left\langle B\frac{v}{v_\parallel}(C_{\rm f}^{\rm PAS}+C_{\rm f}^{\rm ES})\bar{f}_{\rm f}^0\right\rangle=0\tag{34}$$

because of Eq. (13). This condition determines the **v**-space structure of $\bar{f}_{\rm f}^0$ in $0\le\lambda\le 1$ as investigated in Section IV. Note that this $\langle\cdot\rangle$ also is a surface-average keeping constant (v,σ,λ).

For Eqs. (4) and (30) handling the $\partial f_a/\partial t=0$ steady-states, the source term $S_{\rm f}(\mathbf{x},v,\sigma,\lambda)=S_{\mathbf{x}\lambda}(\mathbf{x},\sigma,\lambda)\delta(v-v_{\rm b})/v^2$ in them does not correspond to the number of beam ionization event at each real space position **x** (so-called birth of fast ions), but is defined for a short (but finite) time scale of $2\pi R/v_{\rm b}\ll t\ll\tau_{\rm S}$ just after the ionization by taking into account the initial drift motions in this time scale. Therefore, the balance of the DKE terms at the initial energy is given by

$$S_{\rm f}(\mathbf{x},v,\sigma,\lambda)=-C_{\rm f}^{\rm ES}\bar{f}_{\rm f}^0\quad\text{at}\ |v-v_{\rm b}|/v_{\rm b}\ll 1.\tag{35}$$

In the recent study[1] applying the FIT3D code,[3] for example, results of a Monte Carlo code MCNBI in it, which calculate the initial drift orbit trace in $2\pi R/v_{\rm b}\ll t\ll\tau_{\rm S}$ after the beam ionization handled by HFREYA, are used as the source. The PAS operator in Eq. (32) satisfying $\int_{-1}^{1}(C_{\rm f}^{\rm PAS}\bar{f}_{\rm f}){\rm d}\xi=0$ does not have any essential roles in this particle/energy balance at the energy region of $|v-v_{\rm b}|/v_{\rm b}\ll 1$. Since (as long as aforementioned $\mathbf{v}_{\rm df}\cdot\nabla\bar{f}_{\rm f}^{({\rm even})}$ as the 1st order of $\rho_{\rm p}/L_r$ is excluded) $\bar{f}_{\rm f}^0$ in $\kappa^2>1$ is a function of (s,v,σ,λ) only, the source term in Eq. (35) should be $S_{\mathbf{x}\lambda}(\mathbf{x},\sigma,\lambda)=S_{\mathbf{x}\lambda}(s,\sigma,\lambda)$ except the deeply trapped pitch-angle range $\kappa^2<1$ that we do not need to consider for the tangential NBI operations. This (**x**,**v**) space structure of $S_{\mathbf{x}\lambda}(s,\sigma,\lambda)$ includes also an implicit assumption $\mathbf{b}\cdot\nabla_{(v,\xi)={\rm const}}f_{\rm f}=0$ in Ref. 19 as the $\mathbf{b}\cdot\nabla B\to 0$ limit. Following this conclusion that the generation of $\bar{f}_{\rm f}^{0\,({\rm odd})}$ should be calculated by using $S_{\mathbf{x}\lambda}(\mathbf{x},\sigma,\lambda)=S_{\mathbf{x}\lambda}(s,\sigma,\lambda)$ at least in $0\le\lambda\le 1$, the adjoint equation in Ref. 20 for allowing arbitrary function forms of $S_{\mathbf{x}\lambda}(\mathbf{x},\sigma,\lambda)$ is not used in this present study. Instead of that, we directly solve Eq. (30) for $0\le\lambda\le 1$ in Section IV.

The $\int_{-1}^{1}\cdot{\rm d}\xi=-\sum_\sigma\int_0^{B_{\rm M}/B}\cdot\{\partial(1-\lambda B/B_{\rm M})^{1/2}/\partial\lambda\}{\rm d}\lambda$ integral of Eq. (35) with Eq. (33) indicates that $\int\mathbf{v}_\parallel v^n S_{\rm f}{\rm d}^3\mathbf{v}=\langle B\int v_\parallel v^n S_{\rm f}{\rm d}^3\mathbf{v}\rangle\mathbf{B}/\langle B^2\rangle$ on each flux-surface in Eq. (4) and a $\int\mathbf{v}v^2{\rm d}^3\mathbf{v}$ moment of the Landau equation (A5). By Eq. (21), this constraint leads also to

$$\left\langle\tilde{U}\int v_\parallel v^n S_{\rm f}{\rm d}^3\mathbf{v}\right\rangle=0.\tag{36}$$

Even when the $\mathbf{v}_{\rm df}\cdot\nabla\bar{f}_{\rm f}^{({\rm even})}$ as the 1st order of $\rho_{\rm p}/L_r$ generating poloidal and toroidal variations of $\bar{f}_{\rm f}^{({\rm odd})}$ corresponding to the Pfirsch-Schlüter current in Eq. (26) is included as in Ref. 7, this 1st order variation cannot affect Eq. (35), and consequently Eq. (36). When the 0th order of ρ_p/L_r in $\bar{f}_f^{(even)}$ is obtained by the above procedure, $\bar{f}_{f1}$ as the 1st order of ρ_p/L_r will be determined by $(v_\parallel \mathbf{b}\cdot\nabla - C_f^{PAS} - C_f^{ES})\bar{f}_{f1} = -\mathbf{v}_{df}\cdot\nabla\bar{f}_f^{(even)}$. Although details of this equation and its solution will depend on configurations and the injection conditions $S_f(s,v,\sigma,\lambda)$, for a consistency with Eq. (26), the $\int_{-1}^{1} d\xi$ integral of this equation in $v < v_b$ would commonly be

$$\mathbf{B}\cdot\nabla\left(\int_{-1}^{1}\xi\bar{f}_{f1}d\xi/B\right) = \frac{c}{e_f}\frac{m_f v}{4}\nabla s\times\mathbf{B}\cdot\nabla\frac{1}{B^2}\frac{\partial}{\partial s}\left\langle\int_{-1}^{1}\left(1+\xi^2\right)\bar{f}_f d\xi\right\rangle,$$

giving

$$\int_{-1}^{1}\xi\bar{f}_{f1}d\xi - \left\langle B\int_{-1}^{1}\xi\bar{f}_{f1}d\xi\right\rangle\frac{B}{\langle B^2\rangle} = -\frac{c}{e_f}\frac{m_f v}{4}\tilde{U}\frac{\partial}{\partial s}\left\langle\int_{-1}^{1}\left(1+\xi^2\right)\bar{f}_f d\xi\right\rangle \qquad (37)$$

by Eq. (21). Here, the energy scattering collision $\int_{-1}^{1}(C_f^{ES}\bar{f}_{f1})d\xi$ is neglected by $v_b\tau_S \gg L_c$, and $\int_{-1}^{1}(C_f^{PAS}\bar{f}_{f1})d\xi = 0$ vanishes for general gyro-phase-averaged velocity distributions. Equation (37) as the 1st order of ρ_p/L_r is the contribution of fast ions in Eq. (26), on which it is reported in various experiments that this poloidal variation of $\bar{f}_{f1}^{(l=1)}$ has a non-negligible effect in determining the Shafranov shift. In spite of the fact that this 1st order also may be $\bar{f}_{f1} \propto U(v_b - v)$ at $|v - v_b|/v_b \ll 1$ when the 0th order component is $\bar{f}_f^{(even)} \propto U(v_b - v)$ there, $\bar{f}_{f1}$ cannot be included in Eq. (35) because of a constraint in Sec. II that Eq. (17) should vanish. This kind of perpendicular gradient effects $\mathbf{b}\times\nabla\bar{f}_f^{(even)}$ of the fast ions at $|v - v_b|/v_b \ll 1$ in NBI heated and/or burning plasmas cannot be investigated only by the drift approximation for describing the gyro-phase-averaged velocity distribution functions. When the perpendicular gradient exists, the gyro-phase-dependent part $\tilde{f}_f = \frac{m_f c}{e_f B}\mathbf{v}\cdot\left(\mathbf{b}\times\nabla\bar{f}_f^{(even)}\right)$ exists and it is a cause of $\mathbf{v}_{df}\cdot\nabla\bar{f}_f^{(even)}$ in the gyro-phase-averaged equation. The collision against this component $C_f^{ES}\tilde{f}_f$ (i.e., the collision against the gyro-motion) also diverges as the delta function at $|v - v_b|/v_b \ll 1$ (Generations of the classical diffusions Γ_f^{cl} and $\mathbf{Q}_f^{cl}$ defined in Appendix A have a peaking contribution at this initial velocity in the v-space.), and thus the concept of the collisionless perpendicular guiding center drift velocity $\mathbf{v}_{df}$ for $e_f c^{-1}B/m_f \gg 1/\tau_S$ is violated there. We investigated the constraint on Eq. (35) in the viewpoint of the consistency of the **B**,**J** vector fields for this reason. When the poloidally and toroidally varying parallel flow Eq. (37) as the 1st order of ρ_p/L_r is included in Eq. (35) as a definition of the fast ion source, it corresponds to $P_S(s,-\theta,-\zeta) = -P_S(s,\theta,\zeta) \neq 0$ that is forbidden in Sec. II. Therefore, $\bar{f}_{f1}$ at $|v - v_b|/v_b \ll 1$ cannot be $\propto U(v_b - v)$ nor $\propto U_c(v_b - v)$, but is a function with a continuous derivative $\partial\bar{f}_{f1}/\partial v$ that is determined by Eq. (31). These approximated expressions for $\tilde{f}_f$ and/or $\bar{f}_{f1}^{(l=1)}$ that are caused by $\mathbf{b}\times\nabla\bar{f}_f^{(even)} \neq 0$ can be used only for the aforementioned integrals $\int \mathbf{v}L_j^{(3/2)}(x_a^2)C_{af}(f_{aM},f_f)d^3\mathbf{v}$, $\int \mathbf{v}v^n\sum_{b\neq f}C_{fb}(f_f,f_{bM})d^3\mathbf{v}$ and/or $n_f\mathbf{u}_f \equiv \int \mathbf{v}f_f d^3\mathbf{v}$. (Differential operations $\partial\bar{f}_{f1}/\partial v$, $\partial^2\bar{f}_{f1}/\partial v^2$ will have large uncertainty.) Generations of $\langle\Gamma_f^{PS}\cdot\nabla s\rangle$, $\langle\mathbf{Q}_f^{PS}\cdot\nabla s\rangle$, and modifications of $\langle\Gamma_a^{PS}\cdot\nabla s\rangle$, $\langle\mathbf{Q}_a^{PS}\cdot\nabla s\rangle$ $(a\neq f)$ in Appendix A due to this $\bar{f}_{f1}^{(l=1)}$ are future themes. It also should be noted that this discussion on $\int v_\parallel v^n S_f d^3\mathbf{v}$ assumes the core regions where initial drift orbits crossing the last closed flux surface (LCFS) do not exist.

IV. EIGENFUNCTION METHOD

A. Definition and obtaining method of the eigenfunction

In the circulating pitch-angle $0\le\lambda\le 1$, instead of the local balance of source and collision in uniform magnetic field[19] $\mathbf{b}\cdot\nabla B = 0$, this balance now becomes a surface-averaged balance

$$Z_2\frac{v_c^3}{v^3}\frac{\partial}{\partial\lambda}\lambda\langle(1-\lambda B/B_M)^{1/2}\rangle\frac{\partial\bar{f}_f^0}{\partial\lambda} - \frac{\partial\langle(1-\lambda B/B_M)^{1/2}\rangle}{\partial\lambda}v^{-2}\times\frac{\partial}{\partial v}\left\{\left(v^2 v_{Te}\frac{3\sqrt{\pi}}{2}G(x_e)+v_c^3\right)\bar{f}_f^0\right\} = \frac{\partial\langle(1-\lambda B/B_M)^{1/2}\rangle}{\partial\lambda}\tau_S S_{\mathbf{x}\lambda}(s,\sigma,\lambda)\frac{\delta(v-v_b)}{v^2}, \qquad (38)$$

in which $\mathbf{B}\cdot\nabla$ is eliminated by Eq. (13). Here, $\langle(B/B_M)(1-\lambda B/B_M)^{-1/2}\rangle = -2\partial\langle(1-\lambda B/B_M)^{1/2}\rangle/\partial\lambda$ is used. For solving this equation, the following eigenfunctions $\Lambda_n(\lambda)$ with the eigenvalues κ_n (numbered as $n=1,2,3,.$) are required in describing the pitch-angle (λ) space, instead of the usual Legendre polynomial $P_l(\xi)$:

$$\frac{\partial}{\partial\lambda}\lambda\langle(1-\lambda B/B_M)^{1/2}\rangle\frac{\partial\Lambda_n}{\partial\lambda} = \kappa_n\frac{\partial\langle(1-\lambda B/B_M)^{1/2}\rangle}{\partial\lambda}\Lambda_n \quad \text{in } 0\le\lambda\le1,\ \Lambda_n(0)=1,\ \Lambda_n(1)=0. \qquad (39)$$

This type of eigenfunction is often required when handling the pitch-angle-scattering collision in the toroidal plasmas. Theoretical study on electron cyclotron current drive[26–28] is another application area in addition to the α-particle diffusion[7] and the NB-driven effects.[20] We shall consider here only a determination method for $\bar{f}_f^{0\,(odd)}$ in Eq. (33), and therefore $\Lambda_n(1) = 0$ as the boundary condition at the circulating/trapped boundary $\lambda = 1$ corresponding to a fact $\bar{f}_f^{0\,(odd)}(\lambda = 1) = 0$ is used in this definition. Although the even component $\bar{f}_f^{0\,(even)}$ requires a different boundary condition and a handling of the trapped pitch-angle range $\lambda > 1$ by bounce-integrals instead of the surface-averaging,[29] a practically important component $\langle f_f^{(l=0)}\rangle \propto [v^2 v_{Te}(3\sqrt{\pi}/2)G(x_e)+v_c^3]^{-1}U(v_b - v)$ (the surface-averaged lowest Legendre order $l=0$) in $\bar{f}_f^{0\,(even)}$ is not affected by the **B**-field strength modulation along the field line as discussed in Appendix B. A collision integral $m_a \int v^2 C_{af}(f_{aM}, f_f) d^3\mathbf{v} = -m_f \int v^2 C_{fa}(f_f, f_{aM}) d^3\mathbf{v}$ in Appendix C for power deposition analyses requires only this $\langle f_f^{(l=0)} \rangle$. A main purpose of the present study is $\bar{f}_f^{0\,(\mathrm{odd})}$ determining aforementioned integrals in the common form of $\langle B \int \xi F(v) f_f d^3\mathbf{v} \rangle$. There is a self-adjoint property of this surface-averaged PAS operator for arbitrary functions satisfying this boundary condition $\mathcal{F}(\lambda = 1) = \mathcal{G}(\lambda = 1) = 0$:

$$\int_0^1 \mathcal{G}(\lambda) \frac{\partial}{\partial \lambda} \lambda \langle (1 - \lambda B/B_M)^{1/2} \rangle \frac{\partial \mathcal{F}(\lambda)}{\partial \lambda} d\lambda = \int_0^1 \mathcal{F}(\lambda) \frac{\partial}{\partial \lambda} \lambda \langle (1 - \lambda B/B_M)^{1/2} \rangle \frac{\partial \mathcal{G}(\lambda)}{\partial \lambda} d\lambda. \quad (40)$$

By using this property and the definition Eq. (39), we can immediately find the following orthogonal relation between eigenvalue numbers m, n:

$$\left\langle \int_0^1 \Lambda_m \Lambda_n \frac{\partial (1 - \lambda B/B_M)^{1/2}}{\partial \lambda} d\lambda \right\rangle = 0 \quad \text{for}\, m \neq n. \quad (41)$$

(In non-symmetric toroidal configurations, this type of three dimensional definite integrals in a form of $\int_0^1 \langle \cdot \rangle d\lambda = \langle \int_0^1 \cdot d\lambda \rangle$ is often numerically calculated by surface-averaging the pitch-angle integrals, such as $\int_0^1 \Lambda_m \Lambda_n \{\partial (1 - \lambda B/B_M)^{1/2} / \partial \lambda\} d\lambda$ as function of B, especially when $\lambda B(\theta, \zeta)/B_M = 1$ is a singular point in the 3D space[25] (θ, ζ, λ).) Equation (41) will be used later for orthogonal expanded expressions of Eq. (33) and the source term.

It also should be noted that when our purpose is limited to the integrals in the form of $\langle B \int \xi F(v) f_f d^3\mathbf{v} \rangle$, a complete expression for the full energy range $0 \le v \le v_b$ is not required. Not only the usual Legendre expansion[19] for $\mathbf{b} \cdot \nabla B = 0$ but also the present orthogonal expansion by $\Lambda_n(\lambda)$ for $\mathbf{b} \cdot \nabla B \neq 0$ will require an infinite number of expansion terms ($0 \le l < \infty$ or $1 \le n < \infty$) at high energy regions $v \ge v_c$ or $v \approx v_b$ when the $S_{\mathbf{x}\lambda}(s, \sigma, \lambda)$ in the fast ion source term is a strongly localized function in the pitch-angle (λ) space. However, these higher order ($l > 1$ or $n \gg 1$) pitch-angle space structures will vanish in the integrals $\langle B \int \xi F(v) f_f d^3\mathbf{v} \rangle$. The focusing on these types of integrals rather than f_f itself is motivated especially by the application to the field particle portion $C_{af}(f_{aM}, f_f)$ in thermalized particles kinetic equation. This portion is an integral operator reducing the higher Legendre orders $l \gg 1$ especially when $T_i/m_a \ll (p_f/n_f)/m_f \ll T_e/m_e$ ($a \neq$ e,f).[2,6] Therefore, $1 \le n \le 6$ of Eq. (39) is used here. These eigenfunctions of finite n numbers can be obtained by a numerical shooting method, in which

$$\begin{cases} \dfrac{\partial \Lambda_n}{\partial \lambda} = \dfrac{\Pi_n}{\lambda} \\ \dfrac{\partial \Pi_n}{\partial \lambda} = \dfrac{\partial \ln \langle (1 - \lambda B/B_M)^{1/2} \rangle}{\partial \lambda} (\kappa_n \Lambda_n - \Pi_n) \end{cases} \quad (42)$$

is integrated using an initial condition of

$$\begin{cases} \Lambda_n(\lambda \to 0) = 1 - \dfrac{\lambda}{2} \dfrac{\langle B \rangle}{B_M} \kappa_n \left(1 + \dfrac{\lambda}{8} \dfrac{\langle B \rangle}{B_M} (1 - \kappa_n) \right) \\ \Pi_n(\lambda \to 0) = -\dfrac{\lambda}{2} \dfrac{\langle B \rangle}{B_M} \kappa_n \left(1 + \dfrac{\lambda}{4} \dfrac{\langle B \rangle}{B_M} (1 - \kappa_n) \right) \end{cases} \quad (43)$$

with a guess value of κ_n. When the solution satisfying $\Lambda_n(1) = 0$ is found, the exact eigenvalue κ_n is determined by

$$\kappa_n = -\int_0^1 \lambda \langle (1 - \lambda B/B_M)^{1/2} \rangle \left(\frac{\partial \Lambda_n}{\partial \lambda} \right)^2 d\lambda \Bigg/ \times \left\langle \int_0^1 \Lambda_n^2 \frac{\partial (1 - \lambda B/B_M)^{1/2}}{\partial \lambda} d\lambda \right\rangle, \quad (44)$$

which is obtained by multiplying $\Lambda_n(\lambda)$ to Eq. (39). A Legendre polynomial expansion

$$\Lambda_n(\lambda) = \frac{(1 - \lambda)^{1/2}}{\langle (1 - \lambda B/B_M)^{1/2} \rangle} \sum_{m=1}^{9} a_{nm} P_{2m-1}\left((1 - \lambda)^{1/2} \right), \quad (45)$$

in which the last term $m = 9$ is determined by the boundary condition $\Lambda_n(0) = 1$ as $a_{n9} = -\sum_{m=1}^{8} a_{nm} + 1$ is convenient for this kind of pitch-angle integrals, including the $\langle B \int \xi F(v) f_f d^3\mathbf{v} \rangle$ formula derived below.

To execute this procedure effectively for arbitrary non-symmetric stellarator/heliotron configurations, it is convenient to use results for axisymmetric tokamaks with concentric circular flux geometries[7] as the initial guess κ_n value in an iterative calculation of Eqs.(42) and (43) ($\Lambda_n(1) = 0$ is not guaranteed)

$$\kappa_n^{(\mathrm{new})} = \kappa_n^{(\mathrm{old})} \frac{\int_0^1 \lambda \langle (1 - \lambda B/B_M)^{1/2} \rangle (\partial \Lambda_n / \partial \lambda)^2 d\lambda - \langle (1 - B/B_M)^{1/2} \rangle [\Lambda_n \partial \Lambda_n / \partial \lambda]_{\lambda=1}}{\int_0^1 \lambda \langle (1 - \lambda B/B_M)^{1/2} \rangle (\partial \Lambda_n / \partial \lambda)^2 d\lambda}. \quad (46)$$

In this previous tokamak calculation, the flux-surface coordinates system with the Jacobian and the magnetic field strength of

$$\sqrt{g} = \frac{V'}{4\pi^2}(1 + \varepsilon \cos \theta), \quad B = \frac{B_0}{1 + \varepsilon \cos \theta} \quad (47)$$

was used. The obtained eigenvalues with $1 \le n \le 6$ were (The result in Ref. 7 is extended to include $n = 6$ and $\varepsilon = 0.65$.)

$$\begin{cases} \varepsilon = 0.01;\ \kappa_n = 1.16127, 6.84497, 17.0693, 31.8307, 51.1263, 75.581 \\ \varepsilon = 0.04;\ \kappa_n = 1.36030, 7.88083, 19.5941, 36.4914, 58.5683, 85.809 \\ \varepsilon = 0.09;\ \kappa_n = 1.61375, 9.19224, 22.7862, 42.3856, 67.9864, 99.559 \\ \varepsilon = 1/6;\ \kappa_n = 1.98220, 11.0918, 27.4123, 50.9352, 81.6583, 119.53 \\ \varepsilon = 1/3;\ \kappa_n = 2.93115, 15.9782, 39.3295, 72.9835, 116.940, 171.12 \\ \varepsilon = 0.50;\ \kappa_n = 4.40302, 23.5697, 57.8728, 107.318, 171.905, 251.52 \\ \varepsilon = 0.65;\ \kappa_n = 6.86278, 36.2951, 88.9926, 165.015, 264.164, 386.30. \end{cases} \tag{48}$$

It also is known that the extremely small ε limit is given by

$$\kappa_n \to n(2n-1) + \frac{4n-1}{3}\left(\frac{(2n-1)!!}{(2n-2)!!}\right)^2 1.47\sqrt{\varepsilon} \quad \text{for} \quad \varepsilon \to 0. \tag{49}$$

In these model configurations, a relation between the so-called circulating particles' fraction $f_c \equiv \frac{3}{4}\langle B^2\rangle B_M^{-2}\int_0^1 \lambda\langle(1-\lambda B/B_M)^{1/2}\rangle^{-1} d\lambda$, which routinely appears in the banana regime parallel viscosity in general toroidal configurations,[23,24] and the inverse aspect ratio ε is given by a polynomial fitting

$$\begin{cases} \sqrt{\varepsilon} = f\left(1 - \dfrac{3}{4}\displaystyle\int_0^1 \frac{\lambda d\lambda}{\langle(1-\lambda B/B_M)^{1/2}\rangle}\right) \\ f(x) \equiv x(0.685 + 0.735\times x - 0.133\times x^2 + 2.00\times x^3). \end{cases} \tag{50}$$

When investigating general toroidal configurations, we can know approximated values of κ_n by substituting their f_c integrals into Eq. (50) to convert them to ε for the $\kappa_n(\varepsilon)$ interpolation formulas of Eqs. (48) and (49) in the model tokamak configurations equation (47). After this choice of the initial guess values, only a few iterations of Eq. (46) will immediately find the exact $\Lambda_n(\lambda)$ and κ_n satisfying the boundary condition $\Lambda_n(1) = 0$ and, consequently, the orthogonal relation equation (41).

Although it is mathematically obvious in Eq. (39) that in an extremely small **B**-field strength modulation limit $\langle(1-B/B_M)^{1/2}\rangle \to 0$, the eigenfunction becomes the usual Legendre polynomial $\Lambda_n(\lambda) \to P_{2n-1}((1-\lambda)^{1/2})$, but the numerical scheme in Eqs.(42)–(46) for finite modulation is not suitable for this too simplified situation. If one wants to consider $\langle(1-B/B_M)^{1/2}\rangle \ll 1$ limits (for, e.g., $\varepsilon < 0.005$ in Eq. (50)), it is favorable to use an analytical theory for the $a_{nm}(\varepsilon)$ in $\varepsilon \ll 1$ limits of the model configurations, Eq. (47), for avoiding physically meaningless numerical errors. Reference 7 showed also this asymptotic limit theory, and the essence of it is summarized by Eq. (49) and

$$a_{nm}(\varepsilon) = \frac{4m-1}{3(n-m)(2n+2m-1)} \times \frac{(2m-1)!!(2n-1)!!}{(2m-2)!!(2n-2)!!}(-1)^{n+m} 1.47\sqrt{\varepsilon} \quad \text{for } m \neq n. \tag{51}$$

The non-diagonal coefficients $a_{nm}(\varepsilon)$ with $m \neq n$ should be obtained by interpolations of $a_{nm}(\varepsilon)/\sqrt{\varepsilon}$ as functions of $\sqrt{\varepsilon}$ in this $\varepsilon \ll 1$ limit and $\varepsilon \geq 0.01$, where the coefficients are obtained by the numerical Legendre expansion. The diagonal coefficient should be obtained by extrapolations of numerically obtained $(a_{nn}(\varepsilon)-1)/\sqrt{\varepsilon}$ as functions of $\sqrt{\varepsilon}$ in $0.01 \leq \varepsilon \leq 0.65$.

B. Energy space structure of each eigenvalue numbers

By using Eq. (41), we shall define an orthogonal expansion of arbitrary odd function $\mathcal{F}(v,\sigma,\lambda)$ in $0 \leq \lambda \leq 1$ satisfying $\mathcal{F}(v,-1,\lambda) = -\mathcal{F}(v,1,\lambda)$ and the boundary condition $\mathcal{F}(v,\sigma,\lambda=1) = 0$ as follows:

$$\mathcal{F}(v,\sigma,\lambda) = \sigma\sum_n \mathcal{F}_n(v)\Lambda_n(\lambda)$$
$$\mathcal{F}_n(v) \equiv \sigma\left\langle\int_0^1 \Lambda_n\mathcal{F}(v,\sigma,\lambda)\frac{\partial(1-\lambda B/B_M)^{1/2}}{\partial\lambda}d\lambda\right\rangle \Big/ \left\langle\int_0^1 \Lambda_n^2\frac{\partial(1-\lambda B/B_M)^{1/2}}{\partial\lambda}d\lambda\right\rangle. \tag{52}$$

This pitch-angle integral $\int_0^1 d\lambda$ is performed for only one side of $\sigma = \pm 1$. We shall define also a function for each flux-surfaces by

$$\ln\mathcal{V}(v) \equiv 3v_c^3\int\frac{dv}{v\left\{v^2 v_{Te}(3\sqrt{\pi}/2)G(x_e) + v_c^3\right\}} = \ln\frac{v^3}{v^3+v_{ce}^3} + \frac{4}{3\sqrt{\pi}}\frac{v_{ce}^3}{v_{Te}^3}\ln\left(1+\frac{v^3}{v_{ce}^3}\right)$$
$$v_{ce}^3 \equiv v_c^3\left(1+\frac{4}{3\sqrt{\pi}}\frac{v_c^3}{v_{Te}^3}\right)^{-1}. \tag{53}$$

Here,

$$v_c^3\left\{v^2 v_{Te}\frac{3\sqrt{\pi}}{2}G(x_e) + v_c^3\right\}^{-1} \cong \frac{v_{ce}^3}{v^3+v_{ce}^3}\left(1+\frac{4}{3\sqrt{\pi}}x_e^3\right) \tag{54}$$

given by $G(x) \cong \{(3\sqrt{\pi}/2)/x + 2x^2\}^{-1}$ is substituted to derive the explicit expression using analytical integrals.

Because of a relation $v_c^3/v_{Te}^3 \sim m_e/m_i \ll 1$, this function is $\mathcal{V}(v) \cong v^3/(v^3+v_c^3)$ as assumed in the previous studies[7,19,20,29] even for the low-T_e situations $v_b \sim v_{Te}$. By using the orthogonal expansion of the odd source term $[S_{x\lambda}(s,\sigma,\lambda) - S_{x\lambda}(s,-\sigma,\lambda)]/2$, Eq. (38) for $\bar{f}_f^{0\,(odd)}(v,\sigma,\lambda) = \sigma\sum_n f_n(v)\Lambda_n(\lambda)$ can be rewritten as

$$\frac{d}{dv}\left[f_n(v)\left(v^2 v_{Te}\frac{3\sqrt{\pi}}{2}G(x_e)+v_c^3\right)\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_b)}\right\}^{-\kappa_n Z_2/3}\right] = -\frac{\tau_S\left\langle\int_0^1 \sigma\Lambda_n[S_{x\lambda}(s,\sigma,\lambda)-S_{x\lambda}(s,-\sigma,\lambda)]\left\{\partial(1-\lambda B/B_M)^{1/2}/\partial\lambda\right\}d\lambda\right\rangle}{2\left\langle\int_0^1 \Lambda_n^2\left\{\partial(1-\lambda B/B_M)^{1/2}/\partial\lambda\right\}d\lambda\right\rangle}\delta(v-v_b),$$

and its solution is given by

$$f_n(v) = \frac{\tau_S\left\langle\int_0^1 \sigma\Lambda_n[S_{x\lambda}(s,\sigma,\lambda)-S_{x\lambda}(s,-\sigma,\lambda)]\left\{\partial(1-\lambda B/B_M)^{1/2}/\partial\lambda\right\}d\lambda\right\rangle}{2\left\langle\int_0^1 \Lambda_n^2\left\{\partial(1-\lambda B/B_M)^{1/2}/\partial\lambda\right\}d\lambda\right\rangle} \times\left(v^2 v_{Te}\frac{3\sqrt{\pi}}{2}G(x_e)+v_c^3\right)^{-1}\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_b)}\right\}^{\kappa_n Z_2/3} U(v_b-v). \qquad (55)$$

Although our purpose is not $\bar{f}_f^{0\,(odd)}(v,\sigma,\lambda)$ itself but calculating integrals in the form of $\langle B\int \xi F(v) f_f d^3\mathbf{v}\rangle$ as stated previously, it should be noted that this solution with $1 \le n < \infty$ includes the result in Ref. 19 as a limit of $\langle(1-B/B_M)^{1/2}\rangle \ll 1$ and $v_b/v_{Te} \ll 1$. Due to the use of Eq. (32) instead of Eq. (31), these results do not include $\exp(-m_f v^2/2T_i)$ at $m_f v^2/2 \sim T_i$. This energy space structure is the definition of $\bar{f}_f(\mathbf{x},v,\sigma,\lambda)$ that is stated at the beginning of Sec. III. The omission of the f-e collision in C_f^{PAS} in Eq. (32) is justified by the fact indicated by this resulting **v**-space structure that C_f^{PAS} is substantially effective only in a small velocity range $v \lesssim v_c$ where $n_e[\Phi(x_e) - G(x_e)] / \sum_{b\ne e,f} n_b Z_b^2 [\Phi(x_b) - G(x_b)] \lesssim 4(v_c/v_{Te})/(3\sqrt{\pi}Z_{eff})$. Since we retained the collisional momentum exchange rate of Eq. (C3) approximately, this small underestimation of the PAS rate is compensated by the use of C_f^{ES} for $\partial\bar{f}_f^{0\,(odd)}/\partial v > 0$ in $v \lesssim v_c$, especially when this solution is used for calculating the momentum transfer to target plasma species. The integrals are obtained as follows by truncating the expansion of the source term to include only $1 \le n \le 6$:

$$\left\langle B\int \xi F(v) f_f d^3\mathbf{v}\right\rangle = \pi\frac{\langle B^2\rangle}{B_M}\tau_S\int_0^{v_b}\frac{F(v)v^2}{v^2 v_{Te}(3\sqrt{\pi}/2)G(x_e)+v_c^3} \times\left[\sum_{n=1}^{6}\frac{\left\langle\int_0^1 \sigma\Lambda_n[S_{x\lambda}(\sigma)-S_{x\lambda}(-\sigma)]\left\{\partial(1-\lambda B/B_M)^{1/2}/\partial\lambda\right\}d\lambda\right\rangle\int_0^1\Lambda_n d\lambda}{\left\langle\int_0^1 \Lambda_n^2\left\{\partial(1-\lambda B/B_M)^{1/2}/\partial\lambda\right\}d\lambda\right\rangle}\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_b)}\right\}^{\kappa_n Z_2/3}\right]dv. \qquad (56)$$

In Section V, we apply this formula for $m_a\langle B\int v\xi C_{af}(f_{aM}, f_f)d^3\mathbf{v}\rangle = -m_f\langle B\int v\xi C_{fa}(f_f, f_{aM})d^3\mathbf{v}\rangle$ given by Eq. (C3) in cases of tangential NB injections into non-axisymmetric stellarator/heliotron configurations. It also should be noted that the definition of $\mathcal{V}(v)$ in Eq. (53) gives

$$\int_0^{v_b}\left[1+\frac{Z_2 v_c^3}{n+1}\left\{v^2 v_{Te}\frac{3\sqrt{\pi}}{2}G(x_e)+v_c^3\right\}^{-1}\right] \times\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_b)}\right\}^{Z_2/3} v^n dv = \frac{v_b^{n+1}}{n+1} \qquad (57)$$

for arbitrary integers $n \ge 0$ by integration by parts, which will appear there for the $\langle(1-B/B_M)^{1/2}\rangle \ll 1$ limit giving $\kappa_1 = 1$.

V. MOMENTUM INPUT CALCULATION FOR TANGENTIAL NB INJECTIONS

The MCNBI (Ref. 3) results for the experiments reported in Ref. 1 indicate that the initial pitch-angle of the fast ions is $\langle v_\perp^2\rangle/\langle v_\parallel^2\rangle \simeq 0.2$ at radial positions of $r/a \sim 0.5$. Since it means a localizing of $S_{x\lambda}(s,\sigma,\lambda)$ at $\lambda \ll 1$ in typical tangential NBI operations, we calculate the orthogonal expansion of the odd source term in Eq. (56) and the total momentum input of NB injectors $\langle B\int v\xi S_{\rm f}{\rm d}^3\mathbf{v}\rangle$ by using a delta function approximation

$$\begin{aligned}S_{\mathbf{x}\lambda}(s,\sigma=1,\lambda) &\propto 2\langle(1-\lambda_{\rm b}B/B_{\rm M})^{1/2}\rangle\delta(\lambda-\lambda_{\rm b})\\ &=\frac{B}{B_M}\frac{\langle(1-\lambda_{\rm b}B/B_{\rm M})^{1/2}\rangle}{(1-\lambda_{\rm b}B/B_{\rm M})^{1/2}}\delta\left[\xi-(1-\lambda_{\rm b}B/B_{\rm M})^{1/2}\right]\\ S_{\mathbf{x}\lambda}(s,\sigma=-1,\lambda)&=0\end{aligned} \tag{58}$$

with a fixed value $\lambda_{\rm b}=0.17$. (In other words, when the total momentum and energy inputs $m_{\rm f}\langle B\int v\xi S_{\rm f}{\rm d}^3\mathbf{v}\rangle/\langle B\rangle$ and $(m_{\rm f}/2)\langle\int v^2S_{\rm f}{\rm d}^3\mathbf{v}\rangle$ by NB injectors at $|v-v_{\rm b}|/v_{\rm b}\ll 1$ are given by these kinds of Monte Carlo codes for other experimental conditions, we determine this substantial ionization pitch-angle $\lambda_{\rm b}$ by the relation $\langle\int v^2S_{\rm f}{\rm d}^3\mathbf{v}\rangle/\langle B\int v\xi S_{\rm f}{\rm d}^3\mathbf{v}\rangle=-2v_{\rm b}(B_{\rm M}/\langle B^2\rangle)\partial\langle(1-\lambda_{\rm b}B/B_{\rm M})^{1/2}\rangle/\partial\lambda_{\rm b}$.)

As the assumption regarding the magnetic configurations for calculating Eq. (56), we need only $B(s,\theta,\zeta)/B_0$ given in the Boozer or the Hamada coordinates. The parameters χ',ψ',B_ζ, and B_θ in Eq. (12) are not required, even though the existence of Eq. (12) and its consequences, such as Eq. (13) are implicitly included in the derivation. However, it should be noted that Eq. (56) is applicable only for toroidal configurations with finite rotational transforms $\chi'/\psi'\neq 0$. Here, we use a stellarator/heliotron magnetic field model

$$\begin{aligned}B/B_0=1&-\varepsilon_{\rm t}(s)\cos\theta_{\rm B}\\ &+\varepsilon_{\rm t}(s)\{1-\sigma_{\rm D}(s)\cos\theta_{\rm B}\}\cos(L\theta_{\rm B}-N\zeta_{\rm B})\end{aligned} \tag{59}$$

with $0.01\leq\varepsilon_{\rm t}(s)\leq 0.2$. The poloidal and toroidal period numbers are chosen to be $L=1$ and $N=4$ corresponding to the Heliotron-J device.[1] It is known as "sigma-optimization,"[30] $\sigma_{\rm D}(s)=1$ is a good drift optimization for the ripple-trapped particles in $\kappa^2<1$. Inward shifted configurations in the Large Helical Device (LHD)[31] and high-γ (high-bumpy) configurations in the Heliotron-J (Ref. 32) are often used by aiming this optimization. In these non-axisymmetric devices, the integral $\int_0^1\lambda\langle(1-\lambda B/B_{\rm M})^{1/2}\rangle^{-1}{\rm d}\lambda$ in Eq. (50) governing the trapping effect does not correspond to the geometrical inverse aspect ratio r/R but to this kind of ripple structure. It also should be noted that a modulation amplitude $(B_{\rm M}-B_{\rm min})/(B_{\rm M}+B_{\rm min})=2\varepsilon_{\rm t}/(1+\varepsilon_{\rm t}\sigma_{\rm D})$ for Eq. (59) also is not a good measure for the trapping effect in drift-optimized stellarator/heliotron configurations. This modulation amplitude $(B_{\rm M}-B_{\rm min})/B_0$ is independent of the optimization parameter $\sigma_{\rm D}$, and the normalized amplitude $(B_{\rm M}-B_{\rm min})/(B_{\rm M}+B_{\rm min})$ is reduced by its positive values $\sigma_{\rm D}>0$. However, the increase in $\sigma_{\rm D}$ results in an increase in ε in Eq. (50) and the eigenvalues κ_n as shown in following numerical examples. We investigate the dependence on $\varepsilon_{\rm t}$, $\sigma_{\rm D}$ for (1) reductions of the ratio of the momentum input to the target plasmas due to the friction collision and the total momentum input of the NB injectors

$$1+\frac{\langle BF_{\|{\rm f}1}\rangle}{m_{\rm f}\langle B\int v\xi S_{\rm f}{\rm d}^3\mathbf{v}\rangle}\equiv 1+\frac{\left\langle B\int v\xi\sum_{a\neq{\rm f}}C_{{\rm f}a}(f_{\rm f},f_{a{\rm M}}){\rm d}^3\mathbf{v}\right\rangle}{\langle B\int v\xi S_{\rm f}{\rm d}^3\mathbf{v}\rangle}, \tag{60}$$

and (2) reductions of the momentum transfer to each target plasma particle species in multi-ion-species plasmas

$$1-\frac{\langle B\int v\xi C_{a{\rm f}}(f_{a{\rm M}},f_{\rm f}){\rm d}^3\mathbf{v}\rangle}{\langle B\int v\xi C_{a{\rm f}}(f_{a{\rm M}},f_{\rm f}^{\varepsilon=0}){\rm d}^3\mathbf{v}\rangle}=1-\frac{\langle B\int v\xi C_{{\rm f}a}(f_{\rm f},f_{a{\rm M}}){\rm d}^3\mathbf{v}\rangle}{\langle B\int v\xi C_{{\rm f}a}(f_{\rm f}^{\varepsilon=0},f_{a{\rm M}}){\rm d}^3\mathbf{v}\rangle}. \tag{61}$$

Here, $f_{\rm f}^{\varepsilon=0}$ is the fast ions' velocity distribution for $\langle(1-B/B_{\rm M})^{1/2}\rangle=0$ given by $\Lambda_n(\lambda)=P_{2n-1}((1-\lambda)^{1/2})$, $\kappa_n=n(2n-1)$, and $S_{\mathbf{x}\lambda}(s,\sigma=1,\lambda)=\delta[\xi-(1-\lambda_{\rm b})^{1/2}]$. Before explaining the numerical results for Eq. (60), its $2T_{\rm i}/m_{\rm i}\ll v_{\rm b}^2$ limit in Eq. (C3) giving $G(x_a)\cong(2x_a^2)^{-1}$ for $a\neq$ e,f and consequently

$$\begin{aligned}-\frac{\sum_{a\neq{\rm f}}m_a\left\langle B\int v\xi C_{a{\rm f}}(f_{a{\rm M}},f_{\rm f}){\rm d}^3\mathbf{v}\right\rangle}{m_{\rm f}\langle B\int v\xi S_{\rm f}{\rm d}^3\mathbf{v}\rangle}&=\frac{\left\langle B\int v\xi\sum_{a\neq{\rm f}}C_{{\rm f}a}(f_{\rm f},f_{a{\rm M}}){\rm d}^3\mathbf{v}\right\rangle}{\langle B\int v\xi S_{\rm f}{\rm d}^3\mathbf{v}\rangle}\\ &\cong\frac{1}{2v_{\rm b}}\left\langle\frac{B}{B_{\rm M}(1-\lambda_{\rm b}B/B_{\rm M})^{1/2}}\right\rangle\int_0^{v_{\rm b}}\frac{v^2v_{\rm Te}(3\sqrt{\pi}/2)G(x_e)+v_{\rm c}^3(1+Z_2)}{v^2v_{\rm Te}(3\sqrt{\pi}/2)G(x_e)+v_{\rm c}^3}\\ &\quad\times\left[\sum_{n=1}^{6}\frac{\Lambda_n(\lambda_{\rm b})\int_0^1\Lambda_n{\rm d}\lambda}{\langle\int_0^1\Lambda_n^2\{\partial(1-\lambda B/B_{\rm M})^{1/2}/\partial\lambda\}{\rm d}\lambda\rangle}\left\{\frac{\mathcal{V}(v)}{\mathcal{V}(v_{\rm b})}\right\}^{\kappa_nZ_2/3}\right]{\rm d}v\end{aligned}$$

should be considered. In the $\langle(1-B/B_{\rm M})^{1/2}\rangle\ll 1$ limit giving $\int_0^1\Lambda_1{\rm d}\lambda/\int_0^1\Lambda_1^2\{\partial(1-\lambda B/B_{\rm M})^{1/2}/\partial\lambda\}{\rm d}\lambda=-2$, this ratio is -1 because of Eq. (57). A physical meaning of the deviation Eq. (60) is the parallel viscosity force of fast ions themselves in $\langle\mathbf{B}\cdot\nabla\cdot\pi_{\rm f}\rangle=\langle\mathbf{B}\cdot\mathbf{F}_{{\rm f}1}\rangle+m_{\rm f}\langle\mathbf{B}\cdot\int\mathbf{v}S_{\rm f}{\rm d}^3\mathbf{v}\rangle$ as the surface-averaging of Eq. (4).

The other required assumptions for investigating the parallel momentum exchange by using Eq. (C3) are target plasma parameters n_a,T_a, and the beam injection energy. These are also chosen to be almost equivalent to those at the radial position $r/a=0.5$ in the experimental conditions in Ref. 1. It is reported that the charge exchange spectroscopic measurements were done for $e^- + D^+ + C^{6+}$ multi-ion-species plasmas with $n_e = 1.1 \times 10^{19}\mathrm{m}^{-3}$, $T_e = 230\mathrm{eV}$, $T_i = 110\mathrm{eV}$, and $Z_{\mathrm{eff}} = 1.9$ (at $r/a = 0.5$). A hydrogen beam with injection energy of $m_f v_b^2/2 = 27\mathrm{keV}$ sustained these plasmas. For simplicity in this paper, here we neglect low energy components of 13.5 keV and 9 keV that are produced in the positive ion source injector. The critical velocity and the PAS parameters in Eq. (32) for this condition are $v_c = 680\mathrm{km/s}$ and $Z_2 = 3.69$, respectively. The mean free path of the PAS collision at this critical velocity determined by the slowing down time $\tau_S = 10.2\mathrm{ms}$ is $v_c\tau_S/Z_2 = 1.88\mathrm{km}$, and it corresponds to the banana regime $\nu_D^a/v < 10^{-1}\mathrm{m}^{-1}$ of the viscosity coefficient M^* (parallel viscosity force against parallel flows defined in Ref. 8) in the Heliotron-J configuration.[32] The procedure for $\bar{f}_f^{0(\mathrm{odd})}$ in Sec. IV is applicable for these long mean free path conditions.

Figure 1 shows ε obtained by Eq. (50) for the stellarator model equation (59). The reduction of the total friction equation (60) and that of the momentum transfer equation (61) for electron in these configurations are shown in Figs. 2 and 3, respectively. In the typical injection conditions with $v_b > v_c$ and $\lambda_b \approx 0.1$, the high-energy region $v > v_c$ of $\bar{f}_f(\mathbf{x}, v, \sigma, \lambda)$ is localizing at $\lambda < 1$ of $\sigma = 1$. It is determined mainly by the slowing down collision C_f^{ES}, and is almost irrelative to the fact that $\bar{f}_f^{0(\mathrm{odd})}$ in Eq. (33) vanishes in the trapped pitch-angle range and the κ_n expressing the surface-averaged PAS collision rate is increased by the **B**-field strength modulation $\langle(1 - B/B_M)^{1/2}\rangle \neq 0$. Its integrals $\langle B \int_{v_c}^{v_b} F(v) (\int_{-1}^{1} \xi f_f d\xi) v^2 dv\rangle$ are insensitive to these configuration effects. The total momentum loss of the fast ions (total momentum input to the target plasma species) $-\langle BF_{\|f1}\rangle$ and the momentum exchange between fast ions and electrons $-m_f\langle B \int v\xi C_{fe}(f_f, f_{eM}) d^3\mathbf{v}\rangle = m_e\langle B \int v\xi C_{ef}(f_{eM}, f_f) d^3\mathbf{v}\rangle$ are determined by the full energy range $0 \le v \le v_b$ of $\bar{f}_f^{0(\mathrm{odd})}$. We can see in Figs. 2 and 3 that these friction moments are insensitive to the **B**-field strength modulation. The total momentum loss is reduced only by a factor of $1 - \sqrt{\varepsilon}/2$. This reduction is smaller than that of momentum exchange between fast and thermal ions discussed below. The e-f, f-e momentum exchange is more insensitive. In spite of the modulations of $\sqrt{\varepsilon} < 0.7$ in Fig. 1, the reduction of the momentum exchange is only a few percent or 10%. This characteristic of $\bar{f}_f(\mathbf{x}, v, \sigma, \lambda)$ in $\lambda < 1$ of $v > v_c$ is essentially different from the reduction of the neoclassical parallel conductivity in the banana regime that occurs for nearly isotropic velocity distributions of thermal particles. This result on the e-f, f-e momentum exchange by Eq. (C3) for an initial velocity condition of $v_c < v_b < (3\sqrt{\pi}/4)^{1/3} v_{Te}$ means that beam particle flux $\langle Bn_f u_{\|f}\rangle \equiv \langle B \int v\xi \bar{f}_f d^3\mathbf{v}\rangle$, which was discussed in Ref. 20 and references cited therein, also is insensitive to the **B**-field strength modulation in cases with sufficiently large injection velocities $v_b > v_c$ and sources localizing at $\lambda < 1$. The beam driven parallel particle and heat fluxes of electrons $\langle Bn_e u_{\|e}^{\mathrm{beam}}\rangle$, $\langle Bq_{\|e}^{\mathrm{beam}}\rangle$ will be easily estimated by a 13 M approximation (Eq. (C4) in Ref. 8) neglecting the beam driven

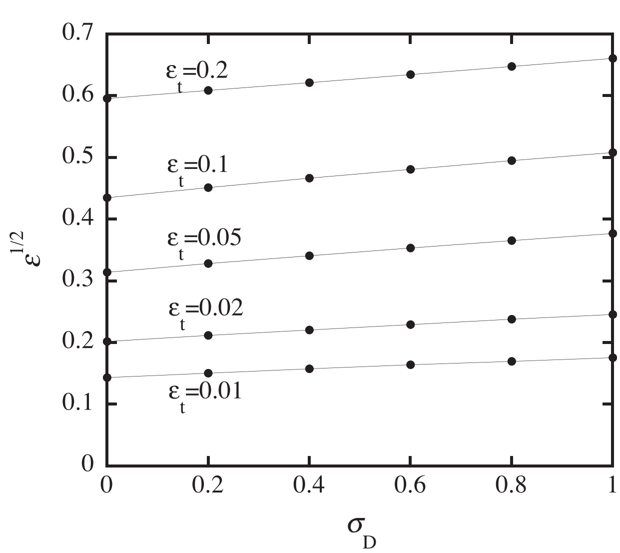

FIG. 1. The **B**-field strength modulation amplitude $\varepsilon^{1/2}$ that is determined by Eq. (50) for the model field equation (59).

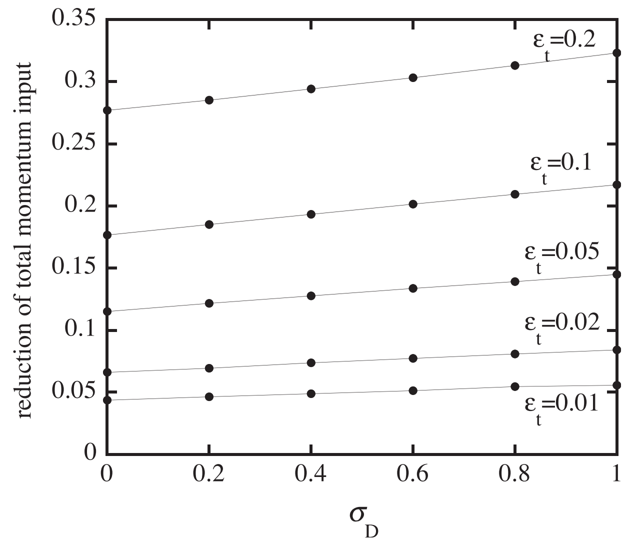

FIG. 2. The reduction of the total friction equation (60) due to the finite **B**-field strength modulation $\langle(1 - B/B_M)^{1/2}\rangle \neq 0$.

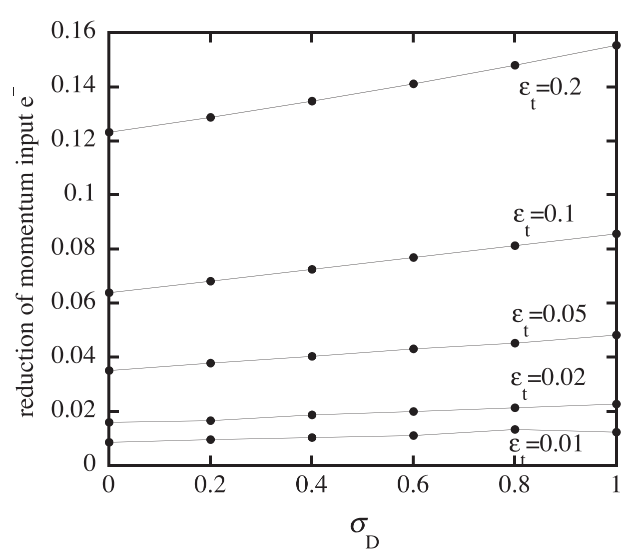

FIG. 3. The reduction of the momentum input equation (61) for electrons due to $\langle(1 - B/B_M)^{1/2}\rangle \neq 0$.

particle fluxes of thermal ions $\langle Bn_a u_{\parallel a}^{\text{beam}} \rangle \ll \langle Bn_e u_{\parallel e}^{\text{beam}} \rangle$ $(a \neq \text{e, f})$, which are an order of[33] $(m_e/m_a)^{1/2}$, and using a relation[2] $\int v\xi L_1^{(3/2)}(x_e^2) C_{\text{ef}}(f_{\text{eM}}, f_f) d^3\mathbf{v} \cong \frac{3}{2} \int v\xi C_{\text{ef}}(f_{\text{eM}}, f_f) d^3\mathbf{v}$.

The configuration effect is more important for the momentum input to thermal ions. The momentum exchange between the fast and thermal ions $m_a \langle B \int v\xi C_{af}(f_{aM}, f_f) d^3\mathbf{v} \rangle = -m_f \langle B \int v\xi C_{fa}(f_f, f_{aM}) d^3\mathbf{v} \rangle$ $(a \neq \text{e,f})$, which is required for calculating the ion flows, is determined only by the fast ions in a low-energy range of $v \lesssim v_c$ following Eqs. (B1) and (C3). In this energy range, the velocity distribution $\bar{f}_f(\mathbf{x}, v, \sigma, \lambda)$ is broadened for the full pitch-angle range $-1 \leq \xi \leq 1$ due to the PAS collision. In this energy range, $\langle B \int_0^{v_c} F(v) (\int_{-1}^{1} \xi f_f d\xi) v^2 dv \rangle$ is reduced by a factor of about $1 - \sqrt{\varepsilon}$ for $\varepsilon \sim 0.2$ as shown in Fig. 4. It is analogous to the reduction of the banana regime neoclassical parallel conductivity. In the neoclassical calculation in Ref. 1, as shown in its Figs. 4(c) and 4(d), the used momentum input was that given by the FIT3D code[3] without taking this configuration effect into account. For approximating the momentum exchange between the fast ions and the target plasma ions, a phenomenological reducing factor $1 - \sqrt{\varepsilon}$ was multiplied to the fast ions friction moment $\langle BF_{\parallel f1} \rangle$ there. However, this method in Ref. 1 is not a systematic method that is applicable to general multi-ion-species plasmas in general toroidal configurations. In particular, $\varepsilon = (B_M - B_{\min})/(B_M + B_{\min})$ used there is not a good measure as the substantial modulation amplitude for Eq. (39). For the drift-optimized stellarator/heliotron magnetic configurations that are modeled by Eq. (59), the qualitative coincidence of Figs. 1 and 4 is obtained when using Eq. (50) as the substantial amplitude.

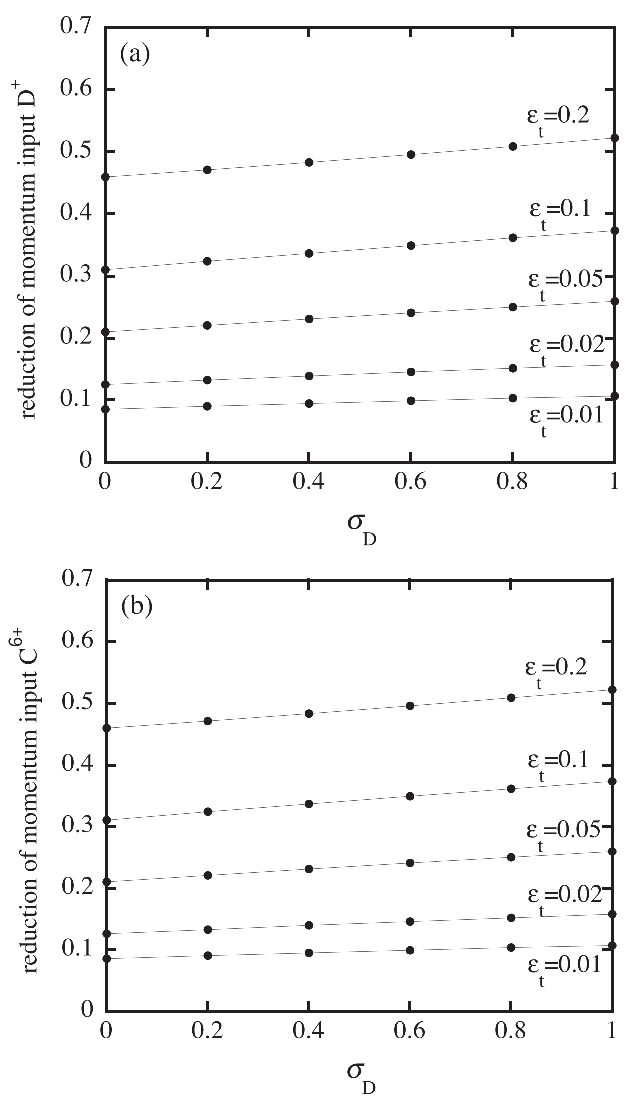

FIG. 4. The reduction of the momentum input equation (61) for the ions due to $\langle (1 - B/B_M)^{1/2} \rangle \neq 0$. (a) D^+ and (b) C^{6+}.

VI. SUMMARY

As pointed out in many experimental[12,13] and theoretical[34] studies on NBI heated plasmas, situations of recent experiments with the external anisotropic heating are different from those assumed in the conventional MHD equilibrium theories[16] and the concept of the flux-surface coordinates[11] based on them using the isotropic pressure. In spite of this fact, previously established methodology for the neoclassical transport is still applicable as long as the contravariant and the covariant expressions of the **B**-field in Eq. (12) exist and their parameters $\chi', \psi', B_\zeta, B_\theta$, and the field strength $B(s, \theta, \zeta)$ are appropriately given. This kind of expression of **J**-vector fields and theorems based on the expression, such as Eq. (15), are not used there.[8,9] It is suggested for practical purposes that isotropic pressure equilibriums reproducing experimentally observed Shafranov shifts, in which the usual scalar pressure moment $\sum_a p_a$ is replaced by $\sum_a (p_{\parallel a} + p_{\perp a})/2$, would give the parameters and the field strength in Eq. (12).[13] Therefore, recent NBI heating experiments in Heliotron-J[1] were analyzed by using thermal particles' DKEs $(a \neq \text{f})$ with an extension to include a collision term $C_{af}(f_a, f_f) \cong C_{af}(f_{aM}, f_f)$, which gives friction (momentum exchange) collision between the species a and the fast ions' slowing down velocity distribution function $f_f(\mathbf{x}, \mathbf{v})$.[2] Following a standard procedure in the moment method shown in Refs. 8–10, this kinetic problem was converted to simultaneous algebraic equations by taking $\langle B \int v\xi L_j^{(3/2)}(x_a^2) d^3\mathbf{v} \rangle$ integrals of the DKEs. Since the non-diagonal coupling terms between the thermal species $C_{ab}(f_{aM}, f_b^{(l=1)})$ are fully included by the Braginskii's matrix elements $\langle n_a/\tau_{ab} \rangle N_{ab}^{jk} \equiv \int v\xi L_j^{(3/2)}(x_a^2) C_{ab}[f_{aM}, (m_b/\langle T_b \rangle) v\xi L_k^{(3/2)}(x_b^2) f_{bM}] d^3\mathbf{v}$, the relation $\sum_a \langle \mathbf{B} \cdot \nabla \cdot \boldsymbol{\pi}_a \rangle = m_f \langle B \int v_\parallel S_f d^3\mathbf{v} \rangle$ due to the momentum conservation $\sum_a \mathbf{F}_{a1} = 0$ is satisfied. However, it should be noted that the charge conservation $\nabla \cdot \mathbf{J} = 0$ in this situation is retained due to a break of the symmetry of the **B**-field strength[8] $c_1 \partial B/\partial\theta + c_2 \partial B/\partial\zeta \neq 0$. These aspects of stellarator/heliotron plasmas with external momentum sources were investigated in Sec. II and Appendix A. Only one inappropriate shortcut in Ref. 1 was a phenomenological reducing factor $1 - \sqrt{\varepsilon}$ for the fast ions' friction $\langle BF_{\parallel f1} \rangle$ based on an analogy of the banana regime neoclassical parallel conductivity. The **v**-space structure of $\bar{f}_f(\mathbf{x}, v, \sigma, \lambda)$ in situations of

$\langle(1-B/B_{\rm M})^{1/2}\rangle \neq 0$ is not correctly taken into account there. However, our purpose is not the fast ions' velocity distribution itself but the parallel friction moments $\langle B\int v\xi L_j^{(3/2)}(x_a^2)C_{a\rm f}(f_{a\rm M},f_{\rm f})d^3\mathbf{v}\rangle$ required in studies on physics of target plasma species. When handling the fast ions' velocity distribution function, this RMJ operator can be calculated by a spherical coordinate expression method for the general Rosenbluth potentials shown in Ref. 35, and then integrations by parts for the energy space give the $\int v\xi L_j^{(3/2)}(x_a^2)C_{a\rm f}(f_{a\rm M},f_{\rm f})d^3\mathbf{v}$ integrals in a common form $\int \xi F(v)f_{\rm f}d^3\mathbf{v}$, such as Eq. (C3). After explaining the drift kinetic equation for this purpose in Sec. III, we applied an idea of eigenfunctions in Ref. 7 for excluding the trapped fast ions from the friction moments in Secs. IV and V. The dependence of these types of integrals having the common form $\langle B\int \xi F(v)f_{\rm f}d^3\mathbf{v}\rangle$ on the ripple amplitude and the drift optimization parameter in Eq. (59), which reduce the fraction of the circulating particles $f_{\rm c}\equiv\frac{3}{4}\langle B^2\rangle B_{\rm M}^{-2}\int_0^1\lambda\langle(1-\lambda B/B_{\rm M})^{1/2}\rangle^{-1}d\lambda$ in Eq. (50), was investigated. It is found that the momentum input to target ions is strongly affected by this configuration effect. As a characteristic of Eqs. (B1) and (C3), these sensitive friction moments are determined only by the lower energy range $v\lesssim v_{\rm c}$ of the fast ion distribution that is strongly affected by the PAS collision. Analogous to the banana regime neoclassical conductivity of thermal particles, this configuration effect is roughly expressed by a reducing factor $1-\sqrt{\varepsilon}$ for $\varepsilon\sim 0.2$ that is the typical ripple amplitude in the experiments reported in Ref. 1. Although this reducing factor was already included in the theoretical calculation in Ref. 1 and the results well explained the experimentally measured ion flow velocity, the method used there was inappropriate in two viewpoints. One is the use of $\varepsilon=(B_{\rm M}-B_{\rm min})/(B_{\rm M}+B_{\rm min})$, which is not a good measure for the **B**-field strength modulation determining the eigenfunction and the eigenvalues in general toroidal configurations, including drift optimized stellarator/heliotron devices. Another is multiplying the factor $1-\sqrt{\varepsilon}$ to the full component of $-\langle BF_{\parallel \rm f1}\rangle=\sum_{a\neq \rm f} m_a\langle B\int v\xi C_{a\rm f}(f_{a\rm M},f_{\rm f})d^3\mathbf{v}\rangle$. Since the momentum exchange between the electrons and the fast ions $-m_{\rm f}\langle B\int v\xi C_{\rm fe}(f_{\rm f},f_{\rm eM})d^3\mathbf{v}\rangle = m_{\rm e}\langle B\int v\xi C_{\rm ef}(f_{\rm eM},f_{\rm f})d^3\mathbf{v}\rangle$ is insensitive to the configuration effects in typical tangential NBI operations, the method in Ref. 1 was not appropriate for calculating the beam driven electron flows corresponding to the so-called shielding current in the Ohkawa current. This insensitivity is also due to a characteristic of Eq. (C3) for the fast ion velocity range $0\le v\le v_{\rm b}$. In future studies on plasma flows and/or current requiring flow calculations of all particle species in more general non-symmetric toroidal configurations, the eigenfunctions investigated in the present work will be useful.

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APPENDIX A: RADIAL TRANSPORT FLUXES OF GENERAL PARTICLE SPECIES

Also in this Appendix for radial particle and energy transport fluxes $\langle\Gamma_a\cdot\nabla s\rangle\equiv\langle n_a\mathbf{u}_a\cdot\nabla s\rangle$, $\langle\mathbf{Q}_a\cdot\nabla s\rangle$ of individual particle species, the symmetric CGL form $\boldsymbol{\pi}_a=(p_{\parallel a}-p_{\perp a})(\mathbf{bb}-\mathbf{I}/3)$ of the viscous tensor in Eq. (4) is assumed. The inertia force is neglected since $m_a n_a u_{\perp a}^2, m_a n_a u_{\perp a}u_{\parallel a}\ll p_a$. In addition to them, $\mathbf{r}_a-r_a\mathbf{I}=(r_{\parallel a}-r_{\perp a})(\mathbf{bb}-\mathbf{I}/3)$ for $\mathbf{r}_a\equiv\frac{m_a}{2}\int v^2\mathbf{vv}f_a d^3\mathbf{v}$, $r_a\equiv{\rm Tr}(\mathbf{r}_a)/3$, $r_{\perp a}\equiv\frac{m_a}{2}\int v^2\frac{v_\perp^2}{2}f_a d^3\mathbf{v}$, and $r_{\parallel a}\equiv\frac{m_a}{2}\int v^2 v_\parallel^2 f_a d^3\mathbf{v}$ also is assumed because of this small perpendicular Mach number. The curvature force in a direction of $\nabla s\times\mathbf{B}$ is calculated by an approximation of

$$\nabla s\times\mathbf{B}\cdot(\mathbf{b}\cdot\nabla\mathbf{b})=\nabla s\times\mathbf{B}\cdot\nabla\ln B+\frac{4\pi}{c}\mathbf{J}\cdot\nabla s \cong \nabla s\times\mathbf{B}\cdot\nabla\ln B\;(\because\mathbf{B}\cdot\nabla s=0)$$

even when $\mathbf{b}\cdot\nabla\sum_a\frac{p_{\parallel a}-p_{\perp a}}{B^2}\neq 0$ and consequently $\mathbf{J}\cdot\nabla s\neq 0$ as discussed in Sec. II. This is a part of the $8\pi\sum_a p_{\perp a}/B^2\ll 1$ approximation in Eq. (6). Therefore,

$$\nabla s\times\mathbf{B}\cdot\nabla\boldsymbol{\pi}_a=\frac{B^3}{3}\nabla s\times\mathbf{B}\cdot\nabla\frac{p_{\perp a}-p_{\parallel a}}{B^3},\quad \mathbf{b}\cdot\nabla\cdot\boldsymbol{\pi}_a=\frac{2}{3}B^{3/2}\mathbf{b}\cdot\nabla\frac{p_{\parallel a}-p_{\perp a}}{B^{3/2}} \tag{A1}$$

and

$$\nabla s\times\mathbf{B}\cdot\nabla\cdot(\mathbf{r}_a-r_a\mathbf{I})=\frac{B^3}{3}\nabla s\times\mathbf{B}\cdot\nabla\frac{r_{\perp a}-r_{\parallel a}}{B^3},\quad \mathbf{b}\cdot\nabla\cdot(\mathbf{r}_a-r_a\mathbf{I})=\frac{2}{3}B^{3/2}\mathbf{b}\cdot\nabla\frac{r_{\parallel a}-r_{\perp a}}{B^{3/2}} \tag{A2}$$

are used. By using Eqs. (4), (A1), and (14),

$$\langle\Gamma_a\cdot\nabla s\rangle\equiv\langle n_a\mathbf{u}_{\perp a}\cdot\nabla s\rangle=\frac{c}{e_a}\left\langle\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\nabla p_a\right\rangle -\frac{1}{3}\frac{c}{e_a}\left\langle(p_{\perp a}-p_{\parallel a})\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\nabla\ln B\right\rangle -c\left\langle n_a\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\mathbf{E}\right\rangle+\frac{c}{e_a}\left\langle\frac{\mathbf{F}_{a1}\times\mathbf{B}}{B^2}\cdot\nabla s\right\rangle$$

is obtained. This first term can be rewritten by using Eq. (23) and the parallel ($\mathbf{b}\cdot$) component of Eq. (4) as in the following:

$$\left\langle\frac{\nabla s\times\mathbf{B}}{B^2}\cdot\nabla p_a\right\rangle=-\langle\tilde{U}\mathbf{b}\cdot\nabla p_a\rangle =-\left\langle(p_{\parallel a}-p_{\perp a})\left(\frac{4}{3}\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\nabla\ln B\right\rangle -\langle\tilde{U}F_{\parallel a1}\rangle-e_a\langle n_a\tilde{U}\mathbf{b}\cdot\mathbf{E}\rangle.$$

Here, $\langle \tilde{U}\mathbf{b}\cdot\nabla(p_{\|a}-p_{\perp a})\rangle$ was rewritten again by using Eq. (23). The parallel momentum input term $\int \mathbf{v}_{\|}S_{\rm f}d^3\mathbf{v}$ is omitted following the conclusions in Secs. II and III expressed in Eq. (36). Substituting it into $\langle\Gamma_a\cdot\nabla s\rangle$ results in

$$\langle\Gamma_a\cdot\nabla s\rangle = \langle\Gamma_a^{\rm bn}\cdot\nabla s\rangle + \langle\Gamma_a^{\rm PS}\cdot\nabla s\rangle + \langle\Gamma_a^{\rm cl}\cdot\nabla s\rangle - c\left\langle n_a\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\mathbf{E}\right\rangle \quad \text{(A3)}$$

with

$$\langle\Gamma_a^{\rm bn}\cdot\nabla s\rangle \equiv -\frac{c}{e_a}\left\langle (p_{\|a}-p_{\perp a})\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\nabla\ln B\right\rangle$$

$$\langle\Gamma_a^{\rm PS}\cdot\nabla s\rangle \equiv -\frac{c}{e_a}\langle\tilde{U}F_{\|a1}\rangle$$

$$\Gamma_a^{\rm cl} \equiv \frac{c}{e_a}\frac{\mathbf{F}_{a1}\times\mathbf{B}}{B^2}.$$

The first term $\langle\Gamma_a^{\rm bn}\cdot\nabla s\rangle$ is the particle flux due to the neoclassical viscosity $\boldsymbol{\pi}_a$ investigated in Refs. 8–10, and the second and third terms $\langle\Gamma_a^{\rm PS}\cdot\nabla s\rangle$, $\Gamma_a^{\rm cl}$ are the Pfirsch-Schlüter and the classical particle fluxes, respectively. The electric field in the fourth term should be separated into electrostatic and inductive fields as

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \equiv -\nabla\Phi + \mathbf{E}^{(\mathbf{A})}. \quad \text{(A4)}$$

As already noted also on Eq. (30), $\mathbf{E}^{(\mathbf{A})}\approx 0$ in present stellarator/heliotron experiments, and the only purpose for retaining it is to confirm the Onsager symmetry between the bootstrap current and the Ware pinch.[10] Therefore, a least function $n_a\mathbf{E}^{(\mathbf{A})} = \langle n_a\rangle\langle\mathbf{B}\cdot\mathbf{E}^{(\mathbf{A})}\rangle\mathbf{B}/\langle B^2\rangle$ as the divergence-free vector field is assumed, and it vanishes by $\nabla s\times\mathbf{B}\cdot\mathbf{B}=0$ and $\langle\tilde{U}B\rangle = 0$ in Eq. (21). In the contribution of the electrostatic field (so-called electrostatic trapping effect[6]) $\langle n_a(\nabla s\times\mathbf{B}/B^2+\tilde{U}\mathbf{b})\cdot\nabla\Phi\rangle = -\langle\Phi(\nabla s\times\mathbf{B}/B^2+\tilde{U}\mathbf{b})\cdot\nabla n_a\rangle$, only poloidal and toroidal variations of the density $\delta n_a\equiv n_a-\langle n_a\rangle$ can remain following Eq. (24). Although the variation of the potential $\nabla s\times\mathbf{B}\cdot\nabla\Phi$, $\mathbf{B}\cdot\nabla\Phi$ may need to be taken into account in calculating an extreme collisional limit of the Pfirsch-Schlüter diffusions,[36] we shall neglect it in the present study.

Analogously, by using a $\int \mathbf{v}v^2d^3\mathbf{v}$ moment of the steady-state Landau equation[16]

$$\nabla\cdot\mathbf{r}_a - \frac{e_a}{m_a}\left[\mathbf{E}\cdot\left(\frac{5}{2}p_a\mathbf{I}+\boldsymbol{\pi}_a+m_an_a\mathcal{O}(u_a^2)\right)+\frac{\mathbf{Q}_a\times\mathbf{B}}{c}\right] = \mathbf{G}_a+\frac{m_a}{2}\int\mathbf{v}_{\|}v^2S_a(\mathbf{x},v,\xi)d^3\mathbf{v} \quad \text{(A5)}$$

with $\mathbf{Q}_a\equiv(m_a/2)\int\mathbf{v}v^2f_ad^3\mathbf{v}$, $\mathbf{G}_a\equiv(m_a/2)\int\mathbf{v}v^2\sum_b C_{ab}(f_a,f_b)d^3\mathbf{v}$, and aforementioned tensor $\mathbf{r}_a$ in a combination with Eqs. (14), (23), (A2), the radial energy transport flux is obtained as follows:

$$\langle\mathbf{Q}_a\cdot\nabla s\rangle = \langle\mathbf{Q}_a^{\rm bn}\cdot\nabla s\rangle + \langle\mathbf{Q}_a^{\rm PS}\cdot\nabla s\rangle + \langle\mathbf{Q}_a^{\rm cl}\cdot\nabla s\rangle - \frac{5}{2}c\left\langle p_a\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\mathbf{E}\right\rangle - \frac{c}{3}\left\langle(p_{\perp a}-p_{\|a})\left(\frac{\nabla s\times\mathbf{B}}{B^2}-2\tilde{U}\mathbf{b}\right)\cdot\mathbf{E}\right\rangle. \quad \text{(A6)}$$

Here, the first three terms are defined by

$$\langle\mathbf{Q}_a^{\rm bn}\cdot\nabla s\rangle \equiv -\frac{m_ac}{e_a}\left\langle(r_{\|a}-r_{\perp a})\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\nabla\ln B\right\rangle$$

$$\langle\mathbf{Q}_a^{\rm PS}\cdot\nabla s\rangle \equiv -\frac{m_ac}{e_a}\langle\tilde{U}G_{\|a}\rangle$$

$$\mathbf{Q}_a^{\rm cl} \equiv \frac{m_ac}{e_a}\frac{\mathbf{G}_a\times\mathbf{B}}{B^2}.$$

These terms are the viscosity-driven neoclassical flux, the Pfirsch-Schlüter flux, and the classical flux, respectively. The contribution of the fast ion source term $\int v_{\|}v^2S_{\rm f}d^3\mathbf{v}$ vanishes following Eq. (36). When the inductive field is $\{\frac{5}{2}p_a+\frac{2}{3}(p_{\|a}-p_{\perp a})\}\mathbf{E}_{\|}^{(\mathbf{A})} = \langle\frac{5}{2}p_a+\frac{2}{3}(p_{\|a}-p_{\perp a})\rangle\langle\mathbf{B}\cdot\mathbf{E}^{(\mathbf{A})}\rangle\mathbf{B}/\langle B^2\rangle$ and $\mathbf{E}_{\perp}^{(\mathbf{A})}=0$ (a least function as the divergence-free vector field for the confirmation of the Onsager symmetry), the electric field driven terms are $\langle p_a(\nabla s\times\mathbf{B}/B^2+\tilde{U}\mathbf{b})\cdot\mathbf{E}\rangle = -\langle p_a(\nabla s\times\mathbf{B}/B^2+\tilde{U}\mathbf{b})\cdot\nabla\Phi\rangle = \langle\Phi(\nabla s\times\mathbf{B}/B^2+\tilde{U}\mathbf{b})\cdot\nabla p_a\rangle$, in which only $\delta p_a\equiv p_a-\langle p_a\rangle$ can remain, and $\langle(p_{\perp a}-p_{\|a})(\nabla s\times\mathbf{B}/B^2-2\tilde{U}\mathbf{b})\cdot\mathbf{E}\rangle = -\langle(p_{\perp a}-p_{\|a})(\nabla s\times\mathbf{B}/B^2-2\tilde{U}\mathbf{b})\cdot\nabla\Phi\rangle$. These effects of $\nabla s\times\mathbf{B}\cdot\nabla\Phi$, $\mathbf{B}\cdot\nabla\Phi$ also are neglected in recent our studies. For the thermalized particles, the radial heat flux can be defined by $\langle\mathbf{q}_a\cdot\nabla s\rangle\equiv\langle\mathbf{Q}_a\cdot\nabla s\rangle-\frac{5}{2}\langle T_a\rangle\langle n_a\mathbf{u}_a\cdot\nabla s\rangle$ and is expressed by using $\theta_a\equiv m_a(\mathbf{r}_a-r_a\mathbf{I})/\langle T_a\rangle-\frac{5}{2}\boldsymbol{\pi}_a$ for $\langle\mathbf{q}_a^{\rm bn}\cdot\nabla s\rangle$, $F_{\|a2}\equiv m_aG_{\|a}/\langle T_a\rangle-\frac{5}{2}F_{\|a1}$ for $\langle\mathbf{q}_a^{\rm PS}\cdot\nabla s\rangle$, and $\mathbf{F}_{\perp a2}\equiv m_a\mathbf{G}_{\perp a}/T_a-\frac{5}{2}\mathbf{F}_{\perp a1}$ for $\langle\mathbf{q}_a^{\rm cl}\cdot\nabla s\rangle$. Although this $\langle\mathbf{q}_a\cdot\nabla s\rangle$ is often used for the Onsager symmetric transport matrix,[8–10] it cannot be considered for the fast ions since their velocity distribution does not include the exponential factor $\exp(-m_av^2/2T_a)$ (does not have the concept of the temperature), and therefore the self-adjoint property of the collision as a basis of the Onsager symmetry does not exist there.

As noted in Sec. II, it is important in the viewpoint of a consistency of the $\mathbf{B}$, $\mathbf{J}$ vector fields to investigate how the ambipolar condition $\langle\mathbf{J}\cdot\nabla s\rangle=0$ is satisfied. This issue is irrelative to how we treat the electric field term $(\nabla s\times\mathbf{B}/B^2+\tilde{U}\mathbf{b})\cdot\mathbf{E}$. By summing Eq. (A3) for all particle species with using the charge neutrality $\sum_a e_an_a=0$ and the momentum conservation $\sum_a\mathbf{F}_{a1}=0$,

$$\langle\mathbf{J}\cdot\nabla s\rangle \equiv \sum_a e_a\langle\Gamma_a\cdot\nabla s\rangle = -c\left\langle\sum_a(p_{\|a}-p_{\perp a})\left(\frac{\nabla s\times\mathbf{B}}{B^2}+\tilde{U}\mathbf{b}\right)\cdot\nabla\ln B\right\rangle, \quad \text{(A7)}$$

which is equivalent to Eq. (16) as noted on Eq. (23), is obtained. The disappeared flux component $\langle\Gamma_a^{\rm PS}\cdot\nabla s\rangle+\langle\Gamma_a^{\rm cl}\cdot\nabla s\rangle-c\langle n_a(\nabla s\times\mathbf{B}/B^2+\tilde{U}\mathbf{b})\cdot\mathbf{E}\rangle$ in Eq. (A3) is called "intrinsically ambipolar" flux. There is an important difference between symmetric configurations, where $c_1\partial B/\partial\theta + c_2\partial B/\partial\zeta = 0$ holds,[8] and non-symmetric configurations, where $c_1\partial B/\partial\theta + c_2\partial B/\partial\zeta \neq 0$. Hereafter, $B_\zeta = B_\zeta^{(\mathrm{Boozer})}$ and $B_\theta = B_\theta^{(\mathrm{Boozer})}$ are used. In the symmetric configurations, $\frac{B_\zeta m + B_\theta n}{\chi' m - \psi' n} = -\frac{c_1 B_\theta + c_2 B_\zeta}{c_1\psi' - c_2\chi'} = \mathrm{const}$ for all Fourier modes (m, n) in Fourier expansions by $\sin(m\theta - n\zeta)$, $\cos(m\theta - n\zeta)$ of arbitrary functions. Therefore, a relation between $(\nabla s \times \mathbf{B}/B^2 + \tilde{U}\mathbf{b})\cdot\nabla\ln B$ and $\mathbf{B}\cdot\nabla\ln B$ is

$$\begin{aligned}\left(\frac{\nabla s\times\mathbf{B}}{B^2} + \tilde{U}\mathbf{b}\right)\cdot\nabla\ln B &= -\frac{1}{\sqrt{g_\mathrm{H}}\langle B^2\rangle}\left(B_\zeta\frac{\partial}{\partial\theta_\mathrm{H}} - B_\theta\frac{\partial}{\partial\zeta_\mathrm{H}}\right)\ln B\\ &= \frac{c_1B_\theta + c_2B_\zeta}{c_1\psi' - c_2\chi'}\frac{1}{\sqrt{g_\mathrm{H}}\langle B^2\rangle}\left(\chi'\frac{\partial}{\partial\theta_\mathrm{H}} + \psi'\frac{\partial}{\partial\zeta_\mathrm{H}}\right)\ln B\\ &= \frac{c_1B_\theta + c_2B_\zeta}{c_1\psi' - c_2\chi'}\frac{\mathbf{B}\cdot\nabla\ln B}{\langle B^2\rangle}\ (\text{symmetric cases}). \qquad (A8)\end{aligned}$$

The radial current Eq. (A7) in this situation has the following relation with the parallel viscous force:

$$\begin{aligned}&\left\langle\sum_a(p_{\parallel a} - p_{\perp a})\left(\frac{\nabla s\times\mathbf{B}}{B^2} + \tilde{U}\mathbf{b}\right)\cdot\nabla\ln B\right\rangle\\ &= \frac{c_1B_\theta + c_2B_\zeta}{c_1\psi' - c_2\chi'}\frac{1}{\langle B^2\rangle}\left\langle\sum_a(p_{\parallel a} - p_{\perp a})\mathbf{B}\cdot\nabla\ln B\right\rangle\\ &= -\frac{c_1B_\theta + c_2B_\zeta}{c_1\psi' - c_2\chi'}\frac{1}{\langle B^2\rangle}\sum_a\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\ (\text{symmetric cases}). \qquad (A9)\end{aligned}$$

In such symmetric configurations, the existence of the external parallel momentum input in Eqs. (4) and (30) directly means a following break of the charge neutrality:

$$\begin{aligned}\langle\mathbf{J}\cdot\nabla s\rangle/c &= \frac{c_1B_\theta + c_2B_\zeta}{c_1\psi' - c_2\chi'}\frac{1}{\langle B^2\rangle}\sum_a\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\\ &= \frac{c_1B_\theta + c_2B_\zeta}{c_1\psi' - c_2\chi'}\frac{m_\mathrm{f}}{\langle B^2\rangle}\left\langle B\int v_\parallel S_\mathrm{f}\mathrm{d}^3\mathbf{v}\right\rangle \neq 0\\ &\quad(\text{symmetric cases}). \qquad (A10)\end{aligned}$$

This problem is caused by a limitation on phases of the local parallel and radial currents $J_\parallel$ and $\mathbf{J}\cdot\nabla s$ that is noted in the end of Sec. II. (This contradiction cannot be removed if Eq. (37) is included in Eq. (35) for generating $\langle\tilde{U}\int v_\parallel S_\mathrm{f}\mathrm{d}^3\mathbf{v}\rangle \neq 0$, since Eq. (37) is only the 1st order of ρ_p/L_r as discussed in Sec. III and thus its contribution is negligibly small in Eq. (A10) determined by the 0th order of ρ_p/L_r.)

However, in non-symmetric stellarator/heliotron configurations, this momentum input is not a serious contradiction to the charge neutrality. In their typical $\mathbf{B}$-field strength $B/B_0 = 1 + \varepsilon_\mathrm{T}(s,\theta) + \varepsilon_\mathrm{H}(s,\theta)\cos[L\theta - N\zeta + \gamma(s,\theta)]$ with $|\chi' L| \ll |\psi' N|$, $|B_\zeta| \gg |B_\theta N|$, the suppression of the radial current Eq. (A7) is suppressing mainly axisymmetric Fourier modes $\sin(m\theta)$ in $\sum_a(p_{\parallel a} - p_{\perp a})$. The pressure perturbation of $\propto\int^l\tilde{U}\mathrm{d}l$, for which we concluded in Secs. II and III that it should vanish in Eq. (17), also has a nearly axisymmetric structure since the modes $n \neq 0$ are strongly suppressed in

$$\begin{aligned}\int^l\tilde{U}\mathrm{d}l &= \frac{V'}{4\pi^2}\sum_{mn}\frac{B_\zeta m + B_\theta n}{(\chi' m - \psi' n)^2}\varepsilon_{mn}^{(\mathrm{Boozer})}\sin(m\theta_\mathrm{B} - n\zeta_\mathrm{B})\\ &= \frac{V'}{4\pi^2}\sum_{mn}\frac{B_\zeta m + B_\theta n}{(\chi' m - \psi' n)^2}\varepsilon_{mn}^{(\mathrm{Hamada})}\sin(m\theta_\mathrm{H} - n\zeta_\mathrm{H})\end{aligned}$$

due to relations of $|\chi'| \ll |\psi' N|$, $|B_\zeta| \gg |B_\theta N|$. On the other hand, the surface-averaged parallel force $\sum_a\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_a\rangle\mathbf{B}/\langle B^2\rangle$ is formed by the non-axisymmetric modes $\sin(m\theta - n\zeta)$ with $n \neq 0$ in the anisotropy. Even when only this parallel force component as a divergence free vector remains in the MHD equilibrium, a consistency of the $\mathbf{B}$, $\mathbf{J}$ vector fields in Sec. II is retained in a ripple-period averaging.

APPENDIX B: THE LOWEST LEGENDRE ORDER OF FAST ION VELOCITY DISTRIBUTION

We discussed in Secs. III–V that the usual Legendre polynomial expansion for the pitch-angle space[19] is not a reasonable expression when the finite $\mathbf{b}\cdot\nabla B$ is included. However, this finite modulation effect is not important in determining $\langle f_\mathrm{f}^{(l=0)}\rangle$ as the surface-averaged lowest Legendre order $(l = 0)$ component $f_\mathrm{f}^{(l=0)} \equiv \frac{1}{2}\int_{-1}^1 f_\mathrm{f}\mathrm{d}\xi$. Since $v_\parallel\mathbf{b}\cdot\nabla$ in Sec. III satisfies $\langle\int_{-1}^1(v_\parallel\mathbf{b}\cdot\nabla f_a)\mathrm{d}\xi\rangle = 0$ for arbitrary velocity distribution functions even when $\mathbf{b}\cdot\nabla B \neq 0$,

$$\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(\langle f_\mathrm{f}^{(l=0)}\rangle, f_{b\mathrm{M}}\right) = -\frac{1}{2}\left\langle\int_{-1}^1 S_\mathrm{f}(\mathbf{x}, v, \sigma, \lambda)\mathrm{d}\xi\right\rangle$$

is obtained by taking $\langle\int_{-1}^1\mathrm{d}\xi\rangle$ integral of Eq. (30). The radial gradient $\frac{\partial}{\partial s}\langle\int_{-1}^1(1+\xi^2)\bar{f}_\mathrm{f}\mathrm{d}\xi\rangle$ in Eq. (37) also vanishes by Eq. (14). From the viewpoint of the Landau equation's $\int v^2\mathrm{d}^3\mathbf{v}$ moment without the drift approximation, this is a neglect of the left hand side (LHS) of

$$\begin{aligned}&\frac{\partial}{\partial V}\langle\mathbf{Q}_\mathrm{f}\cdot\nabla V\rangle + e_\mathrm{f}\langle n_\mathrm{f}\mathbf{u}_\mathrm{f}\cdot\nabla V\rangle\frac{\partial\Phi}{\partial V}\\ &= \frac{m_\mathrm{f}}{2}\int v^2\sum_{b\neq\mathrm{f}}C_{\mathrm{f}b}\left(\langle f_\mathrm{f}^{(l=0)}\rangle, f_{b\mathrm{M}}\right)\mathrm{d}^3\mathbf{v} + \frac{m_\mathrm{f}}{2}\left\langle\int v^2 S_\mathrm{f}\mathrm{d}^3\mathbf{v}\right\rangle,\end{aligned}$$

given by the Gauss' theorem. This LHS being $\mathcal{O}((\rho_\mathrm{p}/L_r)^2)$ also is given by formulas in Appendix A. This determination of $\langle f_\mathrm{f}^{(l=0)}\rangle$ by a balance of its collision and the surface-averaged $l = 0$ source only is identical to that in cases of $\mathbf{b}\cdot\nabla B = 0$.[19] Since the dependence of eigenvalues κ_n on finite $\mathbf{b}\cdot\nabla B \neq 0$ discussed in Sec. IV is caused by the $C_\mathrm{f}^{\mathrm{PAS}}$ operator in solving Eq. (34) or corresponding equation (38), the lowest Legendre order $l = 0$ resulting $C_\mathrm{f}^{\mathrm{PAS}}\langle f_\mathrm{f}^{(l=0)}\rangle = 0$ is irrelative to the $\mathbf{B}$-field strength modulation handled in these equations. When this steady-state solution $\langle f_\mathrm{f}^{(l=0)}\rangle \propto [v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e}) + v_\mathrm{c}^3]^{-1}U(v_\mathrm{b} - v)$ is substituted in the standard RMJ operator Eq. (31),

$$\sum_{b\neq \mathrm{f}} C_{\mathrm{f}b}\left(\langle f_{\mathrm{f}}^{(l=0)}\rangle, f_{b\mathrm{M}}\right) \cong \langle f_{\mathrm{f}}^{(l=0)}\rangle 8\sqrt{\pi}\frac{e_{\mathrm{f}}^2}{m_{\mathrm{f}}}\sum_{b\neq \mathrm{e,f}}\frac{n_b e_b^2 \ln\Lambda_{\mathrm{f}b}}{T_b}\left(\frac{m_b}{2T_b}\right)^{1/2}\exp\left(-x_b^2\right)$$

is obtained at the thermal velocity range $m_{\mathrm{f}}v^2 \sim T_{\mathrm{i}}$, and it satisfies $\int_{m_{\mathrm{f}}v^2\sim T_{\mathrm{i}}}\sum_{b\neq \mathrm{f}} C_{\mathrm{f}b}(\langle f_{\mathrm{f}}^{(l=0)}\rangle, f_{b\mathrm{M}})\mathrm{d}^3\mathbf{v} = \langle\int S_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\rangle$. It corresponds to a particle sink term required for the steady-state solution that is mentioned in Sec. III.

Integrals $\langle\int v^n f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\rangle$ with $n\geq -2$ and/or Rosenbluth potentials[2] $\partial^n H(\langle f_{\mathrm{f}}^{(l=0)}\rangle)/\partial v^n$, $\partial^n G(\langle f_{\mathrm{f}}^{(l=0)}\rangle)/\partial v^n$ for this $\langle f_{\mathrm{f}}^{(l=0)}\rangle$ can be calculated by Eq. (54) and following indefinite integrals or connection formulas of their $v^2/v_{\mathrm{c}}^2 \ll 1$ and $v^2/v_{\mathrm{c}}^2 \gg 1$ asymptotic limit values:

$$\int_0^v \frac{\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} \cong \frac{1}{v_{\mathrm{c}}^2}\left\{\left(\frac{v_{\mathrm{c}}}{v}\right)^{5/2}+\left(\frac{3\sqrt{3}}{2\pi}\right)^{5/2}\right\}^{-2/5}, \tag{B1}$$

$$\int_0^v \frac{v\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} \cong \frac{1}{2v_{\mathrm{c}}}\left(\frac{v_{\mathrm{c}}^2}{v^2}+\frac{3\sqrt{3}}{4\pi}\right)^{-1}, \tag{B2}$$

$$\int_0^v \frac{v^2\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} = \frac{1}{3}\ln\left(1+\frac{v^3}{v_{\mathrm{c}}^3}\right), \tag{B3}$$

$$\int_0^v \frac{v^3\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} \cong v\left\{\left(\frac{4v_{\mathrm{c}}^3}{v^3}\right)^{2/3}+1\right\}^{-3/2}, \tag{B4}$$

$$\int_0^v \frac{v^4\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} \cong \frac{v^2}{2}\left\{\left(\frac{5v_{\mathrm{c}}^3}{2v^3}\right)^{3/4}+1\right\}^{-4/3}, \tag{B5}$$

$$\int_0^v \frac{v^5\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} = \frac{v^3}{3}-\frac{v_{\mathrm{c}}^3}{3}\ln\left(1+\frac{v^3}{v_{\mathrm{c}}^3}\right), \tag{B6}$$

$$\int_0^v \frac{v^n\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} \cong \frac{v^{n-2}}{n-2}\left(\frac{n+1}{n-2}\frac{v_{\mathrm{c}}^3}{v^3}+1\right)^{-1} \quad \text{for } n\geq 6. \tag{B7}$$

Except $\int_0^v \frac{v^2\mathrm{d}v}{v^3+v_{\mathrm{c}}^3}$, numerical calculations of a slow velocity range $v^2/v_{\mathrm{c}}^2 \ll 1$ of mathematically exact integral formulas often cause numerical errors violating an obvious fact $0 < \int_0^v \frac{v^n\mathrm{d}v}{v^3+v_{\mathrm{c}}^3} < \frac{v^{n+1}/v_{\mathrm{c}}^3}{n+1}$. To avoid these errors, they should be replaced by connection formulas as listed here.

Since the second term in Eq. (54) is not important as long as $v_{\mathrm{b}}^2 < (3\sqrt{\pi}/4)^{2/3}2T_{\mathrm{e}}/m_{\mathrm{e}}$, here we regard Eqs. (B1)–(B5) as basic characteristics of $\int_0^v v^n\langle f_{\mathrm{f}}^{(l=0)}\rangle\mathrm{d}v$. While we included fast ions' pressures $p_{\perp\mathrm{f}}$, $p_{\parallel\mathrm{f}}$ in the MHD equilibrium in Sec. II motivated by experimental results suggesting their effects,[12,13] we simultaneously assumed in Sec. III that the density n_{f} is negligible in other summations, such as $\sum_a e_a n_a$ and $\sum_b C_{\mathrm{f}b}(f_{\mathrm{f}}, f_b)$. This assumption is partly based on a characteristic of $\langle\int v^n f_{\mathrm{f}}\mathrm{d}^3\mathbf{v}\rangle$ integrals, in which fast ions in a high-energy range $v > v_{\mathrm{c}}$ do not effectively contribute to them when their n values are small ($n\leq 0$). The Rosenbluth potentials given by these indefinite integrals will be useful for checking whether $C_{a\mathrm{f}}(f_{a1}, \langle f_{\mathrm{f}}^{(l=0)}\rangle)$ is negligibly smaller than $\sum_{b\neq \mathrm{f}} C_{ab}(f_{a1}, f_{b\mathrm{M}})$ in thermal particles' kinetic equations $a\neq \mathrm{f}$. Although formulas in Secs. III–V allow a possibility of $v_{\mathrm{b}}^2 \sim (3\sqrt{\pi}/4)^{2/3}2T_{\mathrm{e}}/m_{\mathrm{e}}$, the averaged fast ion energy given by Eqs. (B3) and (B5) is $\langle p_{\mathrm{f}}\rangle/\langle n_{\mathrm{f}}\rangle/m_{\mathrm{f}} \ll T_{\mathrm{e}}/m_{\mathrm{e}}$ in many practical cases. As long as $3\langle p_{\mathrm{f}}\rangle/\langle n_{\mathrm{f}}\rangle/m_{\mathrm{f}} < (2/\gamma)T_{\mathrm{e}}/m_{\mathrm{e}}$ [$\ln\gamma = 0.57722$: Euler's constant], the Coulomb logarithm $\ln\Lambda_{\mathrm{fe}} = \ln\Lambda_{\mathrm{ef}}$ is that for usual electron-ion temperature relaxation where $|\mathbf{v}-\mathbf{v}'|^2 = (2/\gamma)\langle T_{\mathrm{e}}\rangle/m_{\mathrm{e}}$ is used in the logarithm (Eq. (6.4) in Ref. 37). For collisions between thermal and fast ions, $|\mathbf{v}-\mathbf{v}'|^2 = \min[3\langle p_{\mathrm{f}}\rangle/\langle n_{\mathrm{f}}\rangle/m_{\mathrm{f}}, v_{\mathrm{c}}^2]$ is used in the logarithm $\ln\Lambda_{\mathrm{f}a} = \ln\Lambda_{a\mathrm{f}}$ ($a\neq \mathrm{e}$). This maximum value limit is due to the fact that the high energy range $v > v_{\mathrm{c}}$ does not contribute to Eqs. (C2) and (C3) of the f-i, i-f collisions because of Eqs. (B1) and (B2). It is also a reason for this $\ln\Lambda_{\mathrm{f}a} = \ln\Lambda_{a\mathrm{f}}$ that $\langle f_{\mathrm{f}}^{(l=0)}\rangle$ and $\tilde{f}_{\mathrm{f}}^{0\,(\mathrm{odd})}$ in $v > v_{\mathrm{c}}$ are insensitive to v_{c}^3 and Z_2. This characteristic of Eqs. (B1) and (C3) explains also why the momentum input to target ions is sensitive to the finite $\langle(1-B/B_{\mathrm{M}})^{1/2}\rangle \neq 0$, while the input to electrons is insensitive, as shown in Sec. V.

APPENDIX C: INTEGRAL FORMULAS FOR THE TEST PARTICLE PORTION

Firstly, a partial integral formula for the energy scattering term in Eq. (31) is shown. An integration by parts gives

$$\begin{aligned}\int_0^\infty & v^n\frac{\partial}{\partial v}\left\{G(x_b)v\left(\frac{m_a v}{T_b}+\frac{\partial}{\partial v}\right)f_a\right\}\mathrm{d}v \\ &= \left[v^n G(x_b)\left(\frac{m_a v^2}{T_b}-n+v\frac{\partial}{\partial v}\right)f_a\right]_0^\infty \\ &+n\int_0^\infty v^n\left\{\frac{\Phi(x_b)-G(x_b)}{v^3}+(n-1)\frac{G(x_b)}{v^3}\right. \\ &\left.-\left(1+\frac{m_a}{m_b}\right)\frac{m_b}{T_b}\frac{G(x_b)}{v}\right\}f_a v^2\mathrm{d}v.\end{aligned} \tag{C1}$$

When a low energy limit of $f_a(v,\xi,\phi)$ has a form in which the $v\to 0$ limit of Legendre order l is $\to v^l P_l^m(\xi)$, $f_a(v\to 0)$ in this first term vanishes for energy space weighting of $n\geq -l$. For example, when $n=-1$ is chosen for calculating the lowest Legendre order $l=0$,

$$\left[\frac{G(x_b)}{v}\left(\frac{m_a v^2}{T_b}+1+v\frac{\partial}{\partial v}\right)f_a^{(l=0)}\right]_0^\infty = \frac{2}{3\sqrt{\pi}v_{\mathrm{T}b}}f_a^{(l=0)}(v=0)$$

remains. This kind of energy space weighting should be avoided since velocity distribution functions handled in neoclassical theories are approximated functions in which only a limited number of $\int v^l P_l(\xi)L_j^{(l+1/2)}(x_a^2)C_{ab}(f_a, f_b)\mathrm{d}^3\mathbf{v}$ and $\int v^l P_l(\xi)L_j^{(l+1/2)}(x_a^2)f_a\mathrm{d}^3\mathbf{v}$ integrals are valid. Accuracies of local values in the energy space are not guaranteed. Therefore, Eq. (C1) should be used with the energy space weighting of $n\geq -l$ (Only the second term is used.). General $\int v^n P_l(\xi)C_{ab}(f_a, f_{b\mathrm{M}})\mathrm{d}^3\mathbf{v}$ integrals are obtained by using Eq. (C1) and a relation $\mathcal{L}P_l(\xi) = -l(l+1)P_l(\xi)/2$. For example, together with the partial integral procedure for the field particle portion $C_{ab}(f_{a\mathrm{M}}, f_b)$ shown in Ref. 2, energy/momentum exchange formulas are obtained as follows:

$$m_a \int v^2 C_{ab}(f_{aM}, f_b) d^3\mathbf{v} = -m_b \int v^2 C_{ba}(f_b, f_{aM}) d^3\mathbf{v}$$
$$= 32\pi^2 \frac{n_a (e_a e_b)^2 \ln\Lambda_{ab}}{m_a} \left(\frac{m_a}{2T_a}\right)^{1/2} \int_0^\infty \left\{ x_a G(x_a) - \frac{m_a}{m_b} \frac{1}{\sqrt{\pi}} \exp(-x_a^2) \right\} \left(\int_{-1}^{1} \bar{f}_b d\xi \right) v^2 dv, \quad \text{(C2)}$$

$$m_a \int v\xi C_{ab}(f_{aM}, f_b) d^3\mathbf{v} = -m_b \int v\xi C_{ba}(f_b, f_{aM}) d^3\mathbf{v}$$
$$= 8\pi^2 \frac{n_a (e_a e_b)^2 \ln\Lambda_{ab}}{T_a} \left(1 + \frac{m_a}{m_b}\right) \int_0^\infty G(x_a) \left(\int_{-1}^{1} \xi \bar{f}_b d\xi \right) v^2 dv. \quad \text{(C3)}$$

Here, $\bar{f}_b \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f_b d\phi$ is the gyro-phase-averaged velocity distribution function. More general $m_a \int v\xi L_j^{(3/2)}(x_a^2) C_{ab}(f_{aM}, f_b) d^3\mathbf{v}$ integral formulas for the Legendre order $l = 1$ are listed in Ref. 2.

Next, approximation methods in Eq. (32) for the f-e, f-i collisions are compared with the standard RMJ results Eqs. (C2) and (C3). The approximations give

$$m_f \int v^2 C_{fe}(f_f, f_{eM}) d^3\mathbf{v}$$
$$= -16\pi^2 \frac{n_e (e e_f)^2 \ln\Lambda_{fe}}{T_e} \int_0^\infty G(x_e) \left(\int_{-1}^{1} \bar{f}_f d\xi \right) v^3 dv,$$
$$m_f \int v\xi C_{fe}(f_f, f_{eM}) d^3\mathbf{v}$$
$$= -8\pi^2 \frac{n_e (e e_f)^2 \ln\Lambda_{fe}}{T_e} \int_0^\infty G(x_e) \left(\int_{-1}^{1} \xi \bar{f}_f d\xi \right) v^2 dv$$

for f-e collision, and

$$m_f \int v^2 C_{fa}(f_f, f_{aM}) d^3\mathbf{v}$$
$$= -16\pi^2 \frac{n_a (e_a e_f)^2 \ln\Lambda_{fa}}{m_a} \int_0^\infty \left(\int_{-1}^{1} \bar{f}_f d\xi \right) v dv,$$
$$m_f \int v\xi C_{fa}(f_f, f_{aM}) d^3\mathbf{v}$$
$$= -8\pi^2 \frac{n_a (e_a e_f)^2 \ln\Lambda_{fa}}{m_a} \left(1 + \frac{m_a}{m_f}\right) \int_0^\infty \left(\int_{-1}^{1} \xi \bar{f}_f d\xi \right) dv,$$

corresponding to $G(x) \cong (2x^2)^{-1}$, for f-i collisions ($a \neq$ e,f). It should be noted that these approximations of $C_{fa}(f_f, f_{aM})$ ($a \neq$ f) in Eq. (32) are only methods to obtain the steady-state solution by Eq. (30) with the external source term S_f, as discussed in Sec. III. Since the actual momentum/energy transfer is governed by the standard RMJ formulas (the f_f at $m_f v^2/2 \sim T_e, T_i$ has only a function as a particle source to the thermalized ion species with $m_a = m_f, e_a = e_f$), it should be calculated by substituting the obtained steady-state solution into Eqs. (C2) and (C3).

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