

§41. Development of Extended Finite/ Boundary Node Method

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Introduction Before executing a numerical code based on the Finite Element Method (FEM) or the Boundary Element Method (BEM), a region must be divided into a set of elements. For the purpose of removing the element decomposition from a numerical code, meshless approaches such the Finite Node Method (FNM)^{1, 2, 3, 4} and the Boundary Node Method (BNM)^{5, 6, 7} have been so far proposed. However, they have been plagued by various difficulties. First of all, in the FNM, the Lagrange multiplier method is adopted as the implementation method^{1, 3, 4} of essential boundary conditions. Thus, the boundary value problem of the governing equation must be first rewritten into the equivalent variational form. On the other hand, the number of unknowns for the BNM is twice as many as that for the BEM^{5, 6, 7} and, hence, the computational cost required for the solution of the discretized equation in the BNM becomes about 8 times as much as that in the BEM.

The purpose of the present study is to algebraically formulate the FNM without using the variational principle. In addition, we propose the method for reducing the number of unknowns for the BNM by half.

eXtended FNM For simplicity, we consider the following two-dimensional (2D) Poisson problem on the domain Ω bounded by a simple closed curve $\partial\Omega$:

$$-\nabla^2 u = p \quad \text{in } \Omega, \quad (1)$$

$$u = \bar{u} \quad \text{on } \Gamma_D, \quad (2)$$

$$\frac{\partial u}{\partial n} = \bar{q} \quad \text{on } \Gamma_N, \quad (3)$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$. In addition, \mathbf{n} denotes an outward unit normal on $\partial\Omega$. Furthermore, the superposed bar indicates prescribed boundary values and $p(\mathbf{x})$ is a given function on Ω . As is well known, both eqs. (1) and (3) are satisfied if and only if the following weak form is fulfilled:

$$\forall w \text{ s.t. } w|_{\Gamma_D} = 0 : \int_{\Omega} \nabla w \cdot \nabla u \, d^2 \mathbf{x} = \iint_{\Omega} w p \, d^2 \mathbf{x} + \int_{\Gamma_N} w \bar{q} \, ds = 0. \quad (4)$$

Let both the test function $w(\mathbf{x})$ and the trial function $u(\mathbf{x})$ satisfy $w, u \in \text{span}(\phi_1, \phi_2, \dots, \phi_N)$, and let the function space on Γ_D be limited to $\text{span}(N_1, N_2, \dots, N_K)$. Then, as the discretized form of the weak form (4), the following proposition

is obtained :

$$\forall \mathbf{w} \in \text{Ker } C^T : \mathbf{w}^T (A\mathbf{u} - \mathbf{f}) = 0, \quad (5)$$

where N -dimensional vectors, \mathbf{w} and \mathbf{u} , correspond $w(\mathbf{x})$ and $u(\mathbf{x})$, respectively. In addition, A , C and \mathbf{f} denote an $N \times N$ matrix, an $N \times K$ matrix and an N -dimensional vector, respectively. By taking account of $\text{Ker } C^T = (\text{Im } C)^\perp$, the above proposition (5) is equivalent to $A\mathbf{u} - \mathbf{f} \in \text{Im } C$. Moreover, the discretized form of eq. (2) can be written as $C^T \mathbf{u} = \mathbf{g}$. Hence, the discretized form of the 2D Poisson problem can be written in the form,

$$\begin{bmatrix} A & C \\ C^T & O \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}. \quad (6)$$

Note that eq. (6) coincides with the matrix equation for the FNM. In this way, the FNM is algebraically formulated without employing the variational principle.

Although the 2D Poisson problem is used to demonstrate the algebraic formulation of the FNM, the discretization procedure used in the present study is applicable to general boundary-value problems.

eXtended BNM As mentioned above, the number of unknowns for the BNM is twice as many as that for the BEM. The reason for this is explained as follows. In the conventional BNM, the shape functions $\{\psi_i(s)\}_{i=1}^N$ are determined by means of the Moving-Least Square (MLS) approximation and, therefore, the functions do not at all satisfy the delta function property: $\psi_i(s_j) = \delta_{ij}$. However, it is the delta function property that plays an important role in reducing the number of unknowns in the discretized equation. In contrast, the property is always fulfilled by the shape functions of the BEM. Hence, the number of unknowns for the BEM is equal to the number of nodes on the boundary.

In order to have the delta function property satisfied by the shape functions, we employ not the MLS approximation¹ but the Radial Point Interpolation Method (RPIM)⁸ to the determination of the shape functions. As a result, the CPU time required for the solution of the discretized equation successfully reduces by a factor of eight.

From the above result, we can conclude that the RPIM is useful to accelerate the BNM.

- 1) Belytschko, T. et al.: Int. J. Numer. Meth. Engng. **37** (1994) 229.
- 2) Atluri, S. N. and Zhu, T.: Comput. Mech. **22** (1998) 117.
- 3) Ikuno, S. et al.: IEEE Trans. Magn. **43** (2007) 1501.
- 4) Kamitani, A. et al.: Plasma Fusion Res. **6** (2011) 2401074.
- 5) Mukherjee, Y. X. and Mukherjee, S.: Int. J. Numer. Meth. Engng. **40** (1997) 797.
- 6) Chati, M. K. et al.: Int. J. Numer. Meth. Engng. **50** (2001) 2233.
- 7) Saitoh A. et al.: IEEE Trans. Magn. **47** (2011) 1222.
- 8) Wang, J. G. and Liu, G. R.: Int. J. Numer. Meth. Engng. **54** (2002) 1623.