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Equilibria of Toroidal Plasmas with Toroidal and Poloidal Flow in High-beta Reduced Magnetohydrodynamic Models

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Abstract. A reduced set of magnetohydrodynamic equilibrium equations for high-beta tokamaks is derived from the fluid moment equations for collisionless, magnetized plasmas. Effects of toroidal and poloidal flow comparable to the poloidal sound velocity, two-fluid, ion finite Larmor radius (FLR), pressure anisotropy and parallel heat fluxes are incorporated into the Grad-Shafranov equation by means of asymptotic expansions in terms of the inverse aspect ratio of a torus. The two-fluid effects induce the diamagnetic flows, which result in asymmetry of the equilibria with respect to the sign of the $E \times B$ flow. The gyrovisosity and other FLR effects cause the so-called gyroviscous cancellation of the convection due to the ion diamagnetic flow. The qualitative difference between the equilibria with and without the parallel heat fluxes is shown to stem from characteristics of the sound waves. Higher order terms of quantities like the pressures and the stream functions show the shift of their isosurfaces from the magnetic surfaces due to effects of flow, two-fluid and pressure anisotropy. The reduced form of the diamagnetic current associated with pressure anisotropy is also obtained.

1. Introduction

Magnetohydrodynamic (MHD) equilibria with flow describe macroscopic structure of plasma flows in steady state. Equilibrium models are needed for rigorous stability analysis or nonlinear simulation as initial states. The MHD equations for equilibria with flow reduce to the so-called generalized Grad-Shafranov (GS) equation and the Bernoulli law in axisymmetric systems [1, 2]. Recently, this MHD model for flowing equilibria has been extended to include the hot ion effects such as the ion gyroviscosity and other finite Larmor radius (FLR) effects that are relevant to fusion plasmas as well as two-fluid effects [3, 4, 5]. Those small-scale effects cannot be neglected at the sharp boundary of a well-confined region in magnetically confined plasmas where high-beta is achieved by shear-flow suppression of instability and turbulent transport, while the ion FLR terms had been neglected in previous models of two-fluid equilibria [6, 7, 8, 9]. The fluid formalism of magnetized plasmas suggests that the difference between the dynamics parallel and perpendicular to the magnetic field brings the Chew-Goldberger-Low (CGL) anisotropic pressure [10] to the pressure tensor as well as the gyroviscosity. Strong pressure anisotropy is found in plasma flows externally driven by the neutral beam injection (NBI). The equations for the anisotropic pressures derived in the formalism include the heat fluxes. The system of fluid moment equations for collisionless, magnetized plasmas needs a closure condition to truncate higher-rank fluid moments. In the equilibrium model with flow comparable to the poloidal-sound velocity [4], the parallel heat fluxes, a third order fluid moment, are neglected for simplicity to obtain a closed set of equations with the isotropic, adiabatic pressures for ions and electrons, while they are ordered out in the model with flow comparable to the poloidal Alfvén velocity [3]. In [11], two-fluid equilibria with cold ions and anisotropic pressures for massless electrons are studied. The parallel heat flux equations for massless electrons do not contain parallel heat fluxes and they become the equations for anisotropic electron pressures. The parallel heat fluxes for ions should be retained for more accurate equilibrium models with hot ions.

In this paper we present the formulation for the effects of flow, two-fluid, ion FLR, pressure anisotropy and parallel heat fluxes on equilibria of a high-beta toroidal plasma in the framework of reduced magnetohydrodynamics (MHD) [12, 13], based on the fluid moment equations for collisionless, magnetized plasmas derived from the Vlasov equation [14, 15, 16]. We consider toroidal and poloidal flows comparable to the poloidal sound velocity. Such poloidal-sonic flow is of interest because a transition of equilibrium occurs between sub- and super-poloidal-sonic flow [17, 18, 19, 20, 21, 22, 23] and also because the poloidal-sound velocity is smaller than other characteristic velocities like the sound, Alfvén and poloidal-Alfvén velocities in tokamak plasmas and, therefore, more accessible to flows in experiments. We introduce pressure anisotropy and the parallel heat fluxes that were not taken into account in our previous formulation [4]. By means of asymptotic expansions, we drive a reduced set of equilibrium equations for high-beta tokamaks. The gyrovisosity and other FLR effects cause the so-called gyroviscous cancellation of the convection due to the ion diamagnetic flow induced by the two-fluid effects in the equilibrium equations of momentum balance, pressure and heat fluxes, which is consistent with the FLR fluid theories of [24] and [25]. We obtain the GS-type equations for the first and second order magnetic flux functions. The equations are written in a unified form of the single-fluid, two-fluid (Hall) and FLR two-fluid models with the isotropic and anisotropic pressures with and without the ion parallel heat fluxes. The first-order GS equation is identical to that of the static, single-fluid equilibria. The second-order GS equation includes terms representing the effects of flow, two-fluid, ion FLR and pressure anisotropy. In spite of its complexity, the second-order GS equation is a linear, elliptic partial differential equation and, thus, is easy to solve; the numerical solutions of these equilibria will be shown elsewhere. However, the equations have singularity that arises because higher order terms not negligible in its vicinity are ordered out in the asymptotic expansions. One has to choose the profiles of free functions that do not include the vicinity of singularity where the poloidal velocity equals the phase velocities of sound waves, and get regular solutions. The parallel heat fluxes induce the characteristic velocities of two additional ion sound waves [26], compared to the adiabatic pressure model. It is noted that the kinetic effect, the Landau damping, may resolve this singularity [27], which is left for future work. The impact of the ion diamagnetic flow on the shock formation of trans-poloidal-sonic flow was examined for low-beta tokamaks [28]. The present models show the modifications of regular single-fluid equilibrium solutions in the elliptic regions due to small-scale effects, and can capture the following features neglected in [28]: the modification of the magnetic structure due to the flow and FLR effects, and the shift of isosurfaces of the ion stream function from the magnetic surfaces due to the two-fluid effects while isotropic pressure isosurfaces shift from them due to flow even in the single-fluid model [3, 20]. Pressure anisotropy is identified by measuring the perpendicular (diamagnetic) current in experiment. We also obtain the reduced form of the diamagnetic current.

This paper is organized as follows. In section 2, we introduce the basic steady state equations for two-fluid MHD with hot ion effects, pressure anisotropy and the parallel heat fluxes, and the orderings for the reduced models. In section 3, we derive the reduced set of equations for equilibria with flow velocity comparable to the poloidal sound velocity by means of asymptotic expansions. In section 4, we discuss the modification of the flowing equilibria by the non-ideal effects mentioned above. A summary is shown in section 5.

2. Fluid moment equations in steady state

The equations for two-fluid equilibria with hot ions and pressure anisotropy are obtained from fluid moment equations for collisionless, magnetized plasmas [14, 15, 16]. The equations of continuity and momentum balance are

$$\nabla \cdot (n\mathbf{v}) = 0, \tag{1}$$

$$m_i n\mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{j}\times\mathbf{B} - \sum_{s=i,e}\left[\nabla p_{s\perp} + \mathbf{B}\cdot\nabla\left(\frac{p_{s\parallel}-p_{s\perp}}{B^2}\mathbf{B}\right)\right] - \lambda_i\nabla\cdot\Pi_i^{gv}, \quad (2)$$

where n is the density, $\mathbf{v}\approx\mathbf{v}_i$ is the ion flow velocity, m_i is the ion mass, $\mathbf{B}$ is the magnetic field, $p_{s\{\parallel,\perp\}}$ are the parallel and perpendicular pressures, $s = i, e$ is for ions and electrons respectively, Π_i^{gv} is the ion gyroviscous tensor, and $B = |\mathbf{B}|$. The electron mass m_e is neglected because $m_e \ll m_i$. The electron gyroviscosity is also neglected. The flow velocity is decomposed into the parallel and perpendicular components,

$$\mathbf{v} = v_\parallel\mathbf{b} + \mathbf{v}_\perp \quad (3)$$

where $\mathbf{b} = \mathbf{B}/B$. The current density $\mathbf{j}$ is

$$\mu_0\mathbf{j} = \nabla\times\mathbf{B}. \quad (4)$$

The momentum balance equation for massless electrons gives the generalized Ohm's law,

$$\mathbf{E}+\mathbf{v}\times\mathbf{B} = \frac{\lambda_H}{ne}\left[\mathbf{j}\times\mathbf{B} - \nabla p_{e\perp} - \mathbf{B}\cdot\nabla\left(\frac{p_{e\parallel}-p_{e\perp}}{B^2}\mathbf{B}\right)\right], \quad (5)$$

where $\mathbf{E}$ is the electric field that satisfies Faraday's law,

$$\nabla\times\mathbf{E} = 0. \quad (6)$$

From (2) and (5), the momentum balance equation for ions is

$$\mathbf{E}+\mathbf{v}\times\mathbf{B} = \frac{\lambda_H}{ne}\left[\nabla p_{i\perp} + \mathbf{B}\cdot\nabla\left(\frac{p_{i\parallel}-p_{i\perp}}{B^2}\mathbf{B}\right) + m_i n\mathbf{v}\cdot\nabla\mathbf{v} + \lambda_i\nabla\cdot\Pi_i^{gv}\right]. \quad (7)$$

We have introduced the artificial indices λ_H and λ_i that label the two-fluid, ion FLR terms respectively: $(\lambda_H, \lambda_i) = (0, 0)$ for single-fluid (ideal) MHD, $(1, 0)$ for two-fluid MHD with zero ion Larmor radius (Hall MHD) and $(1, 1)$ for two-fluid MHD with ion FLR.

In the fluid formalism of collisionless magnetized plasmas, the gyroviscosity and other ion FLR effects are incorporated by means of asymptotic expansions in terms of the small parameter $\delta \sim \rho_i/L$, where ρ_i is the ion Larmor radius and L is the macroscopic scale length. We assume the slow dynamics (drift) ordering,

$$v \sim \delta v_{thi}, \quad m_i n v^2 \sim \|\Pi_i^{gv}\| \sim \delta^2 p_{i,e}, \quad q_{i,e} \sim v p_{i,e} \sim \delta v_{thi} p_{i,e},$$

where v and v_{thi} are the flow and ion thermal velocities respectively, $q_{i,e}$ is the ion and electron heat fluxes respectively. The ion FLR terms are much simplified in the reduced models for large-aspect-ratio, high-beta tokamaks [12] after relating δ to the inverse aspect ratio expansion parameter $\varepsilon \equiv a/R_0 \ll 1$, where a and R_0 are the characteristic scale length of the minor and major radii respectively. The following reduced MHD orderings for high-beta tokamaks with strong pressure anisotropy are applied,

$$B_p \sim \varepsilon B_0, \quad p_{s\{\parallel,\perp\}} \sim \left|p_{s\parallel} - p_{s\perp}\right| \sim \varepsilon\left(B_0^2/\mu_0\right), \quad |\nabla_\parallel| \sim 1/R_0, \quad |\nabla_\perp| \sim 1/a.$$

We derive a reduced set of two-fluid equations for axisymmetric equilibria with flow comparable to the poloidal sound velocity. The poloidal-sonic flow $v \sim (B_p/B_0)\, v_{thi}$ can be described by the reduced model with the relations

$$\delta \sim \epsilon \text{ and } a \sim L.$$

While the poloidal-Alfvénic flow analysis follows the standard orderings of reduced MHD for high-beta tokamaks [29, 30, 3], the poloidal-sonic flow analysis does not and higher-order terms must be taken into account. In the following equations, we keep only terms required to the orders considered. The ion gyroviscous force $\nabla \cdot \Pi_i^{gv}$ is expanded as written in Appendix A. The equations for the perpendicular and parallel pressures for ions and electrons are

$$\begin{aligned}&\mathbf{v}\cdot\nabla p_{i\perp} + 2p_{i\perp}\nabla\cdot\mathbf{v} - p_{i\perp}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}) + \lambda_{i\|}\nabla\cdot\left(q_{iT\|}\mathbf{b}\right)\\ &+\lambda_i\nabla\cdot\mathbf{q}_{iT\perp} + 2\lambda_i\mathbf{q}_{iB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \simeq 0,\end{aligned} \tag{8}$$

$$\begin{aligned}&\frac{1}{2}\mathbf{v}\cdot\nabla p_{i\|} + \frac{1}{2}p_{i\|}\nabla\cdot\mathbf{v} + p_{i\|}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}) + \lambda_{i\|}\nabla\cdot\left(q_{iB\|}\mathbf{b}\right)\\ &+\lambda_i\nabla\cdot\mathbf{q}_{iB\perp} - 2\lambda_i\mathbf{q}_{iB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \simeq 0,\end{aligned} \tag{9}$$

$$\begin{aligned}&\mathbf{v}_e\cdot\nabla p_{e\perp} + 2p_{e\perp}\nabla\cdot\mathbf{v}_e - p_{e\perp}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}_e) + \nabla\cdot\left(q_{eT\|}\mathbf{b}\right)\\ &+\lambda_i\nabla\cdot\mathbf{q}_{eT\perp} + 2\lambda_i\mathbf{q}_{eB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \simeq 0,\end{aligned} \tag{10}$$

$$\begin{aligned}&\frac{1}{2}\mathbf{v}_e\cdot\nabla p_{e\|} + \frac{1}{2}p_{e\|}\nabla\cdot\mathbf{v}_e + p_{e\|}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}_e) + \nabla\cdot\left(q_{eB\|}\mathbf{b}\right)\\ &+\lambda_i\nabla\cdot\mathbf{q}_{eB\perp} - 2\lambda_i\mathbf{q}_{eB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \simeq 0,\end{aligned} \tag{11}$$

where $\mathbf{v}_e = \mathbf{v} - (\lambda_H/ne)\,\mathbf{j}$ is the electron flow velocity, and the perpendicular and parallel heat fluxes, $\mathbf{q}_{s\perp}$ and $\mathbf{q}_{s\|}$ respectively, are defined as follows,

$$\mathbf{q}_{s\perp} = \mathbf{q}_{sB\perp} + \mathbf{q}_{sT\perp} \qquad (s = i, e), \tag{12}$$

$$\mathbf{q}_{sB\perp} = \frac{m_s}{2}\int \left(v'_{s\|} - v_{s\|}\right)^2 \left(\mathbf{v}'_{s\perp} - \mathbf{v}_{s\perp}\right) f_s d^3\mathbf{v}'_s, \tag{13}$$

$$\mathbf{q}_{sT\perp} = \frac{m_s}{2}\int \left(v'_{s\perp} - v_{s\perp}\right)^2 \left(\mathbf{v}'_{s\perp} - \mathbf{v}_{s\perp}\right) f_s d^3\mathbf{v}'_s, \tag{14}$$

$$\mathbf{q}_{s\|} = \left(q_{sB\|} + q_{sT\|}\right)\mathbf{b} \qquad (s = i, e), \tag{15}$$

$$q_{sT\|} = \frac{m_i}{2}\int \left(v'_{s\perp} - v_{s\perp}\right)^2 \left(v'_{s\|} - v_{s\|}\right) f_s d^3\mathbf{v}'_s, \tag{16}$$

$$q_{sB\|} = \frac{m_s}{2}\int \left(v'_{s\|} - v_{s\|}\right)^3 f_s d^3\mathbf{v}'_s. \tag{17}$$

The perpendicular heat fluxes are given by

$$\mathbf{q}_{sB\perp} \simeq \frac{1}{e_s B}\mathbf{b}\times\left[\frac{1}{2}p_{s\perp}\nabla\left(\frac{p_{s\|}}{n}\right) + \frac{p_{s\|}\left(p_{s\|} - p_{s\perp}\right)}{n}(\mathbf{b}\cdot\nabla\mathbf{b})\right], \tag{18}$$

$$\mathbf{q}_{sT\perp} \simeq \frac{1}{e_s B}\mathbf{b} \times \left[2p_{s\perp}\nabla\left(\frac{p_{s\perp}}{n}\right)\right]. \tag{19}$$

We have introduced another index $\lambda_{i\parallel}$ where 0 and 1 represent the presence and absence of the ion parallel heat flux respectively. The case where $\lambda_{i\parallel} = 0$ corresponds to the CGL double adiabatic pressure case [10]. The parallel heat flux cannot be neglected when $q_{s\parallel} \sim q_{s\perp} \sim p_s v$. The equations for ion and electron parallel heat fluxes obtained in [16] are written in the present orderings as

$$\nabla \cdot \left[\left(\mathbf{v} + \frac{\lambda_i}{neB}\nabla p_{i\perp} \times \mathbf{b}\right) q_{iT\parallel}\right] + q_{iT\parallel}\nabla \cdot \left(\mathbf{v} + \frac{\lambda_i}{neB}\nabla p_{i\perp} \times \mathbf{b}\right)$$
$$+\frac{p_{i\parallel}}{m_i}\mathbf{b} \cdot \nabla\left(\frac{p_{i\perp}}{n}\right) - \frac{p_{i\perp}\left(p_{i\parallel} - p_{i\perp}\right)}{m_i n B}\mathbf{b} \cdot \nabla B \simeq 0, \tag{20}$$

$$\nabla \cdot \left[\left(\mathbf{v} + \frac{\lambda_i}{neB}\nabla p_{i\perp} \times \mathbf{b}\right) q_{iB\parallel}\right] + \frac{3p_{i\parallel}}{2m_i}\mathbf{b} \cdot \nabla\left(\frac{p_{i\parallel}}{n}\right) \simeq 0, \tag{21}$$

$$\mathbf{B} \cdot \nabla\left(p_{e\parallel}/n\right) \simeq 0, \tag{22}$$

$$\mathbf{B} \cdot \nabla\left[\left(p_{e\parallel}/p_{e\perp} - 1\right)B\right] \simeq 0. \tag{23}$$

We have closed (20) - (23) by evaluating the fourth-rank moments with the shifted bi-Maxwellian distribution function. Hence, the Landau damping and other kinetic effects which arise from the non-Maxwellian part of distribution function are neglected. Since the equations for parallel electron heat flux for massless electrons (22) and (23) do not include parallel heat fluxes, these are the equations for electron pressure [11] and the heat fluxes are, conversely, obtained from the electron pressure equations (10) and (11).

3. Derivation of reduced equations

We derive a reduced set of equilibrium equations from the equations shown in section 2 by extending the previous formulations [3, 4]. Here we consider the corresponding toroidal axisymmetric equilibria, where, in cylindrical coordinates (R, φ, Z), the magnetic field $\mathbf{B}$, the current density $\mathbf{j}$ and the electric field $\mathbf{E}$ can be written as

$$\mathbf{B} = \nabla\psi(R, Z) \times \nabla\varphi + I(R, Z)\nabla\varphi, \tag{24}$$

$$\mathbf{j} = \nabla I \times \nabla\varphi - \Delta^*\psi\nabla\varphi, \tag{25}$$

$$\mathbf{E} = -\nabla\Phi(R, Z), \tag{26}$$

where $\Delta^* \equiv R^2\nabla \cdot [R^{-2}\nabla]$. The projection of the momentum balance equation (2) along $\nabla\psi$, $\mathbf{B}$ and the poloidal magnetic field $\mathbf{B}_p = \nabla\psi \times \nabla\varphi$ yields

$$\mu_0 R^2\nabla\psi \cdot (m_i n\mathbf{v} \cdot \nabla\mathbf{v} + \lambda_i\nabla \cdot \Pi_i^{gv}) + |\nabla\psi|^2\Delta^*\psi + I\nabla\psi \cdot \nabla I$$
$$+\mu_0 R^2 \sum_{s=i,e} \nabla\psi \cdot \left[\nabla p_{s\perp} + \mathbf{B}\cdot\nabla\left(\frac{p_{s\parallel} - p_{s\perp}}{B^2}\mathbf{B}\right)\right] = 0, \tag{27}$$

$$\mathbf{B}\cdot(m_i n\mathbf{v}\cdot\nabla\mathbf{v}+\lambda_i\nabla\cdot\Pi_i^{gv})+\{p_{i\perp}+p_{e\perp},\,\psi\} + \sum_{s=i,e}\mathbf{B}\cdot\left[\mathbf{B}\cdot\nabla\left(\frac{p_{s\parallel}-p_{s\perp}}{B^2}\mathbf{B}\right)\right]=0, \tag{28}$$

$$(\nabla\psi\times\nabla\varphi)\cdot(m_i n\mathbf{v}\cdot\nabla\mathbf{v}+\lambda_i\nabla\cdot\Pi_i^{gv})+(I/\mu_0R^2)\{I,\,\psi\}+\{p_{i\perp}+p_{e\perp},\,\psi\} + \sum_{s=i,e}(\nabla\psi\times\nabla\varphi)\cdot\left[\mathbf{B}\cdot\nabla\left(\frac{p_{s\parallel}-p_{s\perp}}{B^2}\mathbf{B}\right)\right]=0, \tag{29}$$

where $\{a,\,b\}\equiv(\nabla a\times\nabla b)\cdot\nabla\varphi$. The asymptotic expansions are defined in terms of the inverse aspect ratio $\varepsilon\equiv a/R_0\ll 1$. The variables are expanded as

$$\psi=\psi_1+\psi_2+\ldots,$$

$$I=I_0+I_1+I_2+I_3+\ldots,$$

$$p_{s\parallel}=p_{s\parallel 1}+p_{s\parallel 2}+p_{s\parallel 3}+\ldots,$$

$$p_{s\perp}=p_{s\perp 1}+p_{s\perp 2}+p_{s\perp 3}+\ldots,$$

$$n=n_0+n_1+\ldots,$$

$$\Phi=\Phi_1+\Phi_2+\ldots$$

$$R=R_0+x,$$

where $I_0\equiv B_0R_0$. It is noted that we need only the leading order quantities of the parallel flow velocity $v_\parallel$ and the parallel heat fluxes $q_{sT\parallel}$ and $q_{sB\parallel}$.

The leading order of the momentum balance (2) yields

$$p_{i\perp 1}+p_{e\perp 1}+\frac{B_0}{\mu_0R_0}I_1=\text{const}. \tag{30}$$

The perpendicular velocity is obtained from the ion momentum balance equation (7) as

$$\mathbf{v}_\perp\simeq\frac{\mathbf{E}\times\mathbf{B}}{B^2}-\frac{\lambda_H}{neB^2}\left[\nabla p_{i\perp}+\mathbf{B}\cdot\nabla\left(\frac{p_{i\parallel}-p_{i\perp}}{B^2}\mathbf{B}\right)\right]\times\mathbf{B}. \tag{31}$$

The leading order of (31) reads

$$\mathbf{v}_\perp^{(1)}=-\frac{R_0}{B_0}\left(\nabla\,\Phi_1+\frac{\lambda_H}{en_0}\nabla p_{i\perp 1}\right)\times\nabla\varphi, \tag{32}$$

which shows that the leading order of the perpendicular flow consists of the $E\times B$ and the ion diamagnetic flows. Its divergence is

$$(\nabla\cdot\mathbf{v})^{(1)}=-\frac{\lambda_H R_0}{eB_0}\{n_0^{-1},p_{i\perp 1}\}. \tag{33}$$

The perpendicular (diamagnetic) heat fluxes and their divergences to the leading order are

$$\mathbf{q}_{sT\perp}^{(1)}=-2\frac{p_{s\perp 1}R_0}{e_sB_0}\nabla\left(\frac{p_{s\perp 1}}{n_0}\right)\times\nabla\varphi, \tag{34}$$

$$\mathbf{q}^{(1)}_{sB\perp} = -\frac{1}{2}\frac{p_{s\perp 1}R_0}{e_s B_0}\nabla\left(\frac{p_{s\parallel 1}}{n_0}\right)\times\nabla\varphi, \tag{35}$$

$$\nabla\cdot\mathbf{q}^{(1)}_{sT\perp} = -2\frac{p_{s\perp 1}R_0}{e_s B_0 n_0^2}\{n_0,\, p_{s\perp 1}\}, \tag{36}$$

$$\nabla\cdot\mathbf{q}^{(1)}_{iB\perp} = -\frac{1}{2}\frac{p_{s\parallel 1}R_0}{e_s B_0 n_0^2}\{n_0,\, p_{s\perp 1}\} + \frac{1}{2}\frac{R_0}{e_s B_0 n_0}\{p_{s\parallel 1},\, p_{s\perp 1}\}, \tag{37}$$

The leading order of the equation of continuity (1) is

$$-(R_0/B_0)\{n_0,\, \Phi_1\} = 0, \tag{38}$$

which yields

$$n_0 = n_0(\Phi_1). \tag{39}$$

Substituting (33), (36), (37) and (39) into (8) and (9), we obtain the leading order ion pressure equations

$$\left[1 + 2\frac{\lambda_H - \lambda_i}{e n_0^2}n_0'(\Phi_1)p_{i\perp 1}\right]\{p_{i\perp 1},\, \Phi_1\} = 0, \tag{40}$$

$$\left[1 + \frac{\lambda_H - \lambda_i}{e n_0}p_{i\perp 1}'(\Phi_1)\right]\{p_{i\parallel 1},\, \Phi_1\} = 0, \tag{41}$$

from which we choose to obtain

$$p_{i\perp 1} = p_{i\perp 1}(\Phi_1), \tag{42}$$

$$p_{i\parallel 1} = p_{i\parallel 1}(\Phi_1). \tag{43}$$

The leading order of the projection of the ion momentum balance (7) along $\mathbf{B}$ yields

$$\left[1 - \frac{\lambda_H}{e n_0}p_{i\parallel 1}'(\Phi_1)\right]\{\Phi_1,\, \psi_1\} = 0, \tag{44}$$

from which we choose to obtain

$$\Phi_1 = \Phi_1(\psi_1). \tag{45}$$

Then, we obtain the following relations,

$$n_0 = n_0(\psi_1), \tag{46}$$

$$p_{i\parallel 1} = p_{i\parallel 1}(\psi_1), \tag{47}$$

$$p_{i\perp 1} = p_{i\perp 1}(\psi_1), \tag{48}$$

The leading order of the electron pressure equations are

$$n_0^{-1}\{p_{e\parallel 1},\, \psi_1\} = 0, \tag{49}$$

$$p_{e\parallel 1}\{p_{e\perp 1},\, \psi_1\} = 0, \tag{50}$$

which yield

$$p_{e\parallel 1} = p_{e\parallel 1}(\psi_1), \tag{51}$$

$$p_{e\perp 1} = p_{e\perp 1}(\psi_1), \tag{52}$$

From (30), (48) and (52), we obtain

$$I_1 = I_1(\psi_1). \tag{53}$$

The second-order of the generalized Ohm's law (5) is

$$\left\{\Phi_2 - \Phi_1'\psi_2 - \frac{\lambda_H}{en_0}\left[p_{e\parallel 2} - p_{e\parallel 1}'\psi_2 + \left(\frac{x}{R_0}\right)(p_{e\parallel 1} - p_{e\perp 1})\right], \psi_1\right\} = 0. \tag{54}$$

The second order equations for the electron pressures are

$$\left\{n_0(p_{e\parallel 2} - p_{e\parallel 1}'\psi_2) - p_{e\parallel 1}(n_1 - n_0'\psi_2), \psi_1\right\} = 0, \tag{55}$$

$$\left\{p_{e\perp 2} - p_{e\perp 1}'\psi_2 - \frac{p_{e\perp 1}}{n_0}(n_1 - n_0'\psi_2) + \left(\frac{x}{R_0}\right)\frac{p_{e\perp 1}}{p_{e\parallel 1}}(p_{e\parallel 1} - p_{e\perp 1}), \psi_1\right\} = 0, \tag{56}$$

The ion gyroviscous force (A.1) reduces to

$$\nabla \cdot \Pi_i^{gv} \simeq -\frac{m_i p_{i\perp 1}'}{eB_0}\left(R_0\nabla\varphi \times \nabla\psi_1\right)\cdot\nabla\mathbf{v}^{(1)} - \nabla\left(\chi_v + \chi_q\right). \tag{57}$$

The first order of the ion flow velocity now reads

$$\mathbf{v}^{(1)} = -\frac{R_0}{B_0}\left(\Phi_1' + \frac{\lambda_H}{en_0}p_{i\perp 1}'\right)\nabla\psi_1 \times \nabla\varphi + v_\parallel \nabla\varphi. \tag{58}$$

The second order of the ion flow velocity appears in convective terms as

$$\mathbf{v}^{(2)}\cdot\nabla\psi_1 = \frac{R_0}{B_0}\left\{\Phi_2 + \frac{\lambda_H}{en_0}\left[p_{i\perp 2} - \left(\frac{x}{R_0}\right)\left(p_{i\parallel 1} - p_{i\perp 1}\right)\right], \psi_1\right\}. \tag{59}$$

The first order of the divergence of the ion flow velocity vanishes,

$$(\nabla \cdot \mathbf{v})^{(1)} = 0. \tag{60}$$

The second order of the divergence of the ion flow velocity is

$$\begin{aligned}(\nabla\cdot\mathbf{v})^{(2)} = \frac{R_0}{B_0}\Bigg\{&\frac{v_\parallel}{R_0} + \frac{\lambda_H}{en_0^2}(p_{i\perp 1}'n_1 - n_0'p_{i\perp 2}) \\ &-\left(\frac{2x}{R_0}\right)\left[\Phi_1' + \frac{\lambda_H}{2en_0}\left(p_{i\parallel 1}' + p_{i\perp 1}'\right) - \frac{\lambda_H}{2en_0}\frac{n_0'}{n_0}(p_{i\parallel 1} - p_{i\perp 1})\right], \psi_1\Bigg\}.\end{aligned} \tag{61}$$

Similarly, the first and second order of the divergence of the perpendicular heat fluxes read

$$\nabla\cdot\mathbf{q}_{sT\perp}^{(1)} = \nabla\cdot\mathbf{q}_{sB\perp}^{(1)} = 0, \tag{62}$$

$$\begin{aligned}(\nabla\cdot\mathbf{q}_{sB\perp})^{(2)} = \frac{R_0}{2e_s n_0 B_0}\Bigg\{&p_{s\perp 1}'p_{s\parallel 2} - p_{s\parallel 1}'p_{s\perp 2} + \frac{p_{s\parallel 1}}{n_0}\left(n_0'p_{s\perp 2} - p_{s\perp 1}'n_1\right) \\ &-\left(\frac{2x}{R_0}\right)p_{s\parallel 1}\left[2p_{s\parallel 1}' - p_{s\perp 1}' - \frac{p_{s\parallel 1}n_0'}{n_0}\right], \psi_1\Bigg\},\end{aligned} \tag{63}$$

$$(\nabla\cdot\mathbf{q}_{sT\perp})^{(2)} = \frac{2R_0 p_{s\perp 1}}{e_s n_0 B_0}\left\{\frac{1}{n_0}\left(n_0'p_{s\perp 2} - p_{s\perp 1}'n_1\right) - \left(\frac{2x}{R_0}\right)\left(p_{s\perp 1}' - \frac{n_0'}{n_0}p_{s\perp 1}\right), \psi_1\right\}. \tag{64}$$

The parallel momentum balance equation (28) yields

$$\left\{ m_i n_0 R_0 \left(\Phi_1' + \frac{\lambda_H - \lambda_i}{e n_0} p_{i\perp 1}' \right) v_\parallel \right.$$
$$\left. - \sum_{s=i,e} \left[p_{s\parallel 2} - p_{s\parallel 1}' \psi_2 + \left(\frac{x}{R_0} \right) (p_{s\parallel 1} - p_{s\perp 1}) \right], \psi_1 \right\} = 0, \qquad (65)$$

which is the Bernoulli law in the present system. The first order of the equation of continuity (1) is

$$\left\{ \Phi_1'(n_1 - n_0'\psi_2) - n_0'(\Phi_2 - \Phi_1'\psi_2) - \frac{n_0}{R_0} v_\parallel \right.$$
$$\left. + \left(\frac{x}{R_0} \right) n_0 \left[2\Phi_1' + \frac{\lambda_H}{e n_0} (p_{i\perp 1}' + p_{i\parallel 1}') \right], \psi_1 \right\} = 0. \qquad (66)$$

From (8) - (11), the equations for the second-order quantities of the pressures are obtained as

$$\left\{ \left(\Phi_1' + 2 \frac{\lambda_H - \lambda_i}{e n_0} \frac{n_0'}{n_0} p_{i\perp 1} \right) (p_{i\perp 2} - p_{i\perp 1}' \psi_2) - p_{i\perp 1}' (\Phi_2 - \Phi_1' \psi_2) \right.$$
$$- 2 \frac{\lambda_H - \lambda_i}{e n_0} \frac{p_{i\perp 1}' p_{i\perp 1}}{n_0} (n_1 - n_0' \psi_2) - \frac{p_{i\perp 1}}{R_0} v_\parallel - \frac{\lambda_{i\parallel}}{R_0} q_{iT\parallel}$$
$$+ \left(\frac{x}{R_0} \right) \left\{ 3 p_{i\perp 1} \Phi_1' + \frac{2\lambda_H}{e n_0} \left[p_{i\perp 1} \left(p_{i\parallel 1}' + p_{i\perp 1}' \right) + \left(p_{i\perp 1}' - \frac{n_0'}{n_0} p_{i\perp 1} \right) \left(p_{i\parallel 1} - p_{i\perp 1} \right) \right] \right.$$
$$\left. \left. + \frac{\lambda_i}{e n_0} p_{i\perp 1} \left[4 \left(p_{i\perp 1}' - \frac{n_0'}{n_0} p_{i\perp 1} \right) - \left(p_{i\parallel 1}' - \frac{n_0'}{n_0} p_{i\parallel 1} \right) \right] \right\}, \psi_1 \right\} = 0, \qquad (67)$$

$$\left\{ \frac{1}{2} \left(\Phi_1' + \frac{\lambda_H - \lambda_i}{e n_0} p_{i\perp 1}' \right) (p_{i\parallel 2} - p_{i\parallel 1}' \psi_2) \right.$$
$$- \frac{1}{2} \frac{\lambda_H - \lambda_i}{e n_0} \left(p_{i\parallel 1}' - \frac{n_0'}{n_0} p_{i\parallel 1} \right) (p_{i\perp 2} - p_{i\perp 1}' \psi_2) - \frac{p_{i\parallel 1}'}{2} (\Phi_2 - \Phi_1' \psi_2)$$
$$- \frac{1}{2} \frac{\lambda_H - \lambda_i}{e n_0} \frac{p_{i\perp 1}' p_{i\parallel 1}}{n_0} (n_1 - n_0' \psi_2) - \frac{3 p_{i\parallel 1}}{2 R_0} v_\parallel - \frac{\lambda_{i\parallel}}{R_0} q_{iB\parallel}$$
$$+ \left(\frac{x}{R_0} \right) \left\{ 2 p_{i\parallel 1} \left(\Phi_1' + \frac{\lambda_H}{e n_0} p_{i\perp 1}' \right) \right.$$
$$+ \frac{1}{2} \frac{\lambda_H}{e n_0} \left[\left(p_{i\parallel 1}' - \frac{n_0'}{n_0} p_{i\parallel 1} \right) \left(p_{i\parallel 1} - p_{i\perp 1} \right) + p_{i\parallel 1} \left(p_{i\parallel 1}' - p_{i\perp 1}' \right) \right]$$
$$\left. \left. + \frac{\lambda_i}{e n_0} \left[\left(p_{i\parallel 1}' - \frac{n_0'}{n_0} p_{i\parallel 1} \right) \left(p_{i\parallel 1} + p_{i\perp 1} \right) + p_{i\parallel 1} \left(p_{i\parallel 1}' - p_{i\perp 1}' \right) \right] \right\}, \psi_1 \right\} = 0, \qquad (68)$$

$$\left\{ \left(\Phi_1' - 2 \frac{\lambda_H - \lambda_i}{e n_0} \frac{n_0'}{n_0} p_{e\perp 1} \right) (p_{e\perp 2} - p_{e\perp 1}' \psi_2) - p_{e\perp 1}' (\Phi_2 - \Phi_1' \psi_2) \right.$$
$$+ 2 \frac{\lambda_H - \lambda_i}{e n_0} \frac{p_{e\perp 1}' p_{e\perp 1}}{n_0} (n_1 - n_0' \psi_2) - \frac{p_{e\perp 1}}{R_0} v_\parallel - \frac{q_{eT\parallel}}{R_0}$$
$$- \left(\frac{x}{R_0} \right) \{ -3 p_{e\perp 1} \Phi_1'$$

$$+\frac{\lambda_H}{en_0}\left[-p_{e\perp1}\left(p'_{i\|1}+p'_{i\perp1}-p'_{e\|1}\right)+\left(p'_{e\perp1}-2\frac{n'_0}{n_0}p_{e\perp1}\right)\left(p_{e\|1}-p_{e\perp1}\right)\right]$$
$$+4\frac{\lambda_i}{en_0}\left(p'_{e\perp1}-\frac{n'_0}{n_0}p_{e\perp1}\right)p_{e\perp1}\Bigg\},\psi_1\Bigg\}=0, \tag{69}$$

$$\Bigg\{\frac{1}{2}\left(\Phi'_1-\frac{\lambda_H-\lambda_i}{en_0}p'_{e\perp1}\right)\left(p_{e\|2}-p'_{e\|1}\psi_2\right)$$
$$+\frac{1}{2}\frac{\lambda_H-\lambda_i}{en_0}\left(p'_{e\|1}-\frac{n'_0}{n_0}p_{e\|1}\right)\left(p_{e\perp2}-p'_{e\perp1}\psi_2\right)-\frac{p'_{e\|1}}{2}\left(\Phi_2-\Phi'_1\psi_2\right)$$
$$+\frac{1}{2}\frac{\lambda_H-\lambda_i}{en_0}\frac{p'_{e\perp1}p_{e\|1}}{n_0}\left(n_1-n'_0\psi_2\right)-\frac{3p_{e\|1}}{2R_0}v_\|-\frac{q_{eB\|}}{R_0}$$
$$-\left(\frac{x}{R_0}\right)\Big\{-2p_{e\|1}\Phi'_1$$
$$+\frac{1}{2}\frac{\lambda_H}{en_0}\left\{\left(p'_{e\|1}-\frac{n'_0}{n_0}p_{e\|1}\right)\left(p_{e\|1}-p_{e\perp1}\right)-p_{e\|1}\left[3\left(p'_{i\|1}+p'_{i\perp1}\right)+2p'_{e\|1}\right]\right\}$$
$$+\frac{\lambda_i}{en_0}\left[\left(p'_{e\|1}-\frac{n'_0}{n_0}p_{e\|1}\right)\left(p_{e\|1}+p_{e\perp1}\right)+p_{e\|1}\left(p'_{e\|1}-p'_{e\perp1}\right)\right]\Big\},\psi_1\Bigg\}=0. \tag{70}$$

Equations (69) and (70) are used to obtain the electron parallel heat fluxes. The equations for ion parallel heat fluxes (21) and (20) yields

$$\Bigg\{\left(\Phi'_1+\frac{\lambda_H-\lambda_i}{en_0}p'_{i\perp1}\right)q_{iB\|}$$
$$-\frac{3}{2}\frac{p_{i\|1}}{m_in_0R_0}\left[p_{i\|2}-p'_{i\|1}\psi_2-\frac{p_{i\|1}}{n_0}\left(n_1-n'_0\psi_2\right)\right],\psi_1\Bigg\}=0, \tag{71}$$

$$\Bigg\{\left(\Phi'_1+\frac{\lambda_H-\lambda_i}{en_0}p'_{i\perp1}\right)q_{iT\|}-\frac{p_{i\|1}}{m_in_0R_0}\left[p_{i\perp2}-p'_{i\perp1}\psi_2-\frac{p_{i\perp1}}{n_0}\left(n_1-n'_0\psi_2\right)\right]$$
$$-\left(\frac{x}{R_0}\right)\frac{p_{i\perp1}\left(p_{i\|1}-p_{i\perp1}\right)}{m_in_0R_0},\psi_1\Bigg\}=0. \tag{72}$$

In (71) and (72), the contribution of ion diamagnetic flow is cancelled due to the ion FLR effects, which agrees with the reduced equations for parallel heat flux obtained in [24] and [25]. The first order of the poloidal momentum balance (29) is

$$p_{i\perp2}+p_{e\perp2}+\frac{B_0}{\mu_0R_0}I_2-\left(\frac{x}{R_0}\right)\sum_{s=i,e}\left(p_{s\|1}-p_{s\perp1}\right)\equiv g_*\left(\psi_1\right), \tag{73}$$

where g_* is an arbitrary function of ψ_1. Eliminating variables from (54) - (56), (65) - (68), (71) and (72), we obtain the following coupled equations for the perpendicular and parallel pressures for ions ,

$$\left\{A_1\left(\psi_1\right)\left(p_{i\perp2}-p'_{i\perp1}\psi_2\right)+A_2\left(\psi_1\right)\left(p_{i\|2}-p'_{i\|1}\psi_2\right)+(x/R_0)A_3\left(\psi_1\right),\psi_1\right\}=0, \tag{74}$$

$$\left\{B_1\left(\psi_1\right)\left(p_{i\perp2}-p'_{i\perp1}\psi_2\right)+B_2\left(\psi_1\right)\left(p_{i\|2}-p'_{i\|1}\psi_2\right)+(x/R_0)B_3\left(\psi_1\right),\psi_1\right\}=0, \tag{75}$$

where the explicit forms of the coefficients A_1, A_2, A_3, B_1, B_2 and B_3 are shown in Appendix B. Since the double adiabatic ion pressure case, $\lambda_{i\|}=0$, does not coincide with adiabatic pressure even when $p_{i\|1} = p_{i\perp 1}$, we introduce another index δ in order to include the isotropic, adiabatic ion pressure case as

$$p_{i\|1} - p_{i\perp 1} \equiv \delta p_{i\Delta 1}, \tag{76}$$

$$p_{i\|2} - p_{i\perp 2} \equiv \delta p_{i\Delta 2}, \tag{77}$$

where $p_{i\Delta 1}$ and $p_{i\Delta 2}$ satisfy

$$\left\{ p_{i\Delta 2} - p'_{i\Delta 1}\psi_2 + \left(\frac{x}{R_0}\right) \frac{-A_3\left(B_1 + B_2\right) + B_3\left(A_1 + A_2\right)}{A_1 B_2 - A_2 B_1}, \psi_1 \right\} = 0, \tag{78}$$

$$\begin{aligned} \{ &\left(A_1 + B_1 + A_2 + B_2\right)\left(p_{i\perp 2} - p'_{i\perp 1}\psi_2\right) + \delta\left(A_2 + B_2\right)\left(p_{i\Delta 2} - p'_{i\Delta 1}\psi_2\right) \\ &+ (x/R_0)\left(A_3 + B_3\right), \psi_1 \} = 0. \end{aligned} \tag{79}$$

By setting $\delta = 0$, $p_{i\|1} = p_{i\perp 1}$ and $p_{e\|1} = p_{e\perp 1}$, we get FLR two-fluid model with the isotropic, adiabatic pressure for ions and the isothermal pressure for electrons. Substituting (78) into (79), we obtain

$$p_{s\perp 2} = p'_{s\perp 1}\psi_2 + \left(\frac{x}{R_0}\right) C_{s\perp}\left(\psi_1\right) + P_{s\perp 2*}\left(\psi_1\right), \tag{80}$$

$$p_{s\|2} = p'_{s\|1}\psi_2 + \left(\frac{x}{R_0}\right) C_{s\|}\left(\psi_1\right) + P_{s\|2*}\left(\psi_1\right), \tag{81}$$

$$\Phi_2 = \Phi'_1\psi_2 + \left(\frac{x}{R_0}\right) \lambda_H C_{\Phi}\left(\psi_1\right) + \Phi_{2*}\left(\psi_1\right), \tag{82}$$

$$v_\| = \left(\frac{x}{R_0}\right) C_{v\|}\left(\psi_1\right) + v_{\|*}\left(\psi_1\right), \tag{83}$$

$$n_1 = n'_0\psi_2 + \left(\frac{x}{R_0}\right) C_n\left(\psi_1\right) + n_{1*}\left(\psi_1\right), \tag{84}$$

$$q_{sT\|} = \left(\frac{x}{R_0}\right) C_{sqT\|}\left(\psi_1\right) + q_{sT\|*}\left(\psi_1\right), \tag{85}$$

$$q_{sB\|} = \left(\frac{x}{R_0}\right) C_{sqB\|}\left(\psi_1\right) + q_{sB\|*}\left(\psi_1\right), \tag{86}$$

$$I_2 = I'_1\psi_2 + \left(\frac{x}{R_0}\right) C_I\left(\psi_1\right) + \frac{\mu_0 R_0}{B_0}\left[g_*\left(\psi_1\right) - \sum_{s=i,e} P_{s\perp 2*}\left(\psi_1\right)\right], \tag{87}$$

where the explicit forms of the coefficients $C_{s\perp}$, $C_{s\|}$, C_Φ, $C_{v\|}$, C_n, $C_{sqT\|}$, $C_{sqB\|}$ and C_I are shown in (C.1) - (C.12) of Appendix C, while $P_{s\perp 2*}$, $P_{s\|2*}$, Φ_{2*}, $v_{\|*}$, n_{1*}, $q_{sT\|*}$, and $q_{sB\|*}$ are arbitrary functions of ψ_1. Since the terms with (x/R_0) in (80) - (87) represents the poloidal-angle dependence, these higher-order quantities indicate the shift of their isosurfaces from the magnetic surfaces. This dependence does not appear in static equilibria of single-fluid MHD with isotropic pressure. In the presence of flow, isosurfaces of the isotropic pressure are shifted inwards or outwards depending on the poloidal flow velocity relative to the poloidal sound velocity [20]. On the other hand, the anisotropic pressures (80) are not constant on a magnetic surface even in the static case. Equation (82) shows that the poloidal-angle dependence appears due to the two-fluid effect. In section 4.2, we discuss this kind of shift of the isosurfaces in the stream functions for ions and electrons.

The first order of the radial momentum balance (27) gives the equation for ψ_1,

$$\left(\frac{\partial^2}{\partial R^2}+\frac{\partial^2}{\partial Z^2}\right)\psi_1=-\mu_0 R_0^2\left[\left(\frac{x}{R_0}\right)\sum_{s=i,e}(p'_{s\|1}+p'_{s\perp 1})+g'_*\right]-\left(\frac{I_1^2}{2}\right)'. \tag{88}$$

The second order of the poloidal momentum balance (29) yields

$$\frac{B_0 I_3}{\mu_0 R_0}+\sum_{s=i,e}\left[p_{s\perp 3}-\left(\frac{x}{R_0}\right)\left(p_{s\|2}-p_{s\perp 2}\right)+\frac{1}{2}\left(\frac{x}{R_0}\right)^2\left(C_{s\|}+C_{s\perp}-p_{s\|1}+p_{s\perp 1}\right)\right]$$
$$+\frac{I_1}{\mu_0 R_0^2}\left(I_2-I_1'\psi_2\right)-g'_*\psi_2+F\frac{|\nabla\psi_1|^2}{2\mu_0}-\lambda_i(\chi_v+\chi_q)\equiv E_*\left(\psi_1\right), \tag{89}$$

where E_* is an arbitrary function of ψ_1 and

$$F(\psi_1)=\left(\frac{B_0^2}{\mu_0}\right)^{-1}\left[m_i n_0 R_0^2\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p'_{i\perp 1}\right)\left(\Phi_1'+\frac{\lambda_H}{en_0}p'_{i\perp 1}\right)\right.$$
$$\left.+\sum_{s=i,e}(p_{s\|1}-p_{s\perp 1})\right]. \tag{90}$$

From the second order of the radial momentum balance (27), we obtain the equation for ψ_2

$$\left(\frac{\partial^2}{\partial R^2}+\frac{\partial^2}{\partial Z^2}\right)\psi_2+\left[\mu_0 R_0^2\left(\frac{x}{R_0}\right)\sum_{s=i,e}\left(p''_{s\perp 1}+p''_{s\|1}\right)+\mu_0 R_0^2 g''_*+\left(\frac{I_1^2}{2}\right)''\right]\psi_2$$
$$=\left(\frac{1}{R}\frac{\partial}{\partial R}\right)\psi_1+F\left(\frac{\partial^2}{\partial R^2}+\frac{\partial^2}{\partial Z^2}\right)\psi_1+F'\frac{|\nabla\psi_1|^2}{2}$$
$$-\mu_0 R_0^2\left[E'_*+\left(\frac{x}{R_0}\right)\sum_{s=i,e}\left(P'_{s\perp 2*}+P'_{s\|2*}\right)\right.$$
$$\left.+\frac{1}{2}\left(\frac{x}{R_0}\right)^2\sum_{s=i,e}\left(p'_{s\perp 1}+p'_{s\|1}+C'_{s\perp}+C'_{s\|}\right)\right], \tag{91}$$

Equations (88) and (91) form the expanded GS equation in the presence of poloidal-sonic flow, two-fluid and ion FLR effects, and anisotropic ion and electron pressures. The GS equation for ψ_1, (88), is same as for the single-fluid, static case while the one for ψ_2, (91), is modified by the effects of flow, two-fluid and ion FLR. The coefficient F, (90), represents the convection due to the $E\times B$ flow and the ion diamagnetic flow that is cancelled in the presence of the gyroviscosity (gyroviscous cancellation), and pressure anisotropy. The diamagnetic flow causes asymmetry in the equilibrium equations with respect to the sign of the $E\times B$ flow. In spite of its complexity, the GS equation for ψ_2, (91), is a linear, elliptic partial differential equation once the solution for ψ_1 of

(88) is substituted and, thus, is easy to solve in regions where it is regular. There is singularity in the coefficients $C'_{s\perp}$ and $C'_{s\|}$ of (91) where the poloidal flow velocity equals the phase velocity of a sound wave. We discuss this singularity in section 4.1. The other higher order quantities (80) - (87) are determined by ψ_1 and ψ_2. The ordering for poloidal-sonic flow requires the third-order accuracy for the total energy. However, the third-order accuracy is needed in the sum of the pressures plus the magnetic energy as in (89), and the pressures and the magnetic energy themselves are required only up to the second-order [3].

4. Discussion

4.1. Singularity

Singularity in the GS equation for ψ_2 occurs when the denominators of $C'_{s\perp}$ and $C'_{s\|}$ equal zero. It arises because higher order terms not negligible in its vicinity are ordered out in the asymptotic expansions. For the case $\left(\lambda_H, \lambda_i, \lambda_{i\|}\right) = (0, 0, 0)$, single-fluid MHD in the absence of the parallel heat flux for ions, and $\delta = 0$, $p_{i\|1} = p_{i\perp 1}$ and $p_{e\|1} = p_{e\perp 1}$, i.e., the isotropic pressures for both ions and electrons, the singularity appears when the poloidal flow velocity equals the phase velocity of the slow magnetosonic wave with adiabatic ion and isothermal electron pressures,

$$m_i n_0 R_0^2 \Phi_1'^2 = \frac{5}{3} p_{i1} + p_{e1}. \tag{92}$$

The denominators of $C'_{s\perp}$ and $C'_{s\|}$ require that the poloidal flow velocity is well separated from the phase velocities of the sound waves,

$$\left| m_i n_0 R_0^2 \Phi_1'^2 - \left(\frac{5}{3} p_{i1} + p_{e1} \right) \right| \sim m_i n_0 R_0^2 \Phi_1'^2 \sim \frac{5}{3} p_{i1} + p_{e1}, \tag{93}$$

in the single-fluid model for example. This condition excludes the transonic hyperbolic region, $\left| m_i n_0 R_0^2 \Phi_1'^2 - \left(\frac{5}{3} p_{i1} + p_{e1} \right) \right| \leq \varepsilon m_i n_0 R_0^2 \Phi_1'^2$, where a shock solution appears, as well as the singularity itself. Otherwise, higher-order terms, including those of the Hall current that resolve the transonic hyperbolic region and the singularity, have to be taken into account. The present model is to study the extension of regular, elliptic solution for single-fluid MHD equilibria with flow. One has to choose the profiles of free functions that do not include the vicinity of singularity to get regular solutions.

If we induce pressure anisotropy with $\delta = 1$, $p_{i\|1} \neq p_{i\perp 1}$ and $p_{e\|1} \neq p_{e\perp 1}$, the double adiabatic ion pressure and the anisotropic pressure for massless electrons, the difference from (92) is quantitative,

$$m_i n_0 R_0^2 \Phi_1'^2 = 3 p_{i\|1} + p_{e\|1}. \tag{94}$$

In the presence of the parallel heat flux for ions, $\lambda_{i\|} = 1$, the characteristics of the singularity changes qualitatively. The singularity appears when

$$(A_1 B_2 - A_2 B_1) \left[m_i n_0 R_0^2 \left(\Phi_1' + \frac{\lambda_H - \lambda_i}{e n_0} p'_{i\perp 1} \right) \right]^2$$

$$\times\left[m_i n_0 R_0^2\left(\Phi_1' - \frac{\lambda_H}{en_0}\frac{n_0'}{n_0}p_{e\parallel 1}\right)\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp 1}'\right) - p_{e\parallel 1}\right] = 0. \tag{95}$$

For the single-fluid case, $(\lambda_H, \lambda_i) = (0, 0)$, equation (95) gives

$$m_i n_0 R_0^2 \Phi_1'^2 = p_{i\parallel 1}, \tag{96}$$

$$m_i n_0 R_0^2 \Phi_1'^2 = \frac{1}{2}\left(6p_{i\parallel 1} + p_{e\parallel 1} \pm \sqrt{24p_{i\parallel 1}^2 + p_{e\parallel 1}^2}\right), \tag{97}$$

where the poloidal flow velocity equals the phase velocities of either slow magnetosonic or two ion acoustic waves that arise from the heat flux equations. We note that this case coincides with the fluid model in [26] if the electron mass is completely neglected and that the phase velocities in (96) and (97) agree with the wave dispersion relation in that model. Equation (95) suggests that the effects of two-fluid and FLR do not bring additional singular points but shift the ones found in the single-fluid case. For the FLR two-fluid case, $(\lambda_H, \lambda_i) = (1, 1)$, equation (96) remains same due to the gyroviscous cancellation while (97) is modified to

$$m_i n_0 R_0^2 \Phi_1'^2\left(m_i n_0 R_0^2 \Phi_1'^2 - 6p_{i\parallel 1} - p_{e\parallel 1}\right) + 3p_{i\parallel 1}\left(p_{i\parallel 1} + p_{e\parallel 1}\right)$$
$$- m_i n_0 R_0^2 \Phi_1' \frac{p_{e\parallel 1}}{en_0}\left[\frac{n_0'}{n_0}\left(m_i n_0 R_0^2 \Phi_1'^2 - 6p_{i\parallel 1}\right) - p_{i\parallel 1}'\right] = 0. \tag{98}$$

Equation (98) is asymmetric with respect to the sign of the $E \times B$ flow in contrast to (97).

4.2. Toroidal and poloidal flows, diamagnetic current

In the standard formulation for axisymmetric equilibria with flow, the ion flow velocity $\mathbf{v}$ is expressed as

$$n\mathbf{v} = \nabla\Psi \times \nabla\varphi + nRv_\varphi \nabla\varphi. \tag{99}$$

The ion stream function Ψ and the toroidal flow velocity v_φ are also expanded as

$$\Psi = \Psi_1 + \Psi_2 + \ldots,$$

$$v_\varphi = v_{\varphi 1} + \ldots$$

In the leading order, the toroidal flow velocity is equal to the parallel flow velocity,

$$v_{\varphi 1} = v_\parallel. \tag{100}$$

Compared to (58), the projection of the leading order of (99) along $\nabla\psi_1$ yields $\Psi_1 = \Psi_1(\psi_1)$ and, then, the projection along $\nabla\psi_1 \times \nabla\varphi$ gives

$$\Psi_1'(\psi_1) = -\frac{R_0}{B_0}n_0\left(\Phi_1' + \frac{\lambda_H}{en_0}p_{i\perp 1}'\right). \tag{101}$$

Substituting (99) into the projection of the ion momentum balance (7) along $\nabla\psi \times \nabla\varphi$ yields

$$-\{\Phi, \psi\} \simeq \frac{I}{nR^2}\{\Psi, \psi\} + \frac{\lambda_H}{ne}(\nabla\psi \times \nabla\varphi)\cdot\left[\nabla p_{i\perp} + \mathbf{B}\cdot\nabla\left(\frac{p_{i\parallel} - p_{i\perp}}{B^2}\mathbf{B}\right)\right]. \tag{102}$$

The second order of (102) is

$$\left\{\Psi_2 + \frac{n_0 R_0}{B_0}\left[\Phi_2 + \frac{\lambda_H}{en_0}p_{i\perp 2} - \left(\frac{x}{R_0}\right)\frac{\lambda_H}{en_0}(p_{i\|1} - p_{i\perp 1})\right], \psi_1\right\} = 0, \quad (103)$$

which yields

$$\Psi_2 = \Psi_1'\psi_2 + \left(\frac{x}{R_0}\right)\lambda_H C_\Psi(\psi_1) + \Psi_{2*}(\psi_1), \quad (104)$$

where the explicit form of the coefficient C_Ψ is shown in (C.13) of Appendix C. Equation (104) shows that the ion stream function is not constant on a magnetic surface for two-fluid equilibria, $\lambda_H = 1$. This feature is different from the pressure profile mentioned in section 3, and is in common with non-reduced two-fluid equilibria with zero Larmor radius [6, 8, 7, 11], which correspond to the case $(\lambda_H, \lambda_i) = (1, 0)$. The scale length of the shift of the isosurfaces of the ion stream function is the order of εa, which is same as that of the ion Larmor radius in the present orderings.

The electron flow velocity is also defined as

$$n\mathbf{v}_e = \nabla\Psi_e \times \nabla\varphi + nRv_{e\varphi}\nabla\varphi. \quad (105)$$

The electron stream function Ψ_e can be obtained from (5) in analogy to the ion stream function by means of an asymptotic expansion,

$$\Psi_e = \Psi_{e1} + \Psi_{e2} + \dots,$$

$$\Psi_{e1}'(\psi_1) = -\frac{R_0}{B_0}n_0\left(\Phi_1' - \frac{\lambda_H}{en_0}p_{e\perp 1}'\right). \quad (106)$$

$$\left\{\Psi_{e2} + \frac{n_0 R_0}{B_0}\left[\Phi_2 - \frac{\lambda_H}{en_0}p_{e\perp 2} + \left(\frac{x}{R_0}\right)\frac{\lambda_H}{en_0}(p_{e\|1} - p_{e\perp 1})\right], \psi_1\right\} = 0. \quad (107)$$

Equation (107) is rewritten as

$$\left\{\Psi_{e2} - \Psi_{e1}'\psi_2 + \frac{\lambda_H R_0}{eB_0}\left[p_{e\|2} - p_{e\perp 2} - \left(p'_{e\|1} - p'_{e\perp 1}\right)\psi_2 + \left(\frac{2x}{R_0}\right)(p_{e\|1} - p_{e\perp 1})\right]\right.$$
$$\left., \psi_1\right\} = 0, \quad (108)$$

which yields

$$\Psi_{e2} = \Psi_{e1}'\psi_2 + \left(\frac{x}{R_0}\right)\lambda_H C_{\Psi e}(\psi_1) + \Psi_{e2*}(\psi_1), \quad (109)$$

where the explicit form of the coefficient $C_{\Psi e}$ is shown in (C.13) of Appendix C. Equation (108) shows that the isosurfaces of the electron stream function shifts from the magnetic flux surfaces when both of two-fluid effects and pressure anisotropy for electrons exist, which is consistent with its non-reduced form [11]. The toroidal flow velocity for electrons $v_{e\varphi}$ is expanded as

$$v_{e\varphi} = v_{e\varphi 1} + \dots$$

In the leading order, this is equal to the parallel flow velocity for electrons,

$$v_{e\varphi 1} = v_{e\|}, \quad (110)$$

where

$$v_{e\parallel} = v_{\parallel} - \frac{\lambda_H}{ne} j_{\parallel}, \tag{111}$$

$$\begin{aligned}\frac{j_{\parallel}}{ne} &\simeq \frac{1}{\mu_0 e n_0 R_0}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_1 \\ &= -\frac{R_0}{en_0}\left[\left(\frac{x}{R_0}\right)\sum_{s=i,e}(p'_{s\parallel 1} + p'_{s\perp 1}) + g'_* + \frac{1}{\mu_0 R_0^2}\left(\frac{I_1^2}{2}\right)'\right].\end{aligned} \tag{112}$$

Substituting (83) and (112) into (111), we get

$$v_{e\parallel} = \left(\frac{x}{R_0}\right) C_{ve\parallel}(\psi_1) + v_{\parallel *}(\psi_1) + \frac{\lambda_H R_0}{en_0}\left[g'_* + \frac{1}{\mu_0 R_0^2}\left(\frac{I_1^2}{2}\right)'\right], \tag{113}$$

where the explicit form of the coefficient $C_{ve\parallel}$ is shown in (C.15) of Appendix C.

The perpendicular (diamagnetic) current is measured to identify pressure anisotropy in experiment. It is written as

$$\mathbf{j}_{\perp} \simeq \frac{\mathbf{B}}{B^2} \times \sum_{s=i,e}\left[\nabla p_{s\perp} + \mathbf{B}\cdot\nabla\left(\frac{p_{s\parallel} - p_{s\perp}}{B^2}\mathbf{B}\right)\right] \tag{114}$$

The radial component of (114) is given by

$$\mathbf{j}_{\perp} \cdot \nabla\psi \simeq \frac{1}{B_0}\sum_{s=i,e}\left[C_{s\perp} - (p_{s\parallel 1} - p_{s\perp 1})\right]\{x, \psi_1\}. \tag{115}$$

5. Summary

We have derived a set of reduced equilibrium equations for high-beta tokamaks from the fluid moment equations for collisionless, magnetized plasmas. It takes a form of the expanded GS equation including effects of toroidal and poloidal flow comparable to the poloidal sound velocity, two-fluid, ion finite Larmor radius, pressure anisotropy and parallel heat fluxes. The second-order GS equation includes the terms representing the convection due to the $E \times B$ flow and the ion diamagnetic flow that is cancelled in the presence of the gyroviscosity (gyroviscous cancellation) and pressure anisotropy. The two-fluid effects induce the diamagnetic flows, which result in asymmetry of the equilibria with respect to the sign of the $E \times B$ flow. The gyrovisosity and other FLR effects cause the so-called gyroviscous cancellation of the convection due to the ion diamagnetic flow. The study for the singularity with respect to the poloidal flow velocity has shown the qualitative difference between the equilibria with and without the parallel heat fluxes. Higher order terms of quantities like the pressures and the stream functions show the shift of their isosurfaces from the magnetic surfaces due to effects of flow, two-fluid and pressure anisotropy. The reduced form of the diamagnetic current associated with pressure anisotropy is also obtained.

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Appendix A. Gyroviscous force

Ion gyroviscous force was obtained in [15]. Keeping only terms required to the orders considered, it is written as

$$\nabla \cdot \Pi_i^{gv} = \nabla \cdot \left(\sum_{N=1}^{5} \Pi_i^{gvN} \right), \tag{A.1}$$

$$\nabla \cdot \Pi_i^{gv1} \simeq - m_i n \mathbf{v}_{*i} \cdot \nabla \mathbf{v} - \nabla \chi_v - \nabla \times \left[\frac{m_i p_{i\perp}}{2eB^2} (\nabla \cdot \mathbf{v}) \mathbf{B} \right], \tag{A.2}$$

$$\nabla \cdot \Pi_i^{gv2} \simeq -\nabla \chi_q - \nabla \times \left[\frac{m_i}{4eB^2} (\nabla \cdot \mathbf{q}_{iT\perp}) \mathbf{B} \right], \tag{A.3}$$

$$\nabla \cdot \Pi_i^{gv3} \simeq \nabla \times \left\{ \mathbf{B} \times \left[\frac{m_i}{eB^2} (\mathbf{c} + \mathbf{d}) \right] \right\}, \tag{A.4}$$

$$\nabla \cdot \Pi_i^{gv4} \simeq \nabla \cdot \Pi_i^{gv5} \simeq 0 \tag{A.5}$$

where

$$\mathbf{c} \simeq - \left(\frac{p_{i\parallel} - p_{i\perp}}{B} \right) \nabla \times \left(\frac{1}{en} \nabla p_{i\perp} \right), \tag{A.6}$$

$$\mathbf{d} \simeq \nabla \times \left[\left(2\mathbf{q}_{iB\perp} - \frac{1}{2} \mathbf{q}_{iT\perp} \right) \times \mathbf{b} \right], \tag{A.7}$$

$$\mathbf{v}_{*i} = - \frac{1}{en} \nabla \times \left(\frac{p_{i\perp}}{B^2} \mathbf{B} \right), \tag{A.8}$$

$$\chi_v = \frac{m_i p_{i\perp}}{2eB^2} \mathbf{B} \cdot (\nabla \times \mathbf{v}), \tag{A.9}$$

$$\chi_q = \frac{m_i}{4eB^2} \mathbf{B} \cdot (\nabla \times \mathbf{q}_{iT\perp}). \tag{A.10}$$

Appendix B. Coefficients of (74) and (75)

Coefficients of (74) and (75) are written as

$$A_1(\psi_1) = - \left(\Phi_1' + 2 \frac{\lambda_H - \lambda_i}{en_0} \frac{n_0'}{n_0} p_{i\perp 1} \right) + \frac{\lambda_{i\parallel} p_{i\parallel 1}}{m_i n_0 R_0^2 \left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0} p_{i\perp 1}' \right)}, \tag{B.1}$$

$$\begin{aligned} A_2(\psi_1) = &\left[p_{i\perp 1} \left(\Phi_1' + 2 \frac{\lambda_H - \lambda_i}{en_0} p_{i\perp 1}' \right) + \frac{\lambda_H}{en_0} \left(p_{i\perp 1}' - \frac{n_0'}{n_0} p_{i\perp 1} \right) p_{e\parallel 1} \right. \\ &\left. - \frac{\lambda_{i\parallel} p_{i\parallel 1} p_{i\perp 1}}{m_i n_0 R_0^2 \left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0} p_{i\perp 1}' \right)} \right] \\ &\times \left[m_i n_0 R_0^2 \left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0} p_{i\perp 1}' \right) \left(\Phi_1' - \frac{\lambda_H}{en_0} \frac{n_0'}{n_0} p_{e\parallel 1} \right) - p_{e\parallel 1} \right]^{-1}, \end{aligned} \tag{B.2}$$

$$
\begin{aligned}
A_3(\psi_1) = & -3p_{i\perp1}\Phi_1' - 2\frac{\lambda_H}{en_0}\left[\left(p_{i\|1}' + p_{i\perp1}'\right)p_{i\perp1} + \left(p_{i\perp1}' - \frac{n_0'}{n_0}p_{i\perp1}\right)\left(p_{i\|1} - p_{i\perp1}\right)\right] \\
& -\frac{\lambda_i}{en_0}\left[4\left(p_{i\perp1}' - \frac{n_0'}{n_0}p_{i\perp1}\right) - \left(p_{i\|1}' - \frac{n_0'}{n_0}p_{i\|1}\right)\right]p_{i\perp1} \\
& +\frac{\lambda_{i\|}p_{i\perp1}\left(p_{i\|1} - p_{i\perp1}\right)}{m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right)} \\
& -\left\{\left[\left(p_{e\|1} + p_{e\perp1}\right)\Phi_1' + \frac{\lambda_H}{en_0}\left(p_{i\|1}' + p_{i\perp1}'\right)p_{e\|1}\right]\right. \\
& \times\left[m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right)\frac{\lambda_H}{en_0}p_{i\perp1}' + p_{i\perp1}\right] \\
& -\left(p_{i\|1} - p_{i\perp1}\right)\left[p_{i\perp1}\Phi_1' + \frac{\lambda_H}{en_0}\left(p_{i\perp1}' - \frac{n_0'}{n_0}p_{i\perp1}\right)p_{e\|1}\right] \\
& +\left\{m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right)\left[2\Phi_1' + \frac{\lambda_H}{en_0}\left(p_{i\perp1}' + p_{i\|1}'\right)\right.\right. \\
& \left.\left. -\frac{\lambda_H}{en_0}\frac{n_0'}{n_0}\left(p_{e\|1} - p_{e\perp1}\right)\right] - \sum_{s=i,e}\left(p_{s\|1} - p_{s\perp1}\right)\right\} \\
& \left.\times\left[2\frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'p_{i\perp1} - \frac{\lambda_{i\|}p_{i\|1}p_{i\perp1}}{m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right)}\right]\right\} \\
& \times\left[m_i n_0 R_0^2\left(\Phi_1' - \frac{\lambda_H}{en_0}\frac{n_0'}{n_0}p_{e\|1}\right)\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right) - p_{e\|1}\right]^{-1}, \qquad \text{(B.3)}
\end{aligned}
$$

$$
B_1(\psi_1) = \frac{\lambda_H - \lambda_i}{2en_0}\left(p_{i\|1}' - \frac{n_0'}{n_0}p_{i\|1}\right), \qquad \text{(B.4)}
$$

$$
\begin{aligned}
B_2(\psi_1) = & -\frac{1}{2}\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right) + \frac{3}{2}\frac{\lambda_{i\|}p_{i\|1}}{m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right)} \\
& +\frac{1}{2}\left[3p_{i\|1}\left(\Phi_1' - \frac{\lambda_H}{en_0}\frac{n_0'}{n_0}p_{e\|1}\right) + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'p_{i\|1}\right. \\
& \left. +\frac{\lambda_H}{en_0}p_{i\|1}'p_{e\|1} - \frac{3\lambda_{i\|}p_{i\|1}^2}{m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right)}\right] \\
& \times\left[m_i n_0 R_0^2\left(\Phi_1' - \frac{\lambda_H}{en_0}\frac{n_0'}{n_0}p_{e\|1}\right)\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp1}'\right) - p_{e\|1}\right]^{-1}, \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
B_3(\psi_1) = & -2p_{i\|1}\left(\Phi_1' + \frac{\lambda_H}{en_0}p_{i\perp1}'\right) \\
& -\frac{1}{2}\frac{\lambda_H}{en_0}\left[\left(p_{i\|1}' - \frac{n_0'}{n_0}p_{i\|1}\right)\left(p_{i\|1} - p_{i\perp1}\right) + \left(p_{i\|1}' - p_{i\perp1}'\right)p_{i\|1}\right] \\
& -\frac{\lambda_i}{en_0}\left[\left(p_{i\|1}' - \frac{n_0'}{n_0}p_{i\|1}\right)\left(p_{i\|1} + p_{i\perp1}\right) + \left(p_{i\|1}' - p_{i\perp1}'\right)p_{i\|1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\left\{\left[\left(p_{e\|1}+p_{e\perp 1}\right)\Phi_1'+\frac{\lambda_H}{en_0}\left(p_{i\|1}'+p_{i\perp 1}'\right)p_{e\|1}\right]\right.\\
&\times\left[m_i n_0 R_0^2\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'\right)\frac{\lambda_H}{en_0}p_{i\|1}'+3p_{i\|1}\right]\\
&-\left(p_{i\|1}-p_{i\perp 1}\right)\left[3p_{i\|1}\Phi_1'+\frac{\lambda_H}{en_0}\left(p_{i\|1}'-3\frac{n_0'}{n_0}p_{i\|1}\right)p_{e\|1}\right]\\
&+\left\{m_i n_0 R_0^2\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'\right)\right.\\
&\times\left[2\Phi_1'+\frac{\lambda_H}{en_0}\left(p_{i\perp 1}'+p_{i\|1}'\right)-\frac{\lambda_H}{en_0}\frac{n_0'}{n_0}(p_{e\|1}-p_{e\perp 1})\right]\\
&\left.-\sum_{s=i,e}(p_{s\|1}-p_{s\perp 1})\right\}\\
&\left.\times\left[\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'p_{i\|1}-\frac{3\lambda_{i\|}p_{i\|1}^2}{m_i n_0 R_0^2\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'\right)}\right]\right\}\\
&\times\left[m_i n_0 R_0^2\left(\Phi_1'-\frac{\lambda_H}{en_0}\frac{n_0'}{n_0}p_{e\|1}\right)\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'\right)-p_{e\|1}\right]^{-1}. \qquad \text{(B.6)}
\end{aligned}
$$

Appendix C. Coefficients of higher-order quantities

Coefficients of higher-order quantities are written as

$$
\begin{aligned}
C_{i\perp}(\psi_1)=&-\frac{1}{A_1B_2-A_2B_1}\left\{A_3B_2-A_2B_3\right.\\
&\left.+\frac{(\delta-1)(A_2+B_2)}{A_1+A_2+B_1+B_2}\left[A_3(B_1+B_2)-B_3(A_1+A_2)\right]\right\}, \qquad \text{(C.1)}
\end{aligned}
$$

$$
\begin{aligned}
C_{i\|}(\psi_1)=&-\frac{1}{A_1B_2-A_2B_1}\left\{A_1B_3-A_3B_1\right.\\
&\left.-\frac{(\delta-1)(A_1+B_1)}{A_1+A_2+B_1+B_2}\left[A_3(B_1+B_2)-B_3(A_1+A_2)\right]\right\}, \qquad \text{(C.2)}
\end{aligned}
$$

$$
\begin{aligned}
C_{e\perp}(\psi_1)=&-\frac{p_{e\perp 1}}{p_{e\|1}}\left(p_{e\|1}-p_{e\perp 1}\right)+p_{e\perp 1}\left\{C_{i\|}+\sum_{s=i,e}(p_{s\|1}-p_{s\perp 1})\right.\\
&-m_i n_0 R_0^2\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'\right)\\
&\left.\times\left[2\Phi_1'+\frac{\lambda_H}{en_0}(p_{i\perp 1}'+p_{i\|1}')-\frac{\lambda_H}{en_0}\frac{n_0'}{n_0}(p_{e\|1}-p_{e\perp 1})\right]\right\}\\
&\times\left[m_i n_0 R_0^2\left(\Phi_1'-\frac{\lambda_H}{en_0}\frac{n_0'}{n_0}p_{e\|1}\right)\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'\right)-p_{e\|1}\right]^{-1}, \qquad \text{(C.3)}
\end{aligned}
$$

$$
C_{e\|}(\psi_1)=p_{e\|1}\left\{C_{i\|}+\sum_{s=i,e}(p_{s\|1}-p_{s\perp 1})-m_i n_0 R_0^2\left(\Phi_1'+\frac{\lambda_H-\lambda_i}{en_0}p_{i\perp 1}'\right)\right.
$$

$$\times\left[2\Phi_1' + \frac{\lambda_H}{en_0}(p_{i\perp 1}' + p_{i\parallel 1}') - \frac{\lambda_H}{en_0}\frac{n_0'}{n_0}(p_{e\parallel 1} - p_{e\perp 1})\right]\Bigg\}$$

$$\times\left[m_i n_0 R_0^2\left(\Phi_1' - \frac{\lambda_H}{en_0}\frac{n_0'}{n_0}p_{e\parallel 1}\right)\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp 1}'\right) - p_{e\parallel 1}\right]^{-1}, \tag{C.4}$$

$$C_\Phi\left(\psi_1\right) = \frac{1}{en_0}\left(C_{e\parallel} + p_{e\parallel 1} - p_{e\perp 1}\right), \tag{C.5}$$

$$C_{v\parallel}\left(\psi_1\right) = \sum_{s=i,e}\frac{C_{s\parallel} + p_{s\parallel 1} - p_{s\perp 1}}{m_i n_0 R_0\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp 1}'\right)}, \tag{C.6}$$

$$C_n\left(\psi_1\right) = \frac{n_0}{\Phi_1'}\left[\frac{n_0'}{n_0}C_\Phi + \frac{C_{v\parallel}}{R_0} - 2\Phi_1' - \frac{\lambda_H}{en_0}\left(p_{i\perp 1}' + p_{i\parallel 1}'\right)\right], \tag{C.7}$$

$$C_{iqB\parallel}\left(\psi_1\right) = \frac{3}{2}\frac{p_{i\parallel 1}}{m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp 1}'\right)}\left(C_{i\parallel} - \frac{p_{i\parallel 1}}{n_0}C_n\right), \tag{C.8}$$

$$C_{iqT\parallel}\left(\psi_1\right) = \frac{p_{i\parallel 1}}{m_i n_0 R_0^2\left(\Phi_1' + \frac{\lambda_H - \lambda_i}{en_0}p_{i\perp 1}'\right)}\left[C_{i\perp} - \frac{p_{i\perp 1}}{n_0}C_n + \frac{p_{i\perp 1}}{p_{i\parallel 1}}\left(p_{i\parallel 1} - p_{i\perp 1}\right)\right], \tag{C.9}$$

$$\begin{aligned}
C_{eqT\parallel}\left(\psi_1\right) = -R_0\Bigg\{&-\left(\Phi_1' - 2\frac{\lambda_H - \lambda_i}{en_0}\frac{n_0'}{n_0}p_{e\perp 1}\right)C_{e\perp} + p_{e\perp 1}'C_\Phi \\
&+\frac{p_{e\perp 1}}{R_0}C_{v_\parallel} - 2\frac{\lambda_H - \lambda_i}{en_0}\frac{p_{e\perp 1}'p_{e\perp 1}}{n_0}C_n \\
&-3p_{e\perp 1}\Phi_1' - \frac{\lambda_H}{en_0}\left(p_{i\parallel 1}' + p_{i\perp 1}' - p_{e\parallel 1}'\right)p_{e\perp 1} \\
&+\frac{\lambda_H}{en_0}\left(p_{e\perp 1}' - 2\frac{n_0'}{n_0}p_{e\perp 1}\right)\left(p_{e\parallel 1} - p_{e\perp 1}\right) \\
&+4\frac{\lambda_i}{en_0}\left(p_{e\perp 1}' - \frac{n_0'}{n_0}p_{e\perp 1}\right)p_{e\perp 1}\Bigg\},
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
C_{eqB\parallel}\left(\psi_1\right) = -R_0\Bigg\{&-\frac{1}{2}\left(\Phi_1' - \frac{\lambda_H - \lambda_i}{en_0}p_{e\perp 1}'\right)C_{e\parallel} \\
&-\frac{1}{2}\frac{\lambda_H - \lambda_i}{en_0}\left(p_{e\parallel 1}' - \frac{n_0'}{n_0}p_{e\parallel 1}\right)C_{e\perp} + \frac{p_{e\parallel 1}'}{2}C_\Phi \\
&+\frac{3p_{e\parallel 1}}{2R_0}C_{v_\parallel} - \frac{1}{2}\frac{\lambda_H - \lambda_i}{en_0}\frac{p_{e\perp 1}'p_{e\parallel 1}}{n_0}C_n \\
&-2p_{e\parallel 1}\Phi_1' + \frac{1}{2}\frac{\lambda_H}{en_0}\left(p_{e\parallel 1}' - \frac{n_0'}{n_0}p_{e\parallel 1}\right)\left(p_{e\parallel 1} - p_{e\perp 1}\right) \\
&-\frac{1}{2}\frac{\lambda_H}{en_0}p_{e\parallel 1}\left[3\left(p_{i\parallel 1}' + p_{i\perp 1}'\right) + 2p_{e\parallel 1}'\right] \\
&+\frac{\lambda_i}{en_0}\left[\left(p_{e\parallel 1}' - \frac{n_0'}{n_0}p_{e\parallel 1}\right)\left(p_{e\parallel 1} + p_{e\perp 1}\right) + p_{e\parallel 1}\left(p_{e\parallel 1}' - p_{e\perp 1}'\right)\right]\Bigg\},
\end{aligned} \tag{C.11}$$

$$C_I(\psi_1) = \frac{\mu_0 R_0}{B_0} \sum_{s=i,e} \left(-C_{s\perp} + p_{s\|1} - p_{s\perp 1}\right), \tag{C.12}$$

$$C_\Psi(\psi_1) = -\frac{R_0}{eB_0} \left[C_{i\perp} - \left(p_{i\|1} - p_{i\perp 1}\right) + C_{e\|} + \left(p_{e\|1} - p_{e\perp 1}\right)\right], \tag{C.13}$$

$$C_{\Psi e}(\psi_1) = -\frac{R_0}{eB_0} \left[C_{e\|} - C_{e\perp} + 2\left(p_{e\|1} - p_{e\perp 1}\right)\right], \tag{C.14}$$

$$C_{ve\|}(\psi_1) = C_{v\|} + \frac{\lambda_H R_0}{en_0} \sum_{s=i,e} \left(p'_{s\|1} + p'_{s\perp 1}\right). \tag{C.15}$$

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