

Theory for Finite Temperature Effects on Magnetosonic Waves in a Two-Ion-Species Plasma^{*)}

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The theory of magnetosonic waves perpendicular to a magnetic field in a two-ion-species plasma is extended to include finite temperature effects based on the three-fluid model with finite ion and electron pressures. First, the condition for the dispersion relation of the low-frequency mode, the lower branch of magnetosonic waves, to be approximated as a form of weak dispersion is presented. Next, by virtue of this, it is shown that the KdV equation for the low-frequency mode is valid for amplitude $\varepsilon < \varepsilon_{\max}$, where the upper limit of the amplitude ε_{\max} is given as a function of the ratio of the kinetic to magnetic energies, the density ratio, and the cyclotron frequency ratio of two ion species. The finite-temperature effects on linear and nonlinear high-frequency modes and on heavy-ion acceleration by the high-frequency-mode pulse are also discussed.

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1. Introduction

Fusion and astrophysical plasmas usually contain multiple ion species. The behavior of magnetosonic waves in a multi-species plasma is quite different from that in a single-ion-species plasma. For example, in a two-ion-species plasma, the magnetosonic wave propagating perpendicular to a magnetic field has two branches, high- and low-frequency modes. The frequency of the low-frequency mode is in the region $0 < \omega < \omega_{-r}$, where ω_{-r} is the ion-ion hybrid resonance frequency [1] defined as

$$\omega_{-r} = [(\omega_{pa}^2 \Omega_b^2 + \omega_{pb}^2 \Omega_a^2) / (\omega_{pa}^2 + \omega_{pb}^2)]^{1/2}. \quad (1)$$

Here, the subscripts a and b indicate ion species, and Ω_j and ω_{pj} ($j = a$ or b) represent their cyclotron and plasma frequencies, respectively. In the following, we assume that $\Omega_a > \Omega_b$. The frequency of the high-frequency mode is in the region $\omega_{+0} < \omega < \omega_{+r}$. Here, the resonance frequency ω_{+r} is of the order of the lower-hybrid frequency and is given by

$$\omega_{+r}^2 = \Omega_e^2 (\omega_{pa}^2 + \omega_{pb}^2) / \omega_{pe}^2, \quad (2)$$

where Ω_e and ω_{pe} are the electron cyclotron and plasma frequencies, respectively. The cut-off frequency ω_{+0} is given by

$$\omega_{+0} = (\omega_{pa}^2 / \Omega_a^2 + \omega_{pb}^2 / \Omega_b^2) \Omega_a \Omega_b |\Omega_e| / \omega_{pe}^2, \quad (3)$$

which is slightly greater than ω_{-r} .

Although the dispersion curves of the high- and low-frequency modes are quite different in the long-wavelength

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region, the nonlinear behavior of these modes can be described by Korteweg-de Vries (KdV) equations [2]. The characteristic soliton width of the low-frequency mode is of the order of the ion inertial length c/ω_{pi} , whereas that of the high-frequency mode is of the order of the electron skin depth c/ω_{pe} . The normalized frequency gap between ω_{-r} and ω_{+0} defined by

$$\Delta_\omega = (\omega_{+0} - \omega_{-r}) / \omega_{+0}, \quad (4)$$

is an important parameter for the nonlinear development of these modes [3]. In fact, it was analytically found that the KdV equation for the low-frequency mode is valid for amplitude $\varepsilon < 2\Delta_\omega$. Numerical simulations have demonstrated that high-frequency-mode pulses are generated from a low-frequency-mode pulse if $\varepsilon > 2\Delta_\omega$.

When hydrogen is the major ion component, the high-frequency-mode pulse can accelerate heavy ions by the transverse electric field in the pulse [4]. Because of this energy transfer, the high-frequency-mode pulse is gradually damped even when it propagates perpendicular to a magnetic field in a collisionless plasma [5]. This can be important dissipation mechanism in a collisionless multi-ion-species plasma such as in the solar corona.

Although extensive studies have been conducted on magnetosonic waves in multi-ion-species plasmas, their theoretical analysis is mainly based on the cold-fluid theory. This theory is valid when $\Delta_\omega \gg \beta$, where β is the ratio of the kinetic-to-magnetic energy densities. This condition can break down, for example, in the solar corona: $\Delta_\omega = 0.03$ for the H-He plasma with density ratio $n_{\text{He}}/n_{\text{H}} = 0.1$ and $\beta = 0.02$ for the plasma density $n = 10^9 \text{ cm}^{-3}$, temperature $T = 200 \text{ eV}$, and magnetic field $B = 30 \text{ G}$. We therefore extend the linear and nonlinear theories for perpen-

dicular magnetosonic waves in a two-ion-species plasma to include finite temperature effects.

2. Overview of Linear Dispersion Relation

We consider magnetosonic waves in a two-ion-species plasma based on the three-fluid model with finite ion and electron pressures. We assume that waves propagate in the x direction in an external magnetic field that is in the z direction. From the three-fluid equations, we obtained the linear dispersion relation as

$$\left(\sum_j \frac{\omega_{pj}^2 \Omega_j}{\omega^2 - \Omega_j^2 - k^2 c_j^2} \right)^2 - \left(c^2 k^2 + \sum_j \frac{\omega_{pj}^2 (\omega^2 - k^2 c_j^2)}{\omega^2 - \Omega_j^2 - k^2 c_j^2} \right) \times \left(\sum_j \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2 - k^2 c_j^2} \right) = 0. \quad (5)$$

Here, c is the light speed, the subscript j refers to ion species (a or b) or electrons ($j = e$), and $c_j^2 = \Gamma_j P_{j0} / (n_{j0} m_j)$, where Γ_j is the specific-heat ratio, P_{j0} is the equilibrium pressure, and n_{j0} is the equilibrium density.

Figure 1 shows dispersion curves of the low- and high-frequency modes in a H-He plasma with the density ratio $n_H/n_{He} = 10$ and $\beta = 0.02$, where β is defined as

$$\beta = 8\pi(P_{a0} + P_{b0} + P_{e0})/B_0^2, \quad (6)$$

and the temperatures of H, He, and electrons are equal. The two horizontal dotted lines denote ω_{-r} and ω_{+r} , defined by eqs. (1) and (2), respectively. The frequency of the low-frequency mode approaches zero as $k \rightarrow 0$. The cut-off frequency of the high-frequency mode is ω_{+0} , which is independent of β . As $k \rightarrow \infty$, the frequency of the low-frequency mode does not approach ω_{-r} , but increases with k as

$$\omega_-^2 = \omega_{-r}^2 + k^2(\omega_{pa}^2 c_b^2 + \omega_{pb}^2 c_a^2) / (\omega_{pa}^2 + \omega_{pb}^2). \quad (7)$$

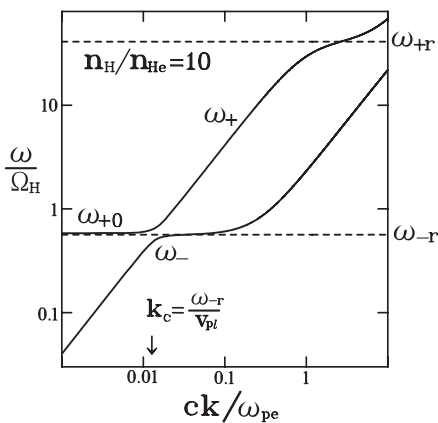


Fig. 1 Dispersion curves for low- and high-frequency modes in H-He plasma with $n_H = 10n_{He}$ and $\beta = 0.02$.

The frequency of the high-frequency mode in the limit $k \rightarrow \infty$ is

$$\omega_+^2 = \omega_{+r}^2 \left(1 + k^2 c_e^2 / \Omega_e^2 \right). \quad (8)$$

3. Low-Frequency Mode

We consider the low-frequency region $\omega \ll \Omega_i$, assuming that $\beta < 1$. Then, the linear dispersion relation of the low-frequency-mode can be approximated as a form of weak dispersion

$$\omega = v_1 k (1 - \mu_1 k^2 / 2). \quad (9)$$

The velocity v_1 is given by

$$v_1^2 = v_A^2 + c_s^2, \quad (10)$$

where v_A and c_s are defined as

$$v_A^2 = B_0^2 / (4\pi\rho_0), \quad (11)$$

$$c_s^2 = (\Gamma_a P_{a0} + \Gamma_b P_{b0} + \Gamma_e P_{e0}) / \rho_0, \quad (12)$$

with $\rho_0 = n_{a0} m_a + n_{b0} m_b + n_{e0} m_e$. We find that the dispersion coefficient μ_1 can be expressed, in terms of Δ_ω , as

$$\mu_1 = (2\Delta_\omega / k_c^2)(1 + r_\beta) + (c^2 / \omega_{pe}^2)(1 + r'_\beta), \quad (13)$$

where r_β and r'_β are of order of β and are given by

$$r_\beta = 2(\Omega_b c_a^2 - \Omega_a c_b^2) / [(\Omega_a - \Omega_b) v_1^2], \quad (14)$$

$$r'_\beta = 2 \frac{\omega_{pe}^4 v_A^6}{\Omega_e^2 c^6} \sum_i \frac{\omega_{pi}^2}{\Omega_i^3} \left(\frac{c_i^2}{\Omega_i v_A^2} - \frac{c_e^2}{\Omega_e v_A^2} \right), \quad (15)$$

The wavenumber k_c is defined as

$$k_c = \omega_{-r} / v_1, \quad (16)$$

where ω_{-r} is given by Eq.(1). In the limit $\beta = 0$, k_c becomes ω_{-r} / v_A , which is the characteristic wavenumber for a cold two-ion-species plasma. The ratio of the first term to the second term on the right-hand side of Eq. (13) is estimated as $\Delta_\omega m_i / m_e$. When $\Delta_\omega \gg m_e / m_i$, we have $\mu_1 \approx (2\Delta_\omega / k_c^2)(1 + r_\beta)$, indicating that μ_1 increases with β .

We now present the condition for which the approximation (9) is valid. We consider the differences between Eq. (9) and the exact dispersion relation ω_{exact} derived from Eq. (5) and define the difference normalized by ω_{exact} as

$$D_{\omega_-} = [v_1(1 - \mu_1 k^2 / 2) - \omega_{\text{exact}}] / \omega_{\text{exact}}. \quad (17)$$

If D_{ω_-} is small, the approximation (9) is valid. The upper panel in Fig. 2 shows D_{ω_-} for the H-He plasma with $n_H = 10n_{He}$ as a function of ck/ω_{pe} . The gray solid, black dashed, and black solid lines are for $\beta = 0, 0.1$, and 0.2 , respectively. As β increases, D_{ω_-} increases faster with ck/ω_{pe} . The lower panel shows D_{ω_-} as a function of k/k_c , where $k_c = \omega_{-r} / v_1$ and includes finite temperature effects through v_1 . Unlike in the upper panel, there are negligible

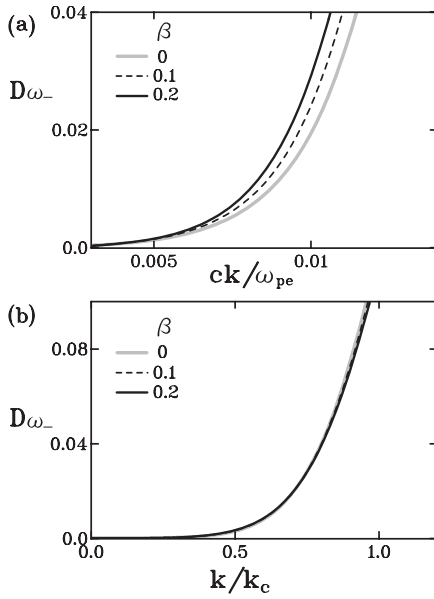


Fig. 2 Differences between the approximated and exact values of the dispersion relation of the low-frequency mode in the H-He plasma ($n_H = 10n_{He}$) with $\beta = 0, 0.1$, and 0.2 as functions of ck/ω_{pe} (upper panel) and k/k_c (lower panel).

differences between the three β cases in the lower one. We can therefore consider k_c defined by Eq. (16) as the characteristic wavenumber including a finite beta effect. Because D_{ω_-} is negligibly small for $k \ll k_c$, we can present the extended form of the condition for the approximation (9) to be valid as

$$k \ll k_c, \quad k_c = \omega_{-r}/v_1. \quad (18)$$

We next consider nonlinear waves. For waves with weak dispersion in the long wavelength region, the KdV equation can be derived with the reductive perturbation method [6]. Using this method, we obtain the KdV equation for the low-frequency mode for the wavenumber region (18) as

$$\frac{\partial B_{z1}}{\partial \tau} + \frac{3}{2}v_1(1 - \alpha_{1\beta})\frac{B_{z1}}{B_0}\frac{\partial B_{z1}}{\partial \xi} + \frac{v_1\mu_1}{2}\frac{\partial^3 B_{z1}}{\partial \xi^3} = 0, \quad (19)$$

where B_{z1} is the perturbation of B_z , ξ and τ are the stretched coordinates, $\xi = \varepsilon^{1/2}(x - v_1t)$, and $\tau = \varepsilon^{3/2}t$, respectively, with ε being the smallness parameter of the order of amplitude $|B_{z1}/B_0|$, and $\alpha_{1\beta}$ is given by

$$\alpha_{1\beta} = \frac{1}{3}\frac{v_A^2}{c^2}\sum_j(2 - \Gamma_j)\frac{\omega_{pj}^2 c_j^2}{\Omega_j^2 v_1^2}, \quad (20)$$

which is of order β .

The soliton solution of this KdV equation is

$$B_{z1}/B_0 = B_n \text{sech}^2[(x - Mv_1t)/D_1], \quad (21)$$

where B_n is the normalized amplitude, M is the Mach number [$M = 1 + (1 - \alpha_{1\beta})B_n/2$], and D_1 is the width,

$$D_1 = 2\sqrt{\mu_1/[(1 - \alpha_{1\beta})B_n]}. \quad (22)$$

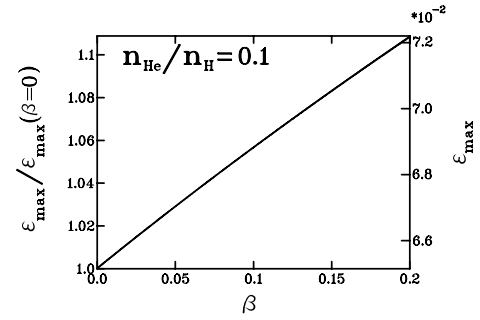


Fig. 3 Upper limit of amplitudes for the KdV equation of the low-frequency mode to be valid as a function of β for the H-He plasma with $n_H = 10n_{He}$.

Because of Eq. (22), the characteristic wavenumber of the solitary wave can be estimated as

$$k \sim 1/D_1 \sim \varepsilon^{1/2}\sqrt{(1 - \alpha_{1\beta})/\mu_1}. \quad (23)$$

The dispersion form (9) is valid in the long-wavelength region (18). Then, using eqs. (18) and (23), we obtain a condition for the amplitude of the low-frequency-mode pulses as

$$\varepsilon \ll \varepsilon_{max}, \quad (24)$$

where ε_{max} is the upper limit of the amplitude defined as

$$\varepsilon_{max} = \mu_1(1 - \alpha_{1\beta})^{-1}k_c^2. \quad (25)$$

Neglecting the quantity of order (m_e/m_i) , we can express ε_{max} in terms of Δ_{ω} as

$$\varepsilon_{max} \approx 2\Delta_{\omega}(1 + r_{\beta} + \alpha_{1\beta}), \quad (26)$$

where the magnitude of $r_{\beta} + \alpha_{1\beta}$ is of the order of β . This indicates that even if $\beta \sim \Delta_{\omega} (\ll 1)$, the finite temperature effects on ε_{max} can be neglected and the dependence of ε_{max} on β is much weaker than on Δ_{ω} .

Figure 3 shows ε_{max} defined by Eq. (25) in the H-He plasma with $n_H/n_{He} = 10$ as a function of β , where ion and electron temperatures are equal. Although ε_{max} increases with β , the difference from the value at $\beta = 0$ is smaller than β ; $[\varepsilon_{max}(\beta) - \varepsilon_{max}(0)]/\varepsilon_{max}(0) < \beta$. We can thus confirm that Δ_{ω} is the essential parameter in nonlinear evolution of the low-frequency mode even in a finite beta plasma.

4. High-Frequency Mode

For a cold two-ion-species plasma [2], the linear dispersion relation of the high-frequency mode for the wavenumber region,

$$\Delta_{\omega}(m_e/m_i)^{1/2} \ll c^2k^2/\omega_{pe}^2 < 1, \quad (27)$$

can be written in a form of weak dispersion as

$$\omega = v_{h0}k[1 - c^2k^2/(2\omega_{pe}^2)], \quad (28)$$

where the velocity v_{h0} is defined as

$$v_{h0}^2 = c^2(\omega_{pa}^2 + \omega_{pb}^2)\Omega_e^2/\omega_{pe}^4. \quad (29)$$

For a finite temperature plasma, we obtain from Eq. (5) the linear dispersion relation of the high-frequency mode for the region (27) as

$$\omega = v_h k(1 - d_h^2 k^2/2). \quad (30)$$

The velocity v_h is given by

$$v_h^2 = (v_{h1}^2 + \sqrt{v_{h1}^4 - 4v_{h2}^4})/2, \quad (31)$$

where v_{h1}^2 and v_{h2}^4 are

$$v_{h1}^2 = v_{h0}^2(1 + \beta_e) + c_a^2 + c_b^2, \quad (32)$$

$$v_{h2}^4 = c^2\Omega_e^2(1 + \beta_e)(\omega_{pa}^2 c_a^2 + \omega_{pb}^2 c_b^2)/\omega_{pe}^4 + c_a^2 c_b^2, \quad (33)$$

with β_e defined as

$$\beta_e = \omega_{pe}^2 c_e^2 / (\Omega_e^2 c^2) = 4\pi\Gamma_e T_e / B_0^2. \quad (34)$$

In the limit $\Delta_\omega = 0$, v_h^2 becomes equal to v_1^2 in the same limit. The characteristic length d_h is given by

$$d_h^2 = \frac{\omega_{pe}^2}{\Omega_e^2} \left(\sum_i \frac{\omega_{pi}^2 v_h^2}{(v_h^2 - c_i^2)^2} \right)^{-1} (1 + \beta_e)^{-2}. \quad (35)$$

In the limit $\Delta_\omega = 0$, d_h^2 and d_1^2 are equal.

Equations (31) and (35) indicate that v_h increases with β and d_h decreases with β . The increment of v_h and the decrement of d_h are both of the order of β . For $\beta \ll 1$, we can approximate v_h and d_h , retaining the first-order term of β , as

$$v_h^2 \simeq v_{h0}^2 \left(1 + \beta_e + \frac{(\omega_{pa}^2 c_a^2 + \omega_{pb}^2 c_b^2)}{(\omega_{pa}^2 + \omega_{pb}^2)v_{h0}^2} \right), \quad (36)$$

$$d_h^2 = (c^2/\omega_{pe}^2)[1 - (v_h^2 - v_{h0}^2)/v_{h0}^2]. \quad (37)$$

We now derive the KdV equation of the high-frequency mode for the region (27) using a scheme slightly different from the conventional reductive perturbation method [2]. We introduce the stretched coordinates, $\xi = \varepsilon^{1/2}(x - v_h t)$ and $\tau = \varepsilon^{3/2}t$ and expand other variables with ε . We also introduce the parameter η defined by

$$\eta = \omega_{pe} v_h / (|\Omega_e| c), \quad (38)$$

which is of the order of $(m_e/m_i)^{1/2}$. To avoid the extremely long-wavelength region where Eq. (30) is not valid and to consider the region (27), we assume that

$$(m_e/m_i)^{1/2} \ll \varepsilon \ll 1, \quad (39)$$

because $kd_h \sim \varepsilon^{1/2}$. We also assume that $(m_e/m_i)^{1/2} \ll \beta \ll 1$. After some manipulation, we obtain the KdV equation

$$\frac{\partial B_1}{\partial \tau} + \frac{3}{2}v_h \alpha_h \frac{B_1}{B_0} \frac{\partial B_1}{\partial \xi} + \frac{1}{2}v_h d_h^2 \frac{\partial^3 B_1}{\partial \xi^3} = 0, \quad (40)$$

where α_h is given by

$$\alpha_h = \frac{\omega_{pe}^2 d_h^2}{3c^2} \sum_i \frac{\omega_{pi}^2 \Omega_i v_h^4}{\eta^4 \omega_{pe}^2 |\Omega_e|} \left(3 + \frac{(1 + \Gamma_i) c_i^2}{(v_h^2 - c_i^2)} \right) \times \frac{(1 + \beta_e)^3}{(v_h^2 - c_i^2)^2} - \frac{4}{3}\beta_e \left(1 - \frac{\Gamma_e}{2} \right). \quad (41)$$

In the limit $\beta = 0$, α_h becomes $1 + 2\Delta_\omega \omega_{pe}^2 / (\Omega_e \Omega_b)$. The soliton solution of Eq. (40) is

$$B_1/B_0 = B_n \text{sech}^2\{[x - (1 + \alpha_h B_n/2)v_h t]/D_h\}, \quad (42)$$

with $D_h = 2d_h/(\alpha_h B_n)$.

We next consider heavy-ion acceleration by the non-linear high-frequency mode. Under the condition that hydrogen is the major ion component, heavy ions are accelerated by the transverse electric field E_y [4]. The y component of the equation of motion for the ions is

$$m_i \frac{dv_{iy}}{dt} = q_i (E_y - v_{ix} B_z/c). \quad (43)$$

In deriving Eq. (42), we have obtained the relation between the lowest order quantities as

$$v_{ix} = v_h \frac{\Omega_i}{|\Omega_e| \eta^2} \frac{v_h^2 (1 + \beta_e) B_1}{(v_h^2 - c_i^2) B_0}, \quad E_y = \frac{v_h}{c} B_1. \quad (44)$$

For a single-ion species plasma, Eq. (44) leads to $E_y - v_{ix} B_z/c \simeq 0$ and $dv_{iy}/dt = 0$. For a multi-ion-species plasma, however, $E_y - v_{ix} B_z/c$ for heavy ions is positive in the pulse region. Therefore, heavy ions are accelerated in the y direction when they pass through the pulse region. Substituting eqs. (42) and (44) in Eq. (43) and integrating over time, we can estimate the maximum speed of accelerated heavy ions as

$$\frac{v_{bym}}{v_h} = \frac{4\sqrt{2}}{\alpha_h \eta} \frac{\Omega_i}{|\Omega_e|} \left(1 - \frac{\Omega_i}{|\Omega_e| \eta^2} \frac{v_h^2 (1 + \beta_e)}{(v_h^2 - c_b^2)} \right) B_n^{1/2}. \quad (45)$$

Figure 4 shows v_{bym} for accelerated He ions in the H-He plasma with $n_H = 10n_{He}$ as a function of β , where B_n is set to 0.1. This indicates that v_{bym} decreases with β and its decrement is of the order of $10^{-2}\beta$.

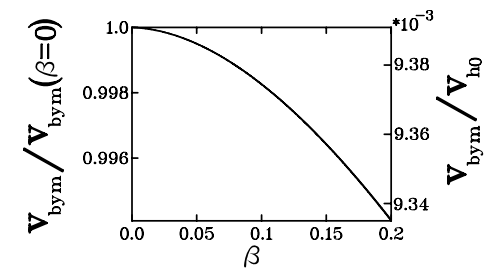


Fig. 4 Maximum speed of He ions accelerated by the high-frequency-mode pulse with $B_n = 0.1$ in the H-He plasma with $n_H = 10n_{He}$ as a function of β .

5. Summary

We have extended the theory of perpendicular magnetosonic waves in a two-ion-species plasma including finite temperature effects using the three-fluid model with finite ion and electron pressures. First, for the low-frequency mode, we presented the condition for which the linear dispersion relation can be written as a form of weak dispersion and showed that the KdV equation is valid $\varepsilon < \varepsilon_{\max}$, where the upper limit of the amplitude ε_{\max} is given as a function of $\Delta\omega$ and β . Next, we presented the dependence of the quantities concerned with the high-frequency mode on β . Heavy-ion acceleration by the high-frequency-mode pulse was also discussed.

We have theoretically shown that the high-frequency-mode pulse accelerates heavy ions perpendicular to the magnetic field in finite beta plasmas. We can therefore expect that if magnetosonic pulses are excited in the solar corona, they would raise ion perpendicular temperature. This feature is consistent with observations that the perpendicular ion temperatures are higher than the parallel

ones in the solar corona [7, 8]. Therefore, this acceleration mechanism would be applicable to ion heating in the solar corona. As for a future work, we will have to compare the theory for the temperature effect on heavy ion acceleration with particle simulations.

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