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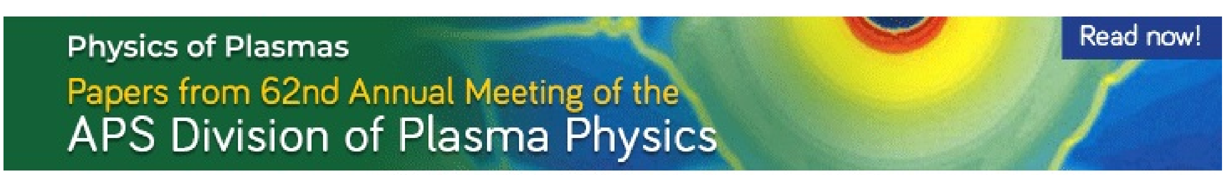

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A Laguerre expansion method for the field particle portion in the linearized Coulomb collision operator

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The spherical coordinates expressions of the Rosenbluth potentials are applied to the field particle portion in the linearized Coulomb collision operator. The Sonine (generalized Laguerre) polynomial expansion formulas for this operator allowing general field particles' velocity distributions are derived. An important application area of these formulas is the study of flows of thermalized particles in NBI-heated or burning plasmas since the energy space structure of the fast ions' slowing down velocity distribution cannot be expressed by usual orthogonal polynomial expansions, and since the Galilean invariant property and the momentum conservation of the collision must be distinguished there. © *2015 AIP Publishing LLC.*
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I. INTRODUCTION

Past kinetic theories on NBI (neutral beam injection)-heated or burning plasmas from the 1970s into the 1990s were constructed for investigating the beam driven currents[1] and the bootstrap currents.[2] After development of the charge exchange recombination spectroscopy,[3,4] the flow velocities of thermalized ions caused by the existence of the fast ions also are regarded as an important physics issue.[4] The application of the neoclassical transport theory for this purpose requires a simultaneous solving of the kinetic equations of all thermal particles, including the Coulomb collision operator $C_{af}(f_{aM},f_f)$ that describes the collision of the thermal particle species a with the fast ions' slowing down velocity distribution function $f_f(\mathbf{x},\mathbf{v})$ (velocity distribution function of proton or deuterium in NBI-heated plasmas or helium in burning plasmas). In a recent theory for general toroidal configurations,[5,6] problems including the field particle portion $C_{ab}(f_{aM},f_b)$ in the linearized collision operator with the local Maxwellian distribution f_{aM} (integro-differential equations) are converted to generalized parallel force balance expressed in an algebraic form by taking $\int v_\parallel L_j^{(3/2)}(x_a^2)\mathrm{d}^3\mathbf{v}$ integrals of the drift kinetic equation. Here, $L_j^{(\alpha)}(K)\equiv(e^K K^{-\alpha}/j!)\mathrm{d}^j(e^{-K}K^{j+\alpha})/\mathrm{d}K^j$ is the Sonine (generalized Laguerre) polynomial,[7] and $x_a\equiv\sqrt{m_a v^2/2T_a}\equiv v/v_{Ta}$. This neoclassical calculation was conducted recently for the NBI-heated plasmas in the Heliotron-J (a non-axisymmetric toroidal device), and its theoretical results successfully explained the experimentally measured impurity flow velocities.[8] In this paper, we report a method used there for obtaining the collision integrals $\int v_\parallel L_j^{(3/2)}(x_a^2)C_{af}(f_{aM},f_f)\mathrm{d}^3\mathbf{v}$.

II. ROSENBLUTH POTENTIAL

We shall start from the well-known RMJ (Rosenbluth-MacDonald-Judd) form of the Coulomb collision operator[9]

$$C_{ab}(f_a,f_b)=-4\pi\left(\frac{e_a e_b}{m_a}\right)^2\ln\Lambda_{ab}\times\frac{\partial}{\partial v_\alpha}\left(\frac{m_a}{m_b}f_a\frac{\partial\mathcal{H}(f_b)}{\partial v_\alpha}-\frac{1}{2}\frac{\partial^2\mathcal{G}(f_b)}{\partial v_\alpha\partial v_\beta}\frac{\partial f_a}{\partial v_\beta}\right). \quad (1)$$

The Rosenbluth potentials in this operator are defined by

$$\mathcal{H}(f_b)\equiv\int\frac{f_b(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|}\mathrm{d}^3\mathbf{v}',\ \mathcal{G}(f_b)\equiv\int f_b(\mathbf{v}')|\mathbf{v}-\mathbf{v}'|\mathrm{d}^3\mathbf{v}', \quad (2)$$

where $\mathbf{v}$ and $\mathbf{v}'$ are velocities of test particles (a) and field particles (b), respectively. The Coulomb logarithm[10] $\ln\Lambda_{ab}=\ln\Lambda_{ba}$ for the colliding species pair a-b in this operator is a constant being independent of the colliding velocity. Hereafter, velocity distribution functions of the field particles $f_b(\mathbf{v})=f_b(v,\theta,\phi)$ given in the spherical harmonic expansion form

$$f_b(\mathbf{v})=\sum_{l=0}^{\infty}\Big[a_l^0(v)P_l(\cos\theta)+\sum_{m=1}^{l}P_l^m(\cos\theta)\{a_l^m(v)\cos(m\phi)+b_l^m(v)\sin(m\phi)\}\Big]\equiv\sum_{l=0}^{\infty}f_b^{(l)}(v,\theta,\phi), \quad (3)$$

are assumed. For the potentials in Eq. (2) including these functions, we shall apply a basic idea in Ref. 11. In this spherical velocity coordinate system, $|\mathbf{v}-\mathbf{v}'|^{-1}$ and $|\mathbf{v}-\mathbf{v}'|$ in Eq. (2) are functions of

$$z=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi'), \quad (4)$$

which is the cosine of the angle between $\mathbf{v}$ and $\mathbf{v}'$, and can be expressed by Legendre polynomial $P_n(z)$ expansions by

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applying the generating function $(1-2hz+h^2)^{-\nu}=\sum_{n=0}^{\infty}C_n^{\nu}(z)h^n$ for the Gegenbauer functions $C_n^{\nu}(z)$ (a special case of it is the Legendre polynomial $C_n^{1/2}(z)=P_n(z)$).[12] Substituting these expansions and the velocity distribution function Eq. (3) into Eq. (2) with the theorem[12]

$$P_n(z)=P_n(\cos\theta)P_n(\cos\theta')+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\{\cos(m\phi)\cos(m\phi')+\sin(m\phi)\sin(m\phi')\},\tag{5}$$

results in spherical harmonic expansion forms of the Rosenbluth potentials and/or their energy differential as follows.

$$\mathcal{H}(f_b)=4\pi\sum_{l=0}^{\infty}\frac{1}{2l+1}\left\{\frac{1}{v^{l+1}}\int_0^v(v')^{l+2}f_b^{(l)}(v',\theta,\phi)\mathrm{d}v'+v^l\int_v^{\infty}\frac{f_b^{(l)}(v',\theta,\phi)}{(v')^{l-1}}\mathrm{d}v'\right\},\tag{6}$$

$$\frac{\partial\mathcal{H}(f_b)}{\partial v}=4\pi\sum_{l=0}^{\infty}\frac{1}{2l+1}\left\{-\frac{l+1}{v^{l+2}}\int_0^v(v')^{l+2}f_b^{(l)}(v',\theta,\phi)\mathrm{d}v'+lv^{l-1}\int_v^{\infty}\frac{f_b^{(l)}(v',\theta,\phi)}{(v')^{l-1}}\mathrm{d}v'\right\},\tag{7}$$

$$\begin{aligned}\frac{\partial\mathcal{G}(f_b)}{\partial v}=4\pi\sum_{l=0}^{\infty}\frac{1}{2l+1}\Bigg\{&-\frac{l+1}{2l+3}\frac{1}{v^{l+2}}\int_0^v(v')^{l+4}f_b^{(l)}(v',\theta,\phi)\mathrm{d}v'+\frac{l-1}{2l-1}\frac{1}{v^l}\int_0^v(v')^{l+2}f_b^{(l)}(v',\theta,\phi)\mathrm{d}v'\\&+\frac{l+2}{2l+3}v^{l+1}\int_v^{\infty}\frac{f_b^{(l)}(v',\theta,\phi)}{(v')^{l-1}}\mathrm{d}v'-\frac{l}{2l-1}v^{l-1}\int_v^{\infty}\frac{f_b^{(l)}(v',\theta,\phi)}{(v')^{l-3}}\mathrm{d}v'\Bigg\},\end{aligned}\tag{8}$$

$$\begin{aligned}\frac{\partial^2\mathcal{G}(f_b)}{\partial v^2}=4\pi\sum_{l=0}^{\infty}\frac{1}{2l+1}\Bigg\{&\frac{(l+1)(l+2)}{2l+3}\frac{1}{v^{l+3}}\int_0^v(v')^{l+4}f_b^{(l)}(v',\theta,\phi)\mathrm{d}v'-\frac{(l-1)l}{2l-1}\frac{1}{v^{l+1}}\int_0^v(v')^{l+2}f_b^{(l)}(v',\theta,\phi)\mathrm{d}v'\\&+\frac{(l+1)(l+2)}{2l+3}v^l\int_v^{\infty}\frac{f_b^{(l)}(v',\theta,\phi)}{(v')^{l-1}}\mathrm{d}v'-\frac{(l-1)l}{2l-1}v^{l-2}\int_v^{\infty}\frac{f_b^{(l)}(v',\theta,\phi)}{(v')^{l-3}}\mathrm{d}v'\Bigg\}.\end{aligned}\tag{9}$$

They are generalizations of the integral formulas for gyro-phase-averaged velocity distributions $f_b(v,\cos\theta)$, which were first derived by Rosenbluth, MacDonald, and Judd in Ref. 13, to arbitrary spherical harmonics (l, m). One application of them may be

$$C_{ab}\left(f_a,f_b^{(l=0)}\right)=4\pi\left(\frac{e_ae_b}{m_a}\right)^2\ln\Lambda_{ab}\left[\frac{1}{v^3}\frac{\partial\mathcal{G}\left(f_b^{(0)}\right)}{\partial v}\mathcal{L}f_a+\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\left(-\frac{m_a}{m_b}\frac{\partial\mathcal{H}\left(f_b^{(0)}\right)}{\partial v}+\frac{1}{2}\frac{\partial^2\mathcal{G}\left(f_b^{(0)}\right)}{\partial v^2}\frac{\partial}{\partial v}\right)f_a\right\}\right],\tag{10}$$

(collision of arbitrary distributions f_a of the test particle species a with isotropic distributions $f_b^{(l=0)}$ of the field particle species b) that is obtained by using $\sum_\alpha v_\alpha\partial F/\partial v_\alpha=v\partial F/\partial v$, $\sum_\alpha\partial(v_\alpha F)/\partial v_\alpha=v^{-2}\partial(v^3F)/\partial v$, and

$$\mathcal{L}\equiv\frac{1}{2}\left(\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right),$$

and will result in the well-known formula of the test particle portion $C_{ab}(f_a,f_{b\mathrm{M}})$ of the linearized Coulomb collision operator when the Maxwellian velocity distribution $f_b^{(l=0)}=f_{b\mathrm{M}}$ is substituted. $[f_{a\mathrm{M}}\equiv n_a\{m_a/(2\pi T_a)\}^{3/2}\exp(-m_av^2/2T_a)$ $\equiv n_a\pi^{-3/2}v_{\mathrm{T}a}^{-3}\exp(-x_a^2)$: Maxwellian distribution with the density $n_a\equiv\int f_a\mathrm{d}^3\mathbf{v}$, the temperature defined by $T_a\equiv p_a/n_a$ using the pressure $p_a\equiv m_a\int v^2f_a\mathrm{d}^3\mathbf{v}/3$ of the species a, and without the velocity moment $\mathbf{u}_a\equiv\int\mathbf{v}f_a\mathrm{d}^3\mathbf{v}/n_a]$. These kinds of indefinite integrals of the Maxwellian can be calculated by

$$\int x^{2n+1}\exp(-a^2x^2)\mathrm{d}x=-\frac{\exp(-a^2x^2)}{2a^2}\sum_{k=0}^{n}\frac{n!}{(n-k)!a^{2k}}x^{2(n-k)},\tag{11}$$

and

$$\begin{aligned}\int x^{2n}\exp(-a^2x^2)\mathrm{d}x&=-\frac{(2n-1)!!}{2a^2}\exp(-a^2x^2)\sum_{k=0}^{n-1}\frac{x^{2n-2k-1}}{2^ka^{2k}(2n-2k-1)!!}+\frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}a^{2n+1}}\Phi(ax)\\&=-\frac{(2n-1)!!}{2a^2}\exp(-a^2x^2)\sum_{k=0}^{n-2}\frac{x^{2n-2k-1}}{2^ka^{2k}(2n-2k-1)!!}+\frac{(2n-1)!!\sqrt{\pi}}{2^na^{2n-1}}x^2G(ax).\end{aligned}\tag{12}$$

Here, $\Phi(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\mathrm{d}y$ and $G(x) \equiv \{\Phi(x) - x\Phi'(x)\}/(2x^2)$ are the error and the Chandrasekhar functions, respectively. They are routinely included in general neoclassical transport analyses. Instead of the already known $C_{ab}(f_a, f_{bM})$, however, we shall investigate here the field particle portion $C_{ab}(f_{aM}, f_b)$ in the linearized collision operator by using Eqs. (6)–(9) in Section III.

III. APPLICATION TO THE FIELD PARTICLE PORTION

The field particle portion is given by[9]

$$C_{ab}(f_{aM},f_b) = 4\pi\left(\frac{e_a e_b}{m_a}\right)^2 \ln\Lambda_{ab} f_{aM}\frac{m_a}{T_a}\left\{\frac{4\pi T_a}{m_b}f_b - \mathcal{H}(f_b) + \left(\frac{m_a}{m_b}-1\right)v\frac{\partial\mathcal{H}(f_b)}{\partial v} + \frac{m_a v^2}{2T_a}\frac{\partial^2\mathcal{G}(f_b)}{\partial v^2}\right\}. \tag{13}$$

This formula is obtained by using the Poisson equations $\nabla_v^2\mathcal{H}(f_b) = -4\pi f_b$ and $\nabla_v^2\mathcal{G}(f_b) = 2\mathcal{H}(f_b)$ with $\nabla_v^2 \equiv \sum_\alpha \partial^2/\partial v_\alpha^2$ for Eq. (1). The field particles' velocity distribution function f_b in this operator may be sometimes that of thermal particles $f_b \simeq f_{bM}$ and, at other time, may be functions with quite different $\mathbf{v}$-space structures, such as the aforementioned fast ions' slowing down velocity distribution $f_\mathrm{f}(\mathbf{x},\mathbf{v})$ which we will discuss in detail later. One common feature of Eq. (13) for these general cases is that $f_{aM} \propto \exp(-x_a^2)$ is multiplied to the full part. Therefore, even when $C_{a\mathrm{f}}(f_{aM}, f_\mathrm{f})$ is included in the collision operator $\sum_b C_{ab}(f_a, f_b)$ of thermal particles $a \neq \mathrm{f}$, we can use the previous solving procedure for the kinetic equations for them[5,6] where the perturbation $\widehat{f_a}$ in the velocity distribution $f_a(\mathbf{x},\mathbf{v}) = f_{aM}(1+\widehat{f}_a)$ must be $|\widehat{f}_a| \ll 1$ in most of the thermal velocity range $m_a v^2 \sim 2T_a$ and $|\widehat{f}_a| \lesssim 1$ even for collisionless high energy range $m_a v^2 \gg 2T_a$. (If $C_{a\mathrm{f}}(f_{aM}, f_\mathrm{f})$ is not $\propto \exp(-x_a^2)$, this constraint on the energy space structure will be strongly violated.) As long as the $C_{ab}(f_{aM}, f_b)$ has this energy space structure of $\propto \exp(-x_a^2)$, the orthogonal expansion of this energy space structure using the Sonine polynomials can be defined for cases of general field particle distributions f_b. General expansion coefficients $\int v^l P_l^m(\xi)\cos\{m(\phi-\phi_0)\}L_j^{(l+1/2)}(x_a^2)C_{ab}(f_{aM}, f_b)\mathrm{d}^3\mathbf{v}$ with $\xi \equiv \cos\theta \equiv v_\parallel/v$ can be derived by integrations by parts for $\int_0^\infty \mathrm{d}v$ using Eqs. (11) and (12). For example, the integrals for the lower Legendre orders $l = 0, 1, 2$ in the gyro-phase-averaged distribution $\bar{f}_b \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f_b \mathrm{d}\phi$ as $m = 0$ are given as follows:

$$m_a\int v^2 C_{ab}(f_{aM},f_b)\mathrm{d}^3\mathbf{v} = -m_b\int v^2 C_{ba}(f_b,f_{aM})\mathrm{d}^3\mathbf{v} = 32\pi^2\frac{n_a(e_a e_b)^2\ln\Lambda_{ab}}{m_a}\left(\frac{m_a}{2T_a}\right)^{1/2}\int_0^\infty\left\{x_a G(x_a) - \frac{m_a}{m_b}\frac{1}{\sqrt{\pi}}\exp(-x_a^2)\right\}\left(\int_{-1}^{1}\bar{f}_b\mathrm{d}\xi\right)v^2\mathrm{d}v, \tag{14}$$

$$\int L_2^{(1/2)}(x_a^2)C_{ab}(f_{aM},f_b)\mathrm{d}^3\mathbf{v} = 32\pi\sqrt{\pi}n_a\left(\frac{e_a e_b}{m_a}\right)^2\ln\Lambda_{ab}\int_0^\infty x_a^2\left\{\frac{3}{2}\frac{m_a}{m_b} - \left(\frac{m_a}{m_b}+1\right)x_a^2\right\}\exp(-x_a^2)\left(\int_{-1}^{1}\bar{f}_b\mathrm{d}\xi\right)\mathrm{d}x_a. \tag{15}$$

$$m_a\int v\xi C_{ab}(f_{aM},f_b)\mathrm{d}^3\mathbf{v} = -m_b\int v\xi C_{ba}(f_b,f_{aM})\mathrm{d}^3\mathbf{v} = 8\pi^2\frac{n_a(e_a e_b)^2\ln\Lambda_{ab}}{T_a}\left(\frac{m_a}{m_b}+1\right)\int_0^\infty G(x_a)\left(\int_{-1}^{1}\xi\bar{f}_b\mathrm{d}\xi\right)v^2\mathrm{d}v, \tag{16}$$

$$m_a\int v\xi L_1^{(3/2)}(x_a^2)C_{ab}(f_{aM},f_b)\mathrm{d}^3\mathbf{v} = 24\pi^2\frac{n_a(e_a e_b)^2\ln\Lambda_{ab}}{T_a}\int_0^\infty\left\{\left(\frac{m_a}{m_b}+1\right)\frac{x_a}{\sqrt{\pi}}\exp(-x_a^2) - G(x_a)\right\}\left(\int_{-1}^{1}\xi\bar{f}_b\mathrm{d}\xi\right)v^2\mathrm{d}v, \tag{17}$$

$$m_a\int v\xi L_2^{(3/2)}(x_a^2)C_{ab}(f_{aM},f_b)\mathrm{d}^3\mathbf{v} = 10\pi^2\frac{n_a(e_a e_b)^2\ln\Lambda_{ab}}{T_a}\int_0^\infty\left\{5\frac{m_a}{m_b}+1-2\left(\frac{m_a}{m_b}+1\right)x_a^2\right\}\frac{x_a}{\sqrt{\pi}}\exp(-x_a^2)\left(\int_{-1}^{1}\xi\bar{f}_b\mathrm{d}\xi\right)v^2\mathrm{d}v, \tag{18}$$

$$\begin{aligned} m_a\int v\xi L_3^{(3/2)}(x_a^2)C_{ab}(f_{aM},f_b)\mathrm{d}^3\mathbf{v} = \frac{7}{3}\pi^2\frac{n_a(e_a e_b)^2\ln\Lambda_{ab}}{T_a}\int_0^\infty&\left[5\left(7\frac{m_a}{m_b}+1\right) - 4\left(7\frac{m_a}{m_b}+4\right)x_a^2 + 4\left(\frac{m_a}{m_b}+1\right)x_a^4\right]\\ &\times\frac{x_a}{\sqrt{\pi}}\exp(-x_a^2)\left(\int_{-1}^{1}\xi\bar{f}_b\mathrm{d}\xi\right)v^2\mathrm{d}v. \end{aligned} \tag{19}$$

$$\begin{aligned} \int x_a^2 P_2(\xi)C_{ab}(f_{aM},f_b)\mathrm{d}^3\mathbf{v} = 16\pi^2 n_a\left(\frac{e_a e_b}{m_a}\right)^2\ln\Lambda_{ab}\int_0^\infty x_a^2&\left[\frac{m_a}{m_b}\left\{\frac{3G(x_a)}{x_a} - \frac{2}{\sqrt{\pi}}\exp(-x_a^2)\right\}\right.\\ &\left.-x_a G(x_a) - \frac{3}{\sqrt{\pi}}\exp(-x_a^2)\right]\left(\int_{-1}^{1}P_2(\xi)\bar{f}_b\mathrm{d}\xi\right)\mathrm{d}x_a, \end{aligned} \tag{20}$$

$$\int x_a^2 L_1^{(5/2)}(x_a^2) P_2(\xi) C_{ab}(f_{aM}, f_b) \mathrm{d}^3\mathbf{v} = -4\pi^2 n_a \left(\frac{e_a e_b}{m_a}\right)^2 \ln\Lambda_{ab} \int_0^\infty \left[99 x_a G(x_a) - 8\left\{3 + 2\left(1 + \frac{m_a}{m_b}\right) x_a^2\right\} \times \frac{x_a^2}{\sqrt{\pi}} \exp(-x_a^2) \right] \left(\int_{-1}^{1} P_2(\xi) \bar{f}_b \mathrm{d}\xi \right) \mathrm{d}x_a. \tag{21}$$

Agreements of integrals of test and field particle portions in Eqs. (14) and (16) are due to the conservation of energy and momentum, respectively. As discussed also in Ref. 14, because of a characteristic of $C_{ab}(f_{aM}, f_b)$ as an integral operator, we shall investigate these lower Legendre orders first. When $\bar{f}_b = x_b^l P_l(\xi) L_j^{(l+1/2)}(x_b^2) f_{bM}$ is substituted into these formulas, the definite integrals $\int_0^\infty x_a^{2n-1} \Phi(x_b) \exp(-x_a^2) \mathrm{d}x_a$ or $\int_0^\infty x_a^{2n-1} G(x_b) \exp(-x_a^2) \mathrm{d}x_a$, which will appear there, correspond to the Gauss hypergeometric function[7] ${}_2F_1$ by

$$\int_0^\infty x_a^{2n-1} \Phi(x_b) \exp(-x_a^2) \mathrm{d}x_a = \frac{v_{\mathrm{T}a}}{v_{\mathrm{T}b}} \frac{(2n-1)!!}{2^n} {}_2F_1\left(\frac{1}{2} + n, \frac{1}{2}; \frac{3}{2}; -\left(\frac{v_{\mathrm{T}a}}{v_{\mathrm{T}b}}\right)^2\right) = \frac{1}{2} \frac{(2n-1)!!}{2^n} \left(\frac{v_{\mathrm{T}b}}{v_{\mathrm{T}a}}\right)^{2n} \int_0^1 \frac{t^{-1/2}}{\left\{(v_{\mathrm{T}b}/v_{\mathrm{T}a})^2 + t\right\}^{1/2+n}} \mathrm{d}t,$$

that can be obtained by a recurrence relation

$$\int_0^1 \frac{t^{-1/2}}{\left\{(v_{\mathrm{T}b}/v_{\mathrm{T}a})^2 + t\right\}^{1/2+n}} \mathrm{d}t = \left(\frac{v_{\mathrm{T}a}}{v_{\mathrm{T}b}}\right)^2 \frac{\left\{1 + (v_{\mathrm{T}b}/v_{\mathrm{T}a})^2\right\}^{1/2-n}}{n - 1/2} + \left(\frac{v_{\mathrm{T}a}}{v_{\mathrm{T}b}}\right)^2 \frac{n-1}{n-1/2} \int_0^1 \frac{t^{-1/2}}{\left\{(v_{\mathrm{T}b}/v_{\mathrm{T}a})^2 + t\right\}^{n-1/2}} \mathrm{d}t,$$

and $\int_0^\infty x_a \Phi(x_b) \exp(-x_a^2) \mathrm{d}x_a = \frac{1}{2}\left\{1 + (v_{\mathrm{T}b}/v_{\mathrm{T}a})^2\right\}^{-1/2}$. Using these integrals, Eqs. (14)–(21) reproduce the well-known Braginskii's matrix elements[9,14–17] for the orders $l = 0$, 1, 2, which were previously obtained by a method to substitute the generating function of the Sonine polynomials into the Landau operator. It also should be noted that Eqs. (6)–(9) and (11)–(13) can be used also for deriving explicit expressions of $C_{ab}(f_{aM}, x_b^l P_l(\xi) L_j^{(l+1/2)}(x_b^2) f_{bM})$ that cannot be obtained by this Braginskii's procedure.

However, the most important application area of the expansion coefficients Eqs. (14)–(21) will not be such already known collision processes between thermal particles, but collisions of the thermal particles with the fast ions in the NBI-heated plasmas or the burning plasmas. For general particle species a, $\int f_a \mathrm{d}^3\mathbf{v}$ (number of particles), $\int \mathbf{v} f_a \mathrm{d}^3\mathbf{v}$ (momentum), and $\int v^2 f_a \mathrm{d}^3\mathbf{v}$ (energy) must be absolutely finite under a constraint of $f_a > 0$. If functions with continuous derivatives $\partial f_a/\partial v$ satisfy this convergence of integrals in the 3-dimensional velocity space, they would have the energy-space structure that is nearly equal to the shifted anisotropic exponential. The $\mathbf{v}$-space structure of the slowing down velocity distribution is an exception to this rule. It has the step function structure[1,2,9,18] $f_{\mathrm{f}}(\mathbf{x}, \mathbf{v}) \propto H(v_{\mathrm{b}} - v)$ at the initial velocity $v \cong v_{\mathrm{b}}$, and its energy space broadening is $m_{\mathrm{f}} v_{\mathrm{b}}^2 \gg 2T_{\mathrm{i}}$. Analogously to these past references, we shall define this $f_{\mathrm{f}}(\mathbf{x}, \mathbf{v})$ also as a function that does not include the exponential structure $\propto \exp(-m_{\mathrm{f}} v^2/2T_{\mathrm{i}})$ at $m_{\mathrm{f}} v^2 \sim 2T_{\mathrm{i}}$ in the energy space, since the distribution function component of $\propto \exp(-m_{\mathrm{f}} v^2/2T_{\mathrm{i}})$ must be handled by the kinetic equation for the thermalized ions with $m_a = m_{\mathrm{f}}$ and $e_a = e_{\mathrm{f}}$ where the self-adjoint property[19] in the following discussion is fully utilized. This energy space structure is a typical example for which we cannot use usual orthogonal expansion methods, and we cannot assume there consequences of the linearization assuming the aforementioned limitations $|\hat{f}_a| \ll 1$ in $m_a v^2 \sim 2T_a$ and $|\hat{f}_a| 1$ even at $m_a v^2 \gg 2T_a$, such as the self-adjoint property. As pointed out previously,[14,20] the Braginskii's matrix expression of collisions also cannot be applicable for this function. Another important feature of this $f_{\mathrm{f}}(\mathbf{x}, \mathbf{v})$ is $p_{\mathrm{f}}/n_{\mathrm{f}}/Z_{\mathrm{f}} \gg T_{\mathrm{e}}, T_{\mathrm{i}}/Z_{\mathrm{i}}$, where Z_{i} is the charge number $Z_a \equiv e_a/e$ of the dominant ion species. Since diamagnetic-driven perpendicular and parallel flow velocities $\mathbf{u}_a$ in toroidal plasmas are determined by $(\partial p_a/\partial r)/n_a/Z_a$ of the species a, and this radial gradient scale length $|\partial \ln p_a/\partial r|^{-1}$ is common for all species, $|u_a| \sim |u_b| \sim |u_c| \sim \ldots. \ll |u_{\mathrm{f}}|$ can be assumed not only for $\langle B\bar{f}_a^{(l=1)} \rangle$ in unbalanced tangential NBIs[1,21] but also for the diamagnetic driven flows due to the radial gradients[2,18] $\partial p_{\perp \mathrm{f}}/\partial r$, $\partial p_{\parallel \mathrm{f}}/\partial r$. (Perpendicular and parallel flow velocities driven by $-c\nabla\Phi \times \mathbf{B}/B^2$ of the ambipolar electrostatic potential being order of $|\nabla\Phi| \sim |(\nabla p_a)/(e_a n_a)|$ $(a \neq \mathrm{f})$ are negligible for fast ions' drift motions and thus not calculated in the kinetic equation for the fast ions.[1,2,18,21]) Therefore, we do not need to retain the Galilean invariant property of the Coulomb collision in Eqs. (1) and (2) so rigorously in the determination of the $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$. It should be emphasized here that, if the self-adjoint property does not exist, the momentum conservation, which should be retained for general colliding species pairs *a*-*b* from the viewpoint of the ambipolarity of classical and neoclassical particle diffusions,[5,14,21] and this Galilean invariant property are inherently different and irrelative concepts. When using the notation in Eqs. (4.5) and (4.6) in Ref. 14 or Eq. (4.30) in Ref. 9 where the test and the field particle portions of the thermal-thermal collision are expressed by M_{ab}^{jk} and N_{ab}^{jk}, respectively, the Galilean invariant property is $M_{ab}^{j0}=-N_{ab}^{j0}$ while the momentum conservation in Eq. (16) is $M_{ab}^{0j}=-T_a v_{\mathrm{T}a}N_{ba}^{0j}/(T_b v_{\mathrm{T}b})$. Because of the self-adjoint property expressed by $M_{ab}^{jk}=M_{ab}^{kj}$ and $N_{ba}^{kj}=T_b v_{\mathrm{T}b}N_{ab}^{jk}/(T_a v_{\mathrm{T}a})$, these two physical laws have been often written in only one relation $M_{ab}^{j0}=M_{ab}^{0j}=-T_a v_{\mathrm{T}a}N_{ba}^{0j}/(T_b v_{\mathrm{T}b})=-N_{ab}^{j0}$ and it has been considered that they should be satisfied simultaneously, in theories for thermal-thermal collisions.[9,14] However, we should distinguish the two laws when handling the fast ions' velocity distribution $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ since the self-adjoint property does not exist there. Based on this distinguishing, following approximations of collisions between fast ions and the thermalized target plasma species are considered. In the kinetic equation for the fast ions, the flow velocities $\mathbf{u}_b$ in the shifted Maxwellians $f_{b\mathrm{M}}(\mathbf{v}-\mathbf{u}_b)$ of target plasma species b can be neglected. In fact, an approximation

$$\sum_b C_{\mathrm{f}b}(f_{\mathrm{f}},f_b)\cong\sum_{b\neq \mathrm{f}} C_{\mathrm{f}b}(f_{\mathrm{f}},f_{b\mathrm{M}}), \tag{22}$$

in which the $\mathbf{u}_b$ are neglected and the non-linear collision operator $C_{\mathrm{ff}}(f_{\mathrm{f}},f_{\mathrm{f}})$ is omitted because of the low-density of fast ions themselves $Z_{\mathrm{f}}^2 n_{\mathrm{f}}\ll Z_{\mathrm{eff}}n_{\mathrm{e}}$ and the momentum/energy conservation of like-particle collisions, has been widely used in past calculations of the $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$.[1,18,21–24] In contrast to the simultaneous algebraic equation for the thermalized particles,[5,6] the algebraic handling of the $\int d^3\mathbf{v}$ integrals of the velocity distribution and the collision term is not required in this independent determination without knowing $\widehat{f}_a(\mathbf{x},\mathbf{v})$ of the thermalized particles. Once one has obtained the $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$, the next step is to solve the kinetic equations for the $\widehat{f}_a(\mathbf{x},\mathbf{v})$, including the newly added $C_{a\mathrm{f}}(f_{a\mathrm{M}},f_{\mathrm{f}})$. Here, $C_{a\mathrm{f}}(\widehat{f}_a f_{a\mathrm{M}},f_{\mathrm{f}})$ should be omitted for retaining the momentum/energy conservation that is realized by Eqs. (14) and (16) in a combined use with Eq. (22). It also is worth to consider here Sonine polynomial expansion coefficients of $C_{ab}(\mathbf{v}\cdot\mathbf{u}L_k^{(3/2)}(x_a^2)f_{a\mathrm{M}},f_b^{(l=0)})$ in the Appendix. We can confirm the unimportance of the thermal particles' flows $n_a\mathbf{u}_a$ and $\mathbf{q}_a\equiv(m_a/2)\int \mathbf{v}v^2 f_a d^3\mathbf{v}-\frac{5}{2}p_a\mathbf{u}_a$ $(a\neq \mathrm{f})$ in the fast ions' friction collision by substituting $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ in Refs. 1, 2, 18, and 21 as $f_b(\mathbf{x},\mathbf{v})$ into Eqs. (16), (17) and (A1)–(A4). The low density $Z_{\mathrm{f}}^2 n_{\mathrm{f}}\ll Z_{\mathrm{eff}}n_{\mathrm{e}}$ giving a relation $|C_{a\mathrm{f}}(\widehat{f}_a f_{a\mathrm{M}},f_{\mathrm{f}})|\ll|\sum_{b\neq \mathrm{f}} C_{ab}(\widehat{f}_a f_{a\mathrm{M}},f_{b\mathrm{M}})|$ also is another reason of this omission. Even for higher Legendre order structures $l\gg 1$ in $\widehat{f}_a$, this $C_{a\mathrm{f}}(\widehat{f}_a f_{a\mathrm{M}},f_{\mathrm{f}})$ is negligible in the total $\sum_b C_{ab}(f_a,f_b)$. In the algebraic conversion of these simultaneous integro-differential equations for $a\neq \mathrm{f}$, direct numerical integrals of Eqs. (16)–(19) are used since we do not use the $\int d^3\mathbf{v}$ integrals of $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$ itself[20] such as $\int v^{1+2n}\xi f_{\mathrm{f}}d^3\mathbf{v}$ in the determination of it, and since we cannot reproduce the energy space structure of it only by such integrals of finite numbers. The recent theoretical calculation for the Heliotron-J experiments[8] adopted this numerical integral method with substituting an analytical solution of the fast ions' collision term Eq. (22)[25] as the $f_{\mathrm{f}}(\mathbf{x},\mathbf{v})$.

IV. CONCLUSION

We have shown a Sonine (generalized Laguerre) polynomial expansion procedure for the field particle portion in the linearized Coulomb collision operator that is applicable for both thermal-thermal collisions[14–17] and thermal particles' collision with fast ions in NBI-heated and/or burning plasmas. In particular, the application to the fast ions' slowing down velocity distribution[1,2,18,21–24] without using its energy-integral moments will be important since (1) this energy space structure cannot be expressed by usual orthogonal polynomial expansions, and (2) the self-adjoint property does not exist for energy-integrals of the distribution function and the collision operator, and thus the Galilean invariant property and the momentum conservation must be distinguished. In this Sonine polynomial expansion procedure, we did not use an assumption of $2T_{\mathrm{i}}/m_{\mathrm{i}}\ll v_{\mathrm{b}}^2\ll 2T_{\mathrm{e}}/m_{\mathrm{e}}$ that had been frequently used in past analytical theories on the fast ions' slowing down process.[1,2,14,18,20,25] From the viewpoint of the field particle portion $C_{a\mathrm{f}}(f_{a\mathrm{M}},f_{\mathrm{f}})$, the assumption $v_{\mathrm{b}}^2\ll 2T_{\mathrm{e}}/m_{\mathrm{e}}$ previously corresponded to the use of the usual small mass ratio approximation for the electron-ion collisions[9] also for the e-f collision $C_{\mathrm{ef}}(f_{\mathrm{eM}},f_{\mathrm{f}})$ in calculations of the shielding current component in the beam driven currents.[14] For future studies requiring the $C_{a\mathrm{f}}(f_{a\mathrm{M}},f_{\mathrm{f}})$ of all thermal particle species $a\neq \mathrm{f}$, however, these kinds of asymptotic limit approximations giving the e-f and the i-f collision formulas separately will be confusing and inconvenient. Therefore, we unified the formulas for electrons and thermal ions based on a derivation procedure allowing arbitrary energy space structures of the field particles' velocity distributions.

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APPENDIX: LAGUERRE EXPANSION OF EQ. (10) FOR THE 13M APPROXIMATION

By integrations by parts using Eqs. (11) and (12) for $\int_0^\infty dv$, Sonine polynomial expansion coefficients of $C_{ab}(\mathbf{v}\cdot\mathbf{u}L_k^{(3/2)}(x_a^2)f_{a\mathrm{M}},f_b^{(l=0)})$, where $f_a=\mathbf{v}\cdot\mathbf{u}L_k^{(3/2)}(x_a^2)f_{a\mathrm{M}}$ is substituted in Eq. (10), are given as follows:

$$m_a \int \mathbf{v} C_{ab}\left(\frac{m_a}{T_a}\mathbf{v}\cdot\mathbf{u}_a f_{a\mathrm{M}}, f_b^{(l=0)}\right)\mathrm{d}^3\mathbf{v} = -m_b \int \mathbf{v} C_{ba}\left(f_b^{(l=0)}, \frac{m_a}{T_a}\mathbf{v}\cdot\mathbf{u}_a f_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}$$
$$= -n_a\mathbf{u}_a \frac{64\pi^{3/2}}{3}(e_a e_b)^2 \ln\Lambda_{ab}\left(\frac{1}{m_a}+\frac{1}{m_b}\right)\int_0^\infty f_b^{(l=0)}\exp(-x_a^2)x_a^2\mathrm{d}x_a, \tag{A1}$$

$$m_a \int \mathbf{v} C_{ab}\left(\frac{m_a}{T_a}\frac{\mathbf{v}\cdot\mathbf{q}_a}{p_a}L_1^{(3/2)}(x_a^2) f_{a\mathrm{M}}, f_b^{(l=0)}\right)\mathrm{d}^3\mathbf{v} = -m_b \int \mathbf{v} C_{ba}\left(f_b^{(l=0)}, \frac{m_a}{T_a}\frac{\mathbf{v}\cdot\mathbf{q}_a}{p_a}L_1^{(3/2)}(x_a^2) f_{a\mathrm{M}}\right)\mathrm{d}^3\mathbf{v}$$
$$= -\frac{\mathbf{q}_a}{T_a}\frac{64\pi^{3/2}}{3}(e_a e_b)^2 \ln\Lambda_{ab}\left(\frac{1}{m_a}+\frac{1}{m_b}\right)\int_0^\infty f_b^{(l=0)}\left(\frac{3}{2}-x_a^2\right)\exp(-x_a^2)x_a^2\mathrm{d}x_a, \tag{A2}$$

$$m_a \int \mathbf{v} L_1^{(3/2)}(x_a^2) C_{ab}\left(\frac{m_a}{T_a}\mathbf{v}\cdot\mathbf{u}_a f_{a\mathrm{M}}, f_b^{(l=0)}\right)\mathrm{d}^3\mathbf{v} = -\, n_a\mathbf{u}_a \frac{64\pi^{3/2}}{3}\frac{(e_a e_b)^2\ln\Lambda_{ab}}{m_a}$$
$$\times \int_0^\infty f_b^{(l=0)}\left[5\sqrt{\pi}x_a G(x_a) + \left\{\frac{3}{2} - \frac{1}{2}\frac{m_a}{m_b} - 3\left(\frac{m_a}{m_b}+1\right)x_a^2\right\}\exp(-x_a^2)\right]x_a^2\mathrm{d}x_a, \tag{A3}$$

$$m_a \int \mathbf{v} L_1^{(3/2)}(x_a^2) C_{ab}\left(\frac{m_a}{T_a}\frac{\mathbf{v}\cdot\mathbf{q}_a}{p_a}L_1^{(3/2)}(x_a^2) f_{a\mathrm{M}}, f_b^{(l=0)}\right)\mathrm{d}^3\mathbf{v} = -\frac{\mathbf{q}_a}{T_a}\frac{64\pi^{3/2}}{3}\frac{(e_a e_b)^2\ln\Lambda_{ab}}{m_a}$$
$$\times \int_0^\infty f_b^{(l=0)}\left\{\frac{13}{4} + \frac{9m_a}{4m_b} - \left(1+4\frac{m_a}{m_b}\right)x_a^2 + 3\left(1+\frac{m_a}{m_b}\right)x_a^4\right\}\exp(-x_a^2)x_a^2\mathrm{d}x_a. \tag{A4}$$

The momentum conservation is used in Eqs. (A1) and (A2). When $f_b^{(l=0)} = f_{b\mathrm{M}}$ is substituted, these formulas reproduce the usual friction matrix elements M_{ab}^{jk} for the 13M approximation in Refs. 9, 14–17. The $M_{jk}^{a/\mathrm{f}}$ matrix discussed in Ref. 20 will be obtained by substituting the surface-averaged lowest Legendre order $l=0$ of the fast ions' velocity distribution[21] $\langle f_\mathrm{f}^{(l=0)}\rangle \propto [v^2 v_{\mathrm{Te}}(3\sqrt{\pi}/2)G(x_\mathrm{e}) + v_\mathrm{c}^3]^{-1}H(v_\mathrm{b}-v)$ as the field particles' velocity distribution $f_b^{(l=0)}$, and taking a limit of $x_\mathrm{e}^2 \ll 1$ and $m_\mathrm{e}/m_\mathrm{f} \ll 1$ (corresponding to the usual small mass ratio approximation of electron-ion collisions[9]) for electrons' friction $a=\mathrm{e}$ and a limit of $v_{\mathrm{Ti}}^3 \ll v_\mathrm{c}^3$ for ions' friction $a=\mathrm{i}$ in calculations of $\exp(-x^2)$ and $G(x) \cong \{(3\sqrt{\pi}/2)/x + 2x^2\}^{-1}$.

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