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ABSTRACT

The Eulerian variational formulation of the gyrokinetic system with electrostatic turbulence is presented in general spatial coordinates by extending our previous work [H. Sugama *et al.*, Phys. Plasmas **25**, 102506 (2018)]. The invariance of the Lagrangian of the system under an arbitrary spatial coordinate transformation is used to derive the local momentum balance equation satisfied by the gyrocenter distribution functions and the turbulent potential, which are given as solutions of the governing equations. In the symmetric background magnetic field, the derived local momentum balance equation gives rise to the local momentum conservation law in the direction of symmetry. This derivation is in contrast to the conventional method using the spatial translation in which the asymmetric canonical pressure tensor generally enters the momentum balance equation. In the present study, the variation of the Lagrangian density with respect to the metric tensor is taken to directly obtain the symmetric pressure tensor, which includes the effect of turbulence on the momentum transport. In addition, it is shown in this work how the momentum balance is modified when the collision and/or external source terms are added to the gyrokinetic equation. The results obtained here are considered useful for global gyrokinetic simulations investigating both neoclassical and turbulent transport processes even in general non-axisymmetric toroidal systems.

I. INTRODUCTION

Gyrokinetics[1–7] has been used for several decades as a basic theoretical framework to study microinstabilities and turbulent processes in magnetized plasmas.[8] A significant number of large-scale simulations are being performed based on gyrokinetic equations to analyze and predict turbulent transport fluxes of particles, heat, and momentum.[9,10] The gyrokinetic equations derived from the Lagrangian and/or Hamiltonian describing the gyrocenter motion possess conservation properties,[4,5] which are suitable for long-time and global transport simulations. Variational formulations based on the action integral of the Lagrangian provide a useful and systematic means to obtain governing equations and conservation laws of energy and momentum for considered systems.[4,5,11–13] The variational formulations are also applied to derive the useful conservative numerical schemes[14–18] in plasma physics for solving the guiding center equations and the ideal magnetohydrodynamics (MHD) equations as well as the Vlasov–Poisson and Vlasov–Maxwell equations.

Because background flow profiles are considered as one of the key factors for improving plasma confinement, momentum transport processes that determine the flow profiles are investigated by large-scale gyrokinetic simulations.[19–23] Thus, the momentum balance equation satisfied by the gyrokinetic model attracts our attention as a basis for theoretically and/or numerically investigating the physical mechanisms in the formation of the flow profiles.[24–29] In our previous work,[30] the Eulerian formulation,[31] which is also called the Euler–Poincaré reduction procedure,[11,13,32–34] is applied to derive the governing equations of the Vlasov–Poisson–Ampère (or Vlasov–Darwin) system and those of the drift kinetic system in the general spatial coordinates, and the momentum balance equations for these systems are obtained by using the invariance of the action integrals for the systems under arbitrary spatial coordinate transformations. In this paper, the previous work is extended to present the Eulerian variational formulation of the gyrokinetic system with electrostatic turbulence in general spatial coordinates and to derive the local momentum balance equation with the symmetric pressure tensor including the effects of electrostatic turbulent fluctuations on the momentum transport.

In the present work, the governing equations of the gyrokinetic system are represented in general spatial coordinates so that they are useful for application to systems with complex geometries such as

stellarator and heliotron plasmas[35] in which it is convenient to employ flux surface coordinates (e.g., Hamada coordinates[36] and Boozer coordinates[37]) to express these equations for analytical and numerical studies. In our formulation, the momentum balance equation is derived using the invariance of the Lagrangian under arbitrary spatial coordinate transformations, which is analogous to the derivation of energy-momentum conservation laws from the invariance of the action integral under arbitrary transformations of spatiotemporal coordinates in the theory of general relativity.[38] In the same way as in the previous work,[30] the symmetric pressure tensor entering our momentum balance equation is directly obtained by taking the variational derivative of the Lagrangian density with respect to the metric tensor, which is in contrast to the conventional technique using the spatial translation transformation for the derivation of the canonical momentum balance including the asymmetric pressure tensor in the presence of the magnetic field. The canonical pressure tensor is asymmetric and gauge-dependent because of the vector potential included in the canonical momentum. In order to obtain the symmetric pressure tensor from the asymmetric canonical pressure tensor, additional complicated procedures of the Belinfante–Rosenfeld type using the angular momentum conservation law derived from the rotational symmetry are required.[39,40] On the other hand, our method can more directly derive the symmetric pressure tensor, which is clearly shown in this work to describe both neoclassical[41,42] and turbulent transport of the momentum in the gyrokinetic system.

We should note here that while employing the Hamiltonian gyrocenter motion equations, an irreversible collision term must be included into the gyrokinetic equation in order to treat neoclassical and turbulent transport simultaneously.[43,44] It is shown in our formulation how the momentum balance equations are modified when the collision and/or external source terms are added into the gyrokinetic equation. This is possible because the momentum balance equations are derived from the invariance of the Lagrangian under the general spatial coordinate transformations, with the help of the gyrocenter motion equations and the gyrokinetic Poisson's equation while we have freedom in choosing the governing equation for the gyrocenter distribution function. The momentum balance equations including the effects of collisions and external sources are useful for checking and analyzing results of global gyrokinetic simulations using the Lagrangian and Hamiltonian equations to investigate neoclassical and turbulent transport in plasmas with particle, momentum, and/or heat sources.

It is valuable to make comparisons of the present work with recent works[45,46] on similar subjects. In Ref. 45, a constrained Eulerian variational principle presented by Brizard[47] is used to derive the governing equations and local energy-momentum conservation laws of the electromagnetic gyrokinetic Vlasov–Maxwell system, for which the gyrocenter motion equations are expressed only in terms of the perturbed electric and magnetic fields by including the perturbed fields in the Poisson-bracket structure. In Ref. 46, the field theory presented by Qin *et al.*[48] for particle-field systems on heterogenous manifolds is applied to derive local energy and momentum conservation laws in the electromagnetic gyrokinetic system. Except for dropping magnetic fluctuations and some of the second or higher order terms with respect to the small perturbation amplitude, the gyrokinetic system treated here is basically the same as seen in other earlier works,[4,5] and it contains full finite gyroradius effects due to electrostatic fluctuations with high wavenumbers, which are not included in Ref. 46. Also, in the same way as in our previous work,[30] the governing equations for the gyrokinetic system are derived in this paper based on the Eulerian (or the Euler–Poincaré) formulation, which is historically older than the methods employed in Refs. 45 and 46. The Euler–Poincaré formulation was used in the pioneering work by Newcomb[31] to derive the ideal MHD equations, and later it was applied to the derivation of the Vlasov–Maxwell equations, the guiding-center (or drift kinetic) system, as well as the gyrokinetic system as shown in Refs. 11, 13, 30, and 32–34. Here, the Eulerian formulation implies that, for the present gyrokinetic case, the phase-space velocity (or the temporal change rate of the gyrocenter coordinates in the phase space) of the gyrocenter is regarded as a field function of time and phase-space coordinates of the gyrocenter at the time when the gyrocenter passes through the considered point. In the formulation presented in Refs. 46 and 48, the gyrocenter (or particle) phase-space velocity is described by not the Eulerian but Lagrangian viewpoint and is coupled with the Eulerian description of the electromagnetic fields. In Refs. 45 and 46, not general but isometric transformations such as spatial translation and rotation are considered to derive local conservation laws of canonical linear and angular momentum in collisionless systems. In the present work, recognizing that the Lagrangian and the governing equations of the gyrokinetic system can be expressed in the invariant form under arbitrary spatial coordinate transformations in the Eulerian framework, this invariance property is used to derive the local momentum balance equation even in the presence of collisions and external sources, and the derived balance equation is shown to give the local momentum conservation law in the direction of symmetry for collisionless systems without external sources.

The rest of this paper is organized as follows: in Sec. II, the Lagrangian for describing the single-particle gyrocenter motion is given, in which the electrostatic potential fluctuation with the wavelength of the order of the gyroradius is included. Next, in Sec. III, we use the general spatial coordinates and define the Lagrangian of the whole system including particles of all species and turbulent electrostatic fields to present the Eulerian variational principle, from which the collisionless gyrokinetic equations for the gyrocenter distribution functions and the gyrokinetic Poisson's equation for the electrostatic potential are derived. In Sec. IV, we make use of the invariance of the Lagrangians for the single-species and whole systems under arbitrary spatial coordinate transformations to derive the momentum balance equations for both systems while allowing the collision and/or external source terms to be included in the gyrokinetic equations. Then, the symmetric pressure tensors, which enter these momentum balance equations, are obtained from the variational derivatives of the Lagrangians with respect to the metric tensor components and they are verified to describe both neoclassical and turbulent transport of the momentum. Finally, conclusions are given in Sec. V. In addition, Appendix A is given to briefly explain covariant derivatives and Christoffel symbols, which are used in the general spatial coordinates. Variations in the functional forms of vector and tensor fields under the infinitesimal transformation of the spatial coordinates and variational derivative with respect to the metric tensor components are described in Appendixes B and C, respectively. In Appendix D, the WKB representation[49] is used to express the turbulent part of the pressure tensor in the form comparable with the one obtained in the past work on the momentum transport. Another derivation of the

momentum balance equations, in which the asymmetric canonical pressure tensors appear, is shown in Appendix E and the energy balance equations are derived in Appendix F. The case of the symmetric background magnetic field is considered in Appendix G where it is shown how the local conservation law of the canonical momentum in the direction of symmetry is derived in the present formulation.

II. THE GYROCENTER LAGRANGIAN

The Lagrangian for describing the gyrocenter motion of the particle with mass m and charge e is given by[4]

$$L_{GY} \equiv \left(\frac{e}{c}\mathbf{A}(\mathbf{X},t) + mv_{\parallel}\mathbf{b}(\mathbf{X},t)\right)\cdot\dot{\mathbf{X}} + \frac{mc}{e}\mu\dot{\vartheta} - H_{GY}, \tag{1}$$

where the gyrocenter phase-space coordinates $\mathbf{X}$, $v_{\parallel}$, $\mu \equiv mv_{\perp}^2/(2B)$, and ϑ denote the gyrocenter position, the velocity component parallel to the magnetic field, the magnetic moment, and the gyrophase angle, respectively, and $\dot{} \equiv d/dt$ represents the time derivative along the trajectory in the phase space. The vector potential and the unit vector parallel to the magnetic field $\mathbf{B}$ are written by $\mathbf{A}$ and $\mathbf{b} \equiv \mathbf{B}/B$, respectively. Here, we suppose that $\mathbf{A}$ can weakly depend on time t and, accordingly, the background magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ is allowed to slowly vary in time. Thus, we can treat the inductive electric field $-c^{-1}\partial\mathbf{A}/\partial t$, which drives the Ohmic current in tokamaks.

The gyrocenter Hamiltonian that appears on the right-hand side of Eq. (1) is defined by

$$H_{GY} \equiv \frac{1}{2}mv_{\parallel}^2 + \mu B + e\Psi \tag{2}$$

and its electrostatic fluctuation part is written as

$$e\Psi \equiv e\langle\phi(\mathbf{X}+\boldsymbol{\rho},t)\rangle_{\vartheta} - \frac{e^2}{2B}\frac{\partial}{\partial\mu}\langle(\tilde{\phi})^2\rangle_{\vartheta}, \tag{3}$$

where the gyroradius vector is denoted by $\boldsymbol{\rho} \equiv \mathbf{b}\times\mathbf{v}/\Omega$, $\mathbf{v}$ is the particle's velocity, $\Omega \equiv eB/(mc)$ is the gyrofrequency, and the average of the electrostatic potential ϕ at the particle position $\mathbf{X}+\boldsymbol{\rho}$ over the gyrophase ϑ is given by

$$\langle\phi(\mathbf{X}+\boldsymbol{\rho})\rangle_{\vartheta} \equiv \oint\frac{d\vartheta}{2\pi}\phi(\mathbf{X}+\boldsymbol{\rho}), \tag{4}$$

and the gyrophase-dependent part is denoted by

$$\tilde{\phi} \equiv \phi(\mathbf{X}+\boldsymbol{\rho}) - \langle\phi(\mathbf{X}+\boldsymbol{\rho})\rangle_{\vartheta}. \tag{5}$$

The last term on the right-hand side of Eq. (3) is of the second order in the gyrokinetic ordering parameter ϵ, where $\epsilon \sim e\tilde{\phi}/(m|\mathbf{v}|^2) \sim \rho/L$ with L representing the equilibrium gradient scale length is assumed. This second-order term is retained because it is necessary for deriving the gyrokinetic Poisson's equation correctly including the polarization effect as shown in Sec. III C. However, other second-order terms shown in Ref. 50 are neglected in the gyrocenter Hamiltonian given by Eqs. (2) and (3). Therefore, rigorously speaking, the accuracy of the present model is up to the first order. The turbulent fluctuations are assumed to have the characteristic wavelength $\sim(k_{\perp})^{-1} \sim \rho$. Then, the fluctuation terms in Eq. (3) are considered to contain all-order terms in $k_{\perp}\rho(\sim 1)$ even though small amplitude terms of higher order in $\epsilon \sim \rho/L(\ll 1)$ are neglected. In Eqs. (4) and (5) as well as in the equations shown below, the time variable t on which ϕ depends is omitted for simplicity.

In this section, the Cartesian spatial coordinates are used and three-dimensional vectors are represented in terms of boldface letters. Then, the electrostatic potential $\phi(\mathbf{X}+\boldsymbol{\rho})$ is Taylor expanded about the gyrocenter position $\mathbf{X}$ as

$$\phi(\mathbf{X}+\boldsymbol{\rho}) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}\rho^{j_1}\cdots\rho^{j_n}\frac{\partial^n\phi(\mathbf{X})}{\partial X^{j_1}\cdots\partial X^{j_n}}, \tag{6}$$

where X^j and ρ^j ($j=1,2,3$) and the Cartesian spatial coordinates of the gyrocenter position vector $\mathbf{X}$ and the gyroradius vector $\boldsymbol{\rho}$, respectively. Substituting Eq. (6) into Eq. (4), we obtain

$$\langle\phi(\mathbf{X}+\boldsymbol{\rho})\rangle_{\vartheta} = \sum_{n=0}^{\infty}\sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}\frac{\alpha^{j_1\cdots j_n}}{n!}\frac{\partial^n\phi(\mathbf{X})}{\partial X^{j_1}\cdots\partial X^{j_n}}, \tag{7}$$

where the gyrophase average of a product of n gyroradius vector components is denoted by

$$\alpha^{j_1\cdots j_n} \equiv \langle\rho^{j_1}\cdots\rho^{j_n}\rangle_{\vartheta}. \tag{8}$$

Obviously, $\alpha^{j_1\cdots j_n}$ is symmetric with respect to arbitrary permutations of the indices $j_1,\ldots,j_n$. It can be shown that

$$\alpha^{j_1\cdots j_n} = 0 \quad \text{for odd } n \tag{9}$$

and

$$\alpha^{j_1\cdots j_{2l}} = \frac{1}{(2l)!}\sum_{\sigma\in\mathfrak{S}_{2l}}\eta^{j_{\sigma(1)}\cdots j_{\sigma(2l)}}, \tag{10}$$

where $\mathfrak{S}_{2l}$ is the symmetric group of permutations of the set $\{1,2,\ldots,2l\}$ and $\eta^{j_1\cdots j_{2l}}$ is defined by

$$\eta^{j_1\cdots j_{2l}} = \frac{(2l)!}{(l!)^2}\left(\frac{\rho}{2}\right)^{2l}h^{j_1j_2}h^{j_3j_4}\cdots h^{j_{2l-1}j_{2l}}, \tag{11}$$

with

$$\rho \equiv \frac{c}{e}\sqrt{\frac{2m\mu}{B}} \tag{12}$$

and

$$h^{ij} \equiv \delta^{ij} - b^ib^j. \tag{13}$$

Here, b^i is the ith contravariant component of $\mathbf{b} \equiv \mathbf{B}/B$ and δ^{ij} represents the Kronecker delta; $\delta^{ij} = 1$ (for $i=j$), 0 (for $i\neq j$). In Eq. (3), we also find that

$$\langle(\tilde{\phi})^2\rangle_{\vartheta} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i_1=1}^{3}\cdots\sum_{i_m=1}^{3}\sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}\frac{\beta^{i_1\cdots i_m;j_1\cdots j_n}}{m!\,n!} \times\frac{\partial^n\phi(\mathbf{X})}{\partial X^{i_1}\cdots\partial X^{i_m}}\frac{\partial^n\phi(\mathbf{X})}{\partial X^{j_1}\cdots\partial X^{j_n}}, \tag{14}$$

where

$$\beta^{i_1\cdots i_m;j_1\cdots j_n} \equiv \alpha^{i_1\cdots i_mj_1\cdots j_n} - \alpha^{i_1\cdots i_m}\alpha^{j_1\cdots j_n} \tag{15}$$

is defined and it satisfies

$$\beta^{i_1\cdots i_m;j_1\cdots j_n} = 0 \quad \text{for odd}(m+n). \tag{16}$$

The expressions given in Eqs. (7), (13), and (14) are valid in the Cartesian spatial coordinates although they can be easily transformed into those in general spatial coordinates as shown in Sec. III.

III. EULERIAN VARIATIONAL PRINCIPLE FOR DERIVATION OF THE COLLISIONLESS GYROKINETIC SYSTEM OF EQUATIONS IN GENERAL SPATIAL COORDINATES

In this section, the governing equations for the distribution functions and the turbulent electrostatic potential in the collisionless gyrokinetic system are derived from the Eulerian variational principle using the Lagrangian density represented in general spatial coordinates.

A. The Lagrangian density represented in general spatial coordinates

The action integral I_{GKF} for the Eulerian variational principle to derive all of the governing equations of the collisionless electrostatic gyrokinetic turbulent system is written as

$$I_{GKF} \equiv \int_{t_1}^{t_2} dt\, L_{GKF} \equiv \int_{t_1}^{t_2} dt \int_V d^3x\, \mathcal{L}_{GKF}, \tag{17}$$

where the Lagrangian density $\mathcal{L}_{GKF}$ is given by

$$\begin{aligned}\mathcal{L}_{GKF} &\equiv \mathcal{L}_{GK} + \mathcal{L}_F \\ &\equiv \sum_a \int d^3v\, F_a(x,v,t) L_{GYa}(x,v,t) \\ &\quad + \frac{\sqrt{g}}{8\pi} g^{ij}(x) (E_L)_i(x,t) (E_L)_j(x,t).\end{aligned} \tag{18}$$

Here, $d^3x \equiv dx^1 dx^2 dx^3$ and $d^3v \equiv dv_\parallel d\mu d\vartheta$ are used and the subscript a represents the particle species with mass m_a and charge e_a. We use $x \equiv (x^i)_{i=1,2,3}$ and $v \equiv (v_\parallel, \mu, \vartheta)$ as the gyrocenter phase-space coordinates. It should be emphasized that, in this section, $x \equiv (x^i)_{i=1,2,3}$ represent general spatial coordinates of the gyrocenter position, which can be either Cartesian or any other curved coordinates. However, here we assume that the spatial position vector $\mathbf{r} = \mathbf{r}(x)$ is a function of only the spatial coordinates $x \equiv (x^i)_{i=1,2,3}$ and it is independent of time t. The gyrocenter distribution function on the (x, v)-space is denoted by F_a, and the number of particles of species a in the phase-space volume element $d^3x d^3v \equiv dx^1 dx^2 dx^3 dv_\parallel d\mu d\vartheta$ at time t is given by $F_a(x,v,t) d^3x d^3v$. The field part $\mathcal{L}_F$ of the Lagrangian density $\mathcal{L}_{GKF}$ in Eq. (18) is written in terms of the covariant components of the electric field due to the electrostatic potential,

$$(E_L)_i \equiv -\frac{\partial \phi(x,t)}{\partial x^i}. \tag{19}$$

In this paper, we employ the summation convention that the same symbol used for a pair of upper and lower indices within a term [such as seen in Eq. (18) as well as in the equations shown below] indicates summation over the range $\{1,2,3\}$ of the symbol index. The contravariant metric tensor components g^{ij} in the general spatial coordinates $x \equiv (x^i)$ are related to the covariant components g_{ij} by $g^{ik} g_{kj} = \delta^i_j$, where δ^i_j represents the Kronecker delta. The determinant of the covariant metric tensor matrix is denoted by

$$g(x) \equiv \det\left[g_{ij}(x)\right]. \tag{20}$$

Note that $g_{ij}(x)$, $g^{ij}(x)$, and $g(x)$ are all independent of time t because the spatial position vector $\mathbf{r}$ is assumed to be given by a function of only the spatial coordinates $x \equiv (x^i)_{i=1,2,3}$, independently of time t as mentioned earlier.

The gyrocenter Lagrangian for species a, which is multiplied by F_a to define $\mathcal{L}_{GK}$ in Eq. (18), is represented in the Eulerian picture by

$$\begin{aligned}L_{GYa} &\equiv \left(\frac{e_a}{c} A_j(x,t) + m_a v_\parallel b^i(x,t) g_{ij}(x)\right) u^j_{ax}(x,v,t) \\ &\quad + \frac{m_a c}{e_a} \mu u_{a\vartheta}(x,v,t) - H_{GYa}(x, v_\parallel, \mu, t),\end{aligned} \tag{21}$$

where A_j is the jth covariant component of the vector potential, $b^i \equiv B^i/B$ is the ith contravariant component of the unit vector parallel to the background magnetic field, and the magnetic field strength is given by

$$B(x,t) \equiv \sqrt{g_{ij}(x) B^i(x,t) B^j(x,t)}. \tag{22}$$

The contravariant components $(B^i)_{i=1,2,3}$ of the magnetic field are expressed in terms of the covariant components $(A_i)_{i=1,2,3}$ of the vector potential as

$$B^i(x,t) = \frac{\epsilon^{ijk}}{\sqrt{g(x)}} \frac{\partial A_k(x,t)}{\partial x^j}, \tag{23}$$

where the Levi–Civita symbol is denoted by

$$\begin{aligned}\epsilon^{ijk} &\equiv \epsilon_{ijk} \\ &\equiv \begin{cases} 1 & ((i,j,k) = (1,2,3),(2,3,1),(3,1,2)) \\ -1 & ((i,j,k) = (1,3,2),(2,1,3),(3,2,1)) \\ 0 & (\text{otherwise}). \end{cases}\end{aligned} \tag{24}$$

The gyrocenter Hamiltonian is written here as

$$H_{GYa}(x, v_\parallel, \mu, t) \equiv \frac{1}{2} m_a v_\parallel^2 + \mu B(x,t) + e_a \Psi_a(x,\mu,t) \tag{25}$$

and the electrostatic fluctuation part is given by

$$\Psi_a(x,\mu,t) \equiv \phi(x,t) + \Psi_{E1a}(x,\mu,t) + \Psi_{E2a}(x,\mu,t), \tag{26}$$

where

$$\Psi_{E1a}(x,\mu,t) \equiv \sum_{n=1}^{\infty} \frac{\alpha_a^{j_1\cdots j_n}(x,\mu,t)}{n!} \nabla_{j_1} \cdots \nabla_{j_n} \phi(x,t) \tag{27}$$

and

$$\begin{aligned}\Psi_{E2a}(x,\mu,t) &\equiv -e_a \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(m+n)}{m!\, n!} \frac{\beta_a^{i_1\cdots i_m;j_1\cdots j_n}(x,\mu,t)}{4\mu B(x,t)} \\ &\quad \times \left(\nabla_{i_1} \cdots \nabla_{i_m} \phi(x,t)\right) \left(\nabla_{j_1} \cdots \nabla_{j_n} \phi(x,t)\right).\end{aligned} \tag{28}$$

Here, $\alpha_a^{j_1\cdots j_n}(x,\mu,t)$ and $\beta_a^{i_1\cdots i_m;j_1\cdots j_n}(x,\mu,t)$ are defined by Eqs. (8)–(12) and (15) in which h^{ij} is represented in the general spatial coordinates $x \equiv (x^i)_{i=1,2,3}$ by

$$h^{ij} \equiv g^{ij} - b^i b^j \tag{29}$$

and the covariant derivative ∇_j is defined in Appendix A.

The Eulerian representation of the gyrocenter Lagrangian L_{GYa} shown in Eq. (21) contains the temporal change rates of the gyrocenter position coordinates, parallel velocity, magnetic moment, and gyrophase at time t, which are represented by the functions $u^i_{ax}(x,v,t)$, $u_{av_\parallel}(x,v,t)$, $u_{a\mu}(x,v,t)$, and $u_{a\vartheta}(x,v,t)$, respectively, in the same way as in Ref. 30. Then, the distribution function $F_a(x,v,t)$ satisfies

$$\frac{\partial F_a}{\partial t} + \frac{\partial}{\partial x^j}(F_a u^j_{ax}) + \frac{\partial}{\partial v_\parallel}(F_a u_{av_\parallel}) + \frac{\partial}{\partial \mu}(F_a u_{a\mu}) + \frac{\partial}{\partial \vartheta}(F_a u_{a\vartheta}) = 0. \tag{30}$$

We find here that the gyrocenter Hamiltonian $H_{GY\,a}$ given in Eq. (25) takes a functional form,

$$H_{GYa} = H_{GYa}\big[v, \partial_j A_i(x,t), \{\partial_J \phi(x,t)\}, \{\partial_J g_{ij}(x)\}\big], \tag{31}$$

which depends on the velocity space coordinates (except for ϑ) as well as the general spatial coordinates $x \equiv (x^i)_{i=1,2,3}$ through the field variables $[\partial_j A_i(x,t), \{\partial_J \phi(x,t)\}, \{\partial_J g_{ij}(x)\}]$. Here, we use the notation $J \equiv (j_1, j_2, \ldots, j_n)$ (for $n = 0, 1, 2, \ldots$ and $j_1, j_2, \ldots, j_n = 1, 2, 3$) to write

$$\partial_J \mathcal{F} \equiv \begin{cases} \mathcal{F} & (n=0) \\ \partial_{j_1 j_2 \cdots j_n} \mathcal{F} \equiv \partial^n \mathcal{F} / \partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_n} & (n \geq 1) \end{cases}, \tag{32}$$

where $\mathcal{F}$ is an arbitrary function of $x = (x^i)_{i=1,2,3}$. Then, the compact notations $\{\partial_J \phi(x,t)\}$ and $\{\partial_J g_{ij}(x)\}$ in Eq. (31) imply

$$\{\partial_J \phi\} \equiv \{\phi, \partial_j \phi, \partial_{jk}\phi, \partial_{jkl}\phi, \ldots\} \tag{33}$$

and

$$\{\partial_J g_{ij}\} \equiv \{g_{ij}, \partial_k g_{ij}, \partial_{kl} g_{ij}, \partial_{klm} g_{ij}, \ldots\}, \tag{34}$$

respectively. Note that the high-order spatial derivatives included in Eqs. (33) and (34) enter the gyrocenter Hamiltonian $H_{GY\,a}$ due to finite gyroradius as seen in Eqs. (27) and (28) where the covariant derivatives contain the spatial derivatives of g_{ij} through the Christoffel symbols defined by Eq. (A4) in Appendix A.

In the same way as in Eq. (31), the functional form of the gyrocenter Lagrangian $L_{GY\,a}$ is written as

$$L_{GYa} = L_{GYa}\big[v, u^i_{ax}(x,v,t), u_{a\vartheta}(x,v,t), A_i(x,t), \partial_j A_i(x,t), \{\partial_J \phi(x)\}, \{\partial_J g_{ij}(x)\}\big], \tag{35}$$

where the Eulerian representations of the temporal change rates of the gyrocenter position and the gyrophase, $u^i_{ax}(x,v,t)$ and $u_{a\vartheta}(x,v,t)$, are additionally included. In Eq. (35), $u_{ax}(x,v,t)$, $u_{a\vartheta}(x,v,t)$, and $\phi(x,t)$ are the functions, the governing equations of which are derived from the variation principle in Sec. III C, while the dependence of $L_{GY\,a}$ on $A_i(x,t)$, $\partial_j A_i(x,t)$, and $\partial_J g_{ij}(x,t)$ is also explicitly shown because their variations need to be taken into account to evaluate the variation of $L_{GY\,a}$ in Sec. IV where, in order to derive the local momentum balance, we consider the general spatial coordinate transformation, which causes the variations in the functional forms of both $(u_{ax}, u_{a\vartheta}, \phi)$ and (A_i, g_{ij}). Here, using Eqs. (19) and (33), we note that $\{\partial_J \phi\}$ can be replaced by

$$\{\phi, \{\partial_J (E_L)_i\}\}, \tag{36}$$

where

$$\{\partial_J (E_L)_i\} \equiv \{(E_L)_i, \partial_j (E_L)_i, \partial_{jk}(E_L)_i, \partial_{jkl}(E_L)_i, \ldots\}. \tag{37}$$

B. The Lagrangian density associated with polarization

We find from Eqs. (21) and (25)–(28) that the part of the Lagrangian density $L_{GY\,a}$, which includes the perturbed potential is given by

$$\mathcal{L}_{\Psi a} \equiv -\int d^3v\, F_a\, e_a \Psi_a = -e_a N^{(g)}_a \phi + \mathcal{L}_{E1a} + \mathcal{L}_{E2a}, \tag{38}$$

where the gyrocenter density $N^{(g)}_a$ is defined by

$$N^{(g)}_a \equiv \int d^3v\, F_a, \tag{39}$$

$\mathcal{L}_{E1a}$ is given in the linear form with respect to the longitudinal electric field $(E_L)_i \equiv -\partial\phi/\partial x^i$ and its covariant derivatives,

$$\mathcal{L}_{E1a} \equiv -\int d^3v\, F_a\, e_a \Psi_{E1a} = \sum_{k=1}^{\infty} Q^{j_1 \cdots j_{2k}}_{0a} \nabla_{j_1} \cdots \nabla_{j_{2k-1}} (E_L)_{j_{2k}}, \tag{40}$$

and $\mathcal{L}_{E2a}$ is given in the quadratic form,

$$\begin{aligned} \mathcal{L}_{E2a} &\equiv -\int d^3v\, F_a\, e_a \Psi_{E2a} \\ &= \frac{1}{2}\sum_{m=1}^{\infty} Q^{i_1 \cdots i_m}_{Ea} \nabla_{i_1} \cdots \nabla_{i_{m-1}} (E_L)_{i_m} \\ &= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \chi^{i_1 \cdots i_m; j_1 \cdots j_n}_a \nabla_{i_1} \cdots \nabla_{i_{m-1}} (E_L)_{i_m} \\ &\quad \times \nabla_{j_1} \cdots \nabla_{j_{n-1}} (E_L)_{j_n}. \end{aligned} \tag{41}$$

In this paper, the longitudinal (irrotational) and transverse (solenoidal) parts of any vector field are represented by the subscripts L and T, respectively.[51] In Eq. (40), there appear the multipole moments $Q^{j_1 \cdots j_{2k}}_{0a}$ of the electric charge distribution[51] of species a induced by finite gyroradius,

$$Q^{j_1 \cdots j_{2k}}_{0a} \equiv e_a \int d^3v\, F_a \frac{\alpha_a^{j_1 \cdots j_{2k}}}{(2k)!}, \tag{42}$$

which can exist even without the electric field $(E_L)_i$. The multipole moments $Q^{i_1 \cdots i_m}_{Ea}$ of the electric charge distribution of species a induced by the longitudinal electric field $(E_L)_i$ are shown in Eq. (41), and they are given in the linear form with respect to $(E_L)_i$ and its covariant derivatives as

$$Q^{i_1 \cdots i_m}_{Ea} \equiv \sum_{n=1}^{\infty} \chi^{i_1 \cdots i_m; j_1 \cdots j_n}_a \nabla_{j_1} \cdots \nabla_{j_{n-1}} (E_L)_{j_n}. \tag{43}$$

Here, the coefficients $\chi^{i_1 \cdots i_m; j_1 \cdots j_n}_a$ are defined by

$$\chi_a^{i_1\cdots i_m;j_1\cdots j_n} \equiv e_a^2 \int d^3v \frac{F_a}{2\mu B} \frac{(m+n)\beta_a^{i_1\cdots i_m;j_1\cdots j_n}}{m!\,n!}, \tag{44}$$

which are regarded as the generalized electric susceptibility produced by finite gyroradius. It is remarked that $Q_{0a}^{i_1\cdots i_m}$ and $Q_{Ea}^{i_1\cdots i_m}$ are both symmetric with respect to arbitrary permutations of the indices $i_1, \ldots, i_m$ because $\alpha_a^{i_1\cdots i_m}$ has the same symmetry.

Taking the summation of Eq. (38) over species a, we obtain

$$\mathcal{L}_\Psi \equiv \sum_a \mathcal{L}_{\Psi a} = -\rho^{(g)}\phi + \mathcal{L}_{E1} + \mathcal{L}_{E2}, \tag{45}$$

where the gyrocenter charge density $\rho^{(g)}$ is given by

$$\rho^{(g)} \equiv \sum_a e_a N_a^{(g)}, \tag{46}$$

and the parts of the Lagrangian density associated with polarization are represented by $\mathcal{L}_{E1}$ and $\mathcal{L}_{E2}$. Here, $\mathcal{L}_{E1}$ is defined by

$$\mathcal{L}_{E1} \equiv \sum_a \mathcal{L}_{E1a} = \sum_{k=1}^{\infty} Q_0^{j_1\cdots j_{2k}} \nabla_{j_1}\cdots\nabla_{j_{2k-1}} (E_L)_{j_{2k}} \tag{47}$$

with the multiple moments induced by finite gyroradius,

$$Q_0^{j_1\cdots j_{2k}} \equiv \sum_a Q_{0a}^{j_1\cdots j_{2k}}, \tag{48}$$

and $\mathcal{L}_{E2}$ is defined by

$$\begin{aligned}\mathcal{L}_{E2} &\equiv \sum_a \mathcal{L}_{E2a} = \frac{1}{2}\sum_{m=1}^{\infty} Q_E^{i_1\cdots i_m} \nabla_{i_1}\cdots\nabla_{i_{m-1}} (E_L)_{i_m} \\ &= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \chi^{i_1\cdots i_m;j_1\cdots j_n} \nabla_{i_1}\cdots\nabla_{i_{m-1}} (E_L)_{i_m} \\ &\quad \times \nabla_{j_1}\cdots\nabla_{j_{n-1}} (E_L)_{j_n},\end{aligned} \tag{49}$$

with the multiple moments induced by the longitudinal electric field $(E_L)_i$ and its covariant derivatives,

$$\begin{aligned}Q_E^{i_1\cdots i_m} &\equiv \sum_a Q_{Ea}^{i_1\cdots i_m} \\ &= \sum_{n=1}^{\infty} \chi^{i_1\cdots i_m;j_1\cdots j_n} \nabla_{j_1}\cdots\nabla_{j_{n-1}} (E_L)_{j_n},\end{aligned} \tag{50}$$

and the generalized electric susceptibility due to finite gyroradius,

$$\chi^{i_1\cdots i_m;j_1\cdots j_n} \equiv \sum_a \chi_a^{i_1\cdots i_m;j_1\cdots j_n}. \tag{51}$$

C. Derivation of the governing equations of the collisionless electrostatic gyrokinetic turbulent system

Here, we virtually allow the phase-space trajectories for all species and the electrostatic potential to vary infinitesimally. Following the same procedure as in Ref. 30, the variations in the gyrocenter position, parallel velocity, magnetic moment, and gyrophase of the phase-space trajectory are represented in the Eulerian picture by $\delta x_{aE}^i(x,v,t)$, $\delta v_{a\|E}(x,v,t)$, $\delta\mu_{aE}(x,v,t)$, and $\delta\vartheta_{aE}(x,v,t)$, respectively. We also denote the variation in the electrostatic potential by $\delta\phi$. Then, in terms of δx_E^i, $\delta v_{\|E}$, $\delta\mu_E$, and $\delta\vartheta_E$, the variations in the functional forms of u_{ax}^i, $u_{av_\|}$, $u_{a\mu}$, and $u_{a\vartheta}$ due to the virtual displacement are written as

$$\begin{aligned}\delta u_{ax}^i &= \left(\frac{\partial}{\partial t} + u_{ax}^j\frac{\partial}{\partial x^j} + u_{av_\|}\frac{\partial}{\partial v_\|} + u_{a\mu}\frac{\partial}{\partial\mu} + u_{a\vartheta}\frac{\partial}{\partial\vartheta}\right)\delta x_{aE}^i \\ &\quad - \left(\delta x_{aE}^j\frac{\partial}{\partial x^j} + \delta v_{a\|E}\frac{\partial}{\partial v_\|} + \delta\mu_{aE}\frac{\partial}{\partial\mu} + \delta\vartheta_{aE}\frac{\partial}{\partial\vartheta}\right)u_{ax}^i, \\ \delta u_{av_\|} &= \left(\frac{\partial}{\partial t} + u_{ax}^j\frac{\partial}{\partial x^j} + u_{av_\|}\frac{\partial}{\partial v_\|} + u_{a\mu}\frac{\partial}{\partial\mu} + u_{a\vartheta}\frac{\partial}{\partial\vartheta}\right)\delta v_{a\|E} \\ &\quad - \left(\delta x_{aE}^j\frac{\partial}{\partial x^j} + \delta v_{a\|E}\frac{\partial}{\partial v_\|} + \delta\mu_{aE}\frac{\partial}{\partial\mu} + \delta\vartheta_{aE}\frac{\partial}{\partial\vartheta}\right)u_{av_\|}, \\ \delta u_{a\mu} &= \left(\frac{\partial}{\partial t} + u_{ax}^j\frac{\partial}{\partial x^j} + u_{av_\|}\frac{\partial}{\partial v_\|} + u_{a\mu}\frac{\partial}{\partial\mu} + u_{a\vartheta}\frac{\partial}{\partial\vartheta}\right)\delta\mu_{aE} \\ &\quad - \left(\delta x_{aE}^j\frac{\partial}{\partial x^j} + \delta v_{a\|E}\frac{\partial}{\partial v_\|} + \delta\mu_{aE}\frac{\partial}{\partial\mu} + \delta\vartheta_{aE}\frac{\partial}{\partial\vartheta}\right)u_{a\mu}, \\ \delta u_{a\vartheta} &= \left(\frac{\partial}{\partial t} + u_{ax}^j\frac{\partial}{\partial x^j} + u_{av_\|}\frac{\partial}{\partial v_\|} + u_{a\mu}\frac{\partial}{\partial\mu} + u_{a\vartheta}\frac{\partial}{\partial\vartheta}\right)\delta\vartheta_{aE} \\ &\quad - \left(\delta x_{aE}^j\frac{\partial}{\partial x^j} + \delta v_{a\|E}\frac{\partial}{\partial v_\|} + \delta\mu_{aE}\frac{\partial}{\partial\mu} + \delta\vartheta_{aE}\frac{\partial}{\partial\vartheta}\right)u_{a\vartheta},\end{aligned} \tag{52}$$

and the variation in the distribution function F_a is given by

$$\begin{aligned}\delta F_a &= -\frac{\partial}{\partial x^j}(F_a\delta x_{aE}^j) - \frac{\partial}{\partial v_{a\|}}(F_a\delta v_{a\|E}) - \frac{\partial}{\partial\mu}(F_a\delta\mu_{aE}) \\ &\quad - \frac{\partial}{\partial\vartheta}(F_a\delta\vartheta_{aE}).\end{aligned} \tag{53}$$

Using Eqs. (17), (18), (45), (52), and (53), the variation in the action integral I_{GKF} is expressed as

$$\begin{aligned}\delta I_{GKF} &= \sum_a \int_{t_1}^{t_2} dt \int_V d^3x \int d^3v\, F_a \\ &\quad \times \left[\left\{\left(\frac{\partial L_{GYa}}{\partial x^i}\right)_u - \left(\frac{d}{dt}\right)_a\left(\frac{\partial L_{GYa}}{\partial u_{ax}^i}\right)\right\}\delta x_{aE}^i \right. \\ &\quad + \left(\frac{\partial L_{GYa}}{\partial v_\|}\right)_u \delta v_{\|aE} + \left(\frac{\partial L_{GYa}}{\partial\mu}\right)_u \delta\mu_{aE} \\ &\quad \left. + \left\{\left(\frac{\partial L_{GYa}}{\partial\vartheta}\right)_u - \left(\frac{d}{dt}\right)_a\left(\frac{\partial L_{GYa}}{\partial u_{a\vartheta}}\right)\right\}\delta\vartheta_{aE}\right] \\ &\quad - \int_{t_1}^{t_2} dt \int_V d^3x \left(\rho^{(g)} - \frac{1}{4\pi}\frac{\partial\left(\sqrt{g}D^j\right)}{\partial x^j}\right)\delta\phi + \text{B.T.},\end{aligned} \tag{54}$$

where D^j represents the electric displacement vector defined later in Eq. (73) and B.T. represents boundary terms that appear due to partial integrals. Here, $(\partial L_{GYa}/\partial x^i)_u$, $(\partial L_{GYa}/\partial v_\|)_u$, $(\partial L_{GYa}/\partial\mu)_u$, and $(\partial L_{GYa}/\partial\vartheta)_u$ denote the derivatives of $L_{GY\,a}$ in x^i, $v_\|$, μ, and ϑ, respectively, with $(u_{ax}^i, u_{a\vartheta})$ kept fixed in $L_{GY\,a}$, and the time derivative along the phase-space trajectory is represented by

$$\left(\frac{d}{dt}\right)_a \equiv \frac{\partial}{\partial t} + u_{ax}^k\frac{\partial}{\partial x^k} + u_{av_\|}\frac{\partial}{\partial v_\|} + u_{a\mu}\frac{\partial}{\partial\mu} + u_{a\vartheta}\frac{\partial}{\partial\vartheta}. \tag{55}$$

We now employ the Eulerian variation principle, which implies that the collisionless gyrokinetic equations for the distribution functions

of all species and the gyrokinetic Poisson's equation for the electrostatic potential can be derived from the condition that $\delta I_{GKF} = 0$ for arbitrary variations δx^i_{aE}, $\delta v_{a\|E}$, $\delta\mu_{aE}$, $\delta\vartheta_{aE}$, and $\delta\phi$, which vanish on the boundaries of the integral region to make B.T. disappear in Eq. (54).

We first use $\delta I_{GKF}/\delta x^i_{aE} = 0$ to obtain

$$\left(\frac{d}{dt}\right)_a p_{ai} = \left(\frac{\partial L_{GYa}}{\partial x^i}\right)_u, \tag{56}$$

where p_{ai} represents the covariant vector component of the canonical momentum defined by

$$p_{ai} \equiv \frac{\partial L_{GYa}}{\partial u^i_{ax}} = \frac{e_a}{c}A_i(x,t) + m_a v_\| b_i(x,t) \equiv \frac{e_a}{c}A^*_{ai}(x, v_\|, t). \tag{57}$$

We should note that the distribution function F_a is included as a factor in $\delta I_{GKF}/\delta x^i_{aE} = 0$ although it is omitted from Eq. (56) for simplicity. This omission of F_a is also performed in the other equations obtained below from $\delta I_{GKF}/\delta v_{a\|E} = 0$, $\delta I_{GKF}/\delta\mu_{aE} = 0$, and $\delta I_{GKF}/\delta\vartheta_{aE} = 0$ although it does not make a difference in deriving the resultant collisionless gyrokinetic equation in Eq. (69). We can rewrite Eq. (56) as

$$m_a u_{av_\|} b_i = e_a\left(-\frac{\partial\Psi_a}{\partial x^i} - \frac{1}{c}\frac{\partial A^*_{ai}}{\partial t} + \frac{1}{c}\sqrt{g}\epsilon_{ijk}u^j_x B^{*k}\right) - \mu\frac{\partial B}{\partial x^i}, \tag{58}$$

where the modified magnetic field is defined by

$$B^{*i}_a \equiv \frac{\epsilon^{ijk}}{\sqrt{g}}\frac{\partial A^*_{ak}}{\partial x^j}, \tag{59}$$

respectively.

Next, $\delta I_{GKF}/\delta v_{a\|E} = 0$ is used to obtain

$$\left(\frac{\partial L_{GYa}}{\partial v_\|}\right)_u = m_a\left(u^i_{ax}b_i - v_\|\right) = 0, \tag{60}$$

from which we have

$$u^i_{ax}b_i = v_\|. \tag{61}$$

Furthermore, $\delta I_{GKF}/\delta\mu_{aE} = 0$ and $\delta I_{GKF}/\delta\vartheta_{aE} = 0$ yield

$$\left(\frac{\partial L_{GYa}}{\partial\mu}\right)_u = \frac{m_a c}{e_a}u_{a\vartheta} - B - e_a\frac{\partial\Psi_a}{\partial\mu} = 0 \tag{62}$$

and

$$\left(\frac{d}{dt}\right)_a\left(\frac{\partial L_{GYa}}{\partial u_{a\vartheta}}\right) = \frac{m_a c}{e_a}u_{a\mu} = \left(\frac{\partial L_{GYa}}{\partial\vartheta}\right)_u = 0, \tag{63}$$

respectively.

Equations (58), (61), (62), and (63) are rewritten as

$$u^i_{ax} = \frac{1}{B^*_{a\|}}\left[v_\|B^{*i}_a + c\frac{\epsilon^{ijk}}{\sqrt{g}}b_j\left(\frac{\mu}{e_a}\frac{\partial B}{\partial x^k} + \frac{\partial\Psi_a}{\partial x^k} + \frac{1}{c}\frac{\partial A^*_{ak}}{\partial t}\right)\right], \tag{64}$$

$$m_a u_{av_\|} = -\frac{B^{*i}_a}{B^*_{a\|}}\left[\mu\frac{\partial B}{\partial x^i} + e_a\left(\frac{\partial\Psi_a}{\partial x^i} + \frac{1}{c}\frac{\partial A^*_{ai}}{\partial t}\right)\right], \tag{65}$$

$$u_{a\mu} = 0, \tag{66}$$

and

$$u_{a\vartheta} = \Omega_a + \frac{e_a^2}{m_a c}\frac{\partial\Psi_a}{\partial\mu}, \tag{67}$$

where $\Omega_a \equiv e_a B/(m_a c)$ and

$$B^*_{a\|} \equiv B^{*i}_a b_i. \tag{68}$$

Equations (64) and (65) are obtained by taking the vector and scalar products between the magnetic field and Eq. (58), respectively. We can verify that the right-hand sides of Eqs. (64)–(67) are all independent of ϑ and that the magnetic moment μ is an invariant of motion as seen from Eq. (66).

Substituting Eqs. (64)–(67) into Eqs. (30) and taking its average with respect to the gyrophase ϑ, the collisionless gyrokinetic equation is derived as

$$\frac{\partial\bar{F}_a}{\partial t} + \frac{\partial}{\partial x^i}\left[\bar{F}_a\frac{1}{B^*_{a\|}}\left\{v_\|B^{*i}_a + c\frac{\epsilon^{ijk}}{\sqrt{g}}b_j\left(\frac{\mu}{e_a}\frac{\partial B}{\partial x^k} + \frac{\partial\Psi_a}{\partial x^k} + \frac{1}{c}\frac{\partial A^*_{ak}}{\partial t}\right)\right\}\right] + \frac{\partial}{\partial v_\|}\left[\bar{F}_a\frac{B^{*i}_a}{m_a B^*_{a\|}}\left\{-e_a\left(\frac{\partial\Psi_a}{\partial x^i} + \frac{1}{c}\frac{\partial A^*_{ai}}{\partial t}\right) - \mu\frac{\partial B}{\partial x^i}\right\}\right] = 0, \tag{69}$$

where $\bar{F}_a$ denotes the gyrophase-averaged distribution function,

$$\bar{F}_a \equiv \langle F_a\rangle_\vartheta \equiv \oint\frac{d\vartheta}{2\pi}F_a. \tag{70}$$

The remaining governing equation of the system, namely, the gyrokinetic Poisson's equation is derived from the condition that the variational derivative of the action integral I_{GKF} with respect to the electrostatic potential ϕ vanishes, $\delta I_{GKF}/\delta\phi = 0$. Since the time derivative of ϕ never appears in the Lagrangian density $\mathcal{L}_{GKF}$, the above-mentioned condition can be replaced using the Lagrangian L_{GKF} instead of I_{GKF} by

$$\frac{\delta L_{GKF}[\phi]}{\delta\phi}(x) \equiv \frac{d}{d\epsilon}L_{GKF}\left[\phi + \epsilon\delta^3_x\right]\Big|_{\epsilon=0} = 0, \tag{71}$$

where δ^3_x with the subscript $x = (x^i)_{i=1,2,3}$ represents the function that takes a value $\delta^3_x(y) = \delta(y^1 - x^1)\delta(y^2 - x^2)\delta(y^3 - x^3)$ at $y = (y^i)_{i=1,2,3}$. Equation (71) gives the gyrokinetic Poisson's equation,

$$\frac{\partial\left(\sqrt{g}D^i\right)}{\partial x^i} = 4\pi\rho^{(g)}, \tag{72}$$

where the electric displacement vector D^i is written as

$$\sqrt{g}D^i \equiv \sqrt{g}E^i_L + 4\pi P^i_G. \tag{73}$$

Here, the generalized polarization vector density P^i_G is defined by

$$\begin{aligned} P^i_G &\equiv \frac{\delta L_{GK}}{\delta(E_L)_i} \equiv \frac{d}{d\epsilon}L_{GK}\left[(E_L)_i + \epsilon\delta^3_x\right]\Big|_{\epsilon=0} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m-1}\nabla_{i_1}\cdots\nabla_{i_{m-1}}Q^{ii_1\cdots i_{m-1}} \\ &= -\sum_{k=1}^{\infty}\nabla_{j_1}\cdots\nabla_{j_{2k-1}}Q_0^{j_1\cdots j_{2k-1}i} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m-1} \\ &\quad\times\nabla_{i_1}\cdots\nabla_{i_{m-1}}\left[\chi^{ii_1\cdots i_{m-1}j_1\cdots j_n}\nabla_{j_1}\cdots\nabla_{j_{n-1}}(E_L)_{j_n}\right], \end{aligned} \tag{74}$$

in which not only the dipole moment but also other multipole moments[51] occurring due to finite gyroradius are included. The multipole moments $Q^{i_1\cdots i_m}$ in Eq. (74) are generally given by the sum of the two parts,

$$Q^{i_1\cdots i_m} \equiv Q_0^{i_1\cdots i_m} + Q_E^{i_1\cdots i_m}, \tag{75}$$

where $Q_0^{i_1\cdots i_m}$ and $Q_E^{i_1\cdots i_m}$ are defined by Eqs. (48) and (50), respectively. It should be noted that $Q_0^{i_1\cdots i_m}$ vanishes unless m is an even integer. Taking only the contribution of $m=1$ to the summation over m in Eq. (75), we obtain the dipole moment density, which is denoted by

$$P_D^i \equiv Q_E^i \equiv \frac{\partial \mathcal{L}_{GK}}{\partial (E_L)_i}. \tag{76}$$

The presence of the infinite series due to the finite gyroradius in Eq. (74) and other places has an analogy with that for the case of macroscopic electromagnetism described in detail in Ref. 51 where the macroscopic charge density is evaluated for the system consisting of molecules. The contribution of each molecule to the macroscopic charge density is calculated by spatially averaging the microscopic charge density (given by the point charges constituting the molecule) around the center of mass of the molecule. Then, the resultant expression of the macroscopic charge density is given by the series expansion associated with the multipole moments due to the finite distance of each point charge from the center of mass of the molecule. The local spatial average of the microscopic charge density in the system of molecules is replaced by the phase-space integration in the present case of the gyrokinetic system to represent the macroscopic charge density as

$$\begin{aligned}
&\rho^{(g)}(\mathbf{x},t) - \nabla\cdot\mathbf{P}_G(\mathbf{x},t) \\
&\quad= \frac{\delta}{\delta\phi(\mathbf{x})}\left[\sum_a e_a \int d^3x' \int d^3v\, F_a(\mathbf{x}',v,t)\Psi_a(\mathbf{x}',v,t)\right] \\
&\quad= \sum_a e_a \int d^3x' \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty d\mu \int_0^{2\pi} d\vartheta\, \bar{F}_a(\mathbf{x}',v_\parallel,\mu) \\
&\qquad\times\left[\delta(\mathbf{x}'+\boldsymbol{\rho}_a-\mathbf{x}) - \frac{e_a}{B}\frac{\partial}{\partial\mu}\left\{\tilde{\phi}(\mathbf{x}',\mu,\vartheta)\delta(\mathbf{x}'+\boldsymbol{\rho}_a-\mathbf{x})\right\}\right],
\end{aligned} \tag{77}$$

which contains the polarization effect due to the finite gyroradius and the microscopic electrostatic fluctuations. In Eq. (77), the Cartesian spatial coordinates are used and three-dimensional vectors are represented in terms of boldface letters. In the square brackets on the right-hand side of Eq. (77), the first and second terms give the point charge density of the particle at $\mathbf{x}=\mathbf{x}'+\boldsymbol{\rho}_a$ and its correction due to the electrostatic fluctuation, respectively. We can see that the effect of the finite gyroradius (or the finite distance between the particle and gyrocenter positions) is included in the delta functions, which cause the infinite series expansions to appear in Eq. (74) and other equations related to the polarization (or dipole and multipole moments).

IV. DERIVATION OF THE MOMENTUM BALANCE

In this section, we use the invariance of the Lagrangian under arbitrary infinitesimal transformations of spatial coordinates to derive the momentum balance equation for the single-particle-species system and that for the whole system including all species and the field.

A. Invariance of the Lagrangian under arbitrary spatial coordinate transformations

We now consider the infinitesimal transformation of the spatial coordinates from $x=(x^i)_{i=1,2,3}$ to $x'=(x'^i)_{i=1,2,3}$, which is written as

$$x'^i = x^i + \xi^i(x). \tag{78}$$

Here, the infinitesimal variation in the spatial coordinate x^i is denoted by $\xi^i(x)$, which is an arbitrary function of only the spatial coordinates $x=(x^i)_{i=1,2,3}$ and independent of time t.

From Eq. (18), the gyrokinetic part of the Lagrangian for species a is found to be given by

$$L_{GKa} \equiv \int_V d^3x\, \mathcal{L}_{GKa} \equiv \int_V d^3x \int d^3v\, F_a L_{GYa}, \tag{79}$$

where the gyrocenter Lagrangian $L_{GY\ a}$ is defined in Eq. (21). Since L_{GKa} shown in Eq. (79) is a scalar constant, which is invariant under arbitrary spatial coordinate transformations including the infinitesimal transformation given by Eq. (78), we have

$$\bar{\delta}L_{GKa} \equiv \int_V d^3x \left(\frac{\partial(\xi^i\mathcal{L}_{GKa})}{\partial x^i} + \bar{\delta}\mathcal{L}_{GKa}\right) = 0. \tag{80}$$

Note that here and hereafter we use $\bar{\delta}\cdots$ to represent the variation associated with the infinitesimal spatial coordinate transformation, which should be distinguished from the variation $\delta\cdots$ due to the virtual displacement treated in Sec. III. The divergence term in the integrand of Eq. (80) appears due to the difference between the domains of integrations in $x=(x^i)_{i=1,2,3}$ and $x'=(x'^i)_{i=1,2,3}$, while $\bar{\delta}\mathcal{L}_{GKa}$ in the integrand represents the variation in the spatial functional form of $\mathcal{L}_{GKa}$ due to the infinitesimal spatial coordinate transformation.

As shown in Appendix B, the variation in the spatial functional form of an arbitrary tensor field (as well as an arbitrary tensor field density) due to the infinitesimal spatial coordinate transformation given in Eq. (78) is written as the opposite sign of its Lie derivative[52] with respect to the generating vector field ξ^i, and it is represented by $\bar{\delta}=-L_\xi$. It is also noted that since the Lagrangian density behaves as a scalar field density under arbitrary spatial coordinate transformations, the Lie derivative of $\mathcal{L}_{GKa}$ is given by

$$L_\xi\mathcal{L}_{GKa} = \frac{\partial(\xi^i\mathcal{L}_{GKa})}{\partial x^i}. \tag{81}$$

Therefore, the integrand in Eq. (80) is simply rewritten as $L_\xi\mathcal{L}_{GKa} - L_\xi\mathcal{L}_{GKa}(=0)$, from which its spatial integral $\bar{\delta}L_{GKa}$ is naturally found to vanish as shown in Eq. (80).

We now use Eq. (79) and the Leibniz rule for the derivative operation by $\bar{\delta}=-L_\xi$ to write the variation in the spatial functional form of the Lagrangian density $\mathcal{L}_{GKa}$ as

$$\begin{aligned}
\bar{\delta}\mathcal{L}_{GKa} &= \int d^3v\, \bar{\delta}(F_a L_{GYa}) \\
&= \int d^3v\, (\bar{\delta}F_a \cdot L_{GYa} + F_a \cdot \bar{\delta}L_{GYa}).
\end{aligned} \tag{82}$$

Then, using Eqs. (81), (82), and $\bar{\delta}=-L_\xi$, Eq. (80) is rewritten as

$$\begin{aligned}\bar{\delta}L_{GKa} &= \int_V d^3x \int d^3v \left[-\bar{\delta}(F_a L_{GYa}) + \bar{\delta}(F_a L_{GYa})\right] \\ &= \int_V d^3x \int d^3v\, F_a(-\bar{\delta}L_{GYa} + \bar{\delta}L_{GYa}) \\ &= \int_V d^3x \int d^3v\, F_a \Bigg(\xi^i \frac{\partial L_{GYa}}{\partial x^i} + \frac{\partial L_{GYa}}{\partial u^i_{ax}} \bar{\delta} u^i_{ax} \\ &\quad + \frac{\partial L_{GYa}}{\partial u^i_{a\vartheta}} \bar{\delta} u^i_{a\vartheta} + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J A_i)} \bar{\delta}(\partial_J A_i) \\ &\quad + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J \phi)} \bar{\delta}(\partial_J \phi) + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J g_{ij})} \bar{\delta}(\partial_J g_{ij}) \Bigg) \\ &= 0. \end{aligned} \tag{83}$$

Equation (83) is found to hold by noting that $L_\xi L_{GYa} = \xi^i \partial L_{GYa}/\partial x^i$ (because $L_{GY\,a}$ behaves as a scalar field under arbitrary spatial coordinate transformations) and that the chain rule is applied to the derivative operation $\bar{\delta} = -L_\xi$ on $L_{GYa}[u^i_{ax}, u^i_{a\vartheta}, \{\partial_J A_i\}, \{\partial_J \phi\}, \{\partial_J g_{ij}\}]$ as

$$\begin{aligned}\bar{\delta}L_{GYa} &= -\xi^i \frac{\partial L_{GYa}}{\partial x^i} \\ &= \frac{\partial L_{GYa}}{\partial u^i_{ax}} \bar{\delta} u^i_{ax} + \frac{\partial L_{GYa}}{\partial u^i_{a\vartheta}} \bar{\delta} u^i_{a\vartheta} + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J A_i)} \bar{\delta}(\partial_J A_i) \\ &\quad + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J \phi)} \bar{\delta}(\partial_J \phi) + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J g_{ij})} \bar{\delta}(\partial_J g_{ij}), \end{aligned} \tag{84}$$

where $\partial L_{GYa}/\partial(\partial_J A_i) = 0$ when the order of J is greater than or equal to two [see Eq. (35)].

We next consider the invariance of the Lagrangian L_{GKF} of the whole system under the infinitesimal spatial coordinate transformation, which is written in the same way as in Eq. (80) by

$$\bar{\delta}L_{GKF} \equiv \int_V d^3x \left(\frac{\partial(\xi^i \mathcal{L}_{GKF})}{\partial x^i} + \bar{\delta}\mathcal{L}_{GKF} \right) = 0, \tag{85}$$

where L_{GKF} consists of the gyrokinetic and field parts as shown in Eqs. (17) and (18). Procedures similar to those leading to Eq. (83) can be made to obtain

$$\begin{aligned}\bar{\delta}L_{GKF} &= \int_V d^3x \Bigg[\sum_a \int d^3v\, F_a \Bigg(\xi^i \frac{\partial L_{GYa}}{\partial x^i} + \frac{\partial L_{GYa}}{\partial u^i_{ax}} \bar{\delta} u^i_{ax} \\ &\quad + \frac{\partial L_{GYa}}{\partial u^i_{a\vartheta}} \bar{\delta} u^i_{a\vartheta} + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J A_i)} \bar{\delta}(\partial_J A_i) \\ &\quad + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J \phi)} \bar{\delta}(\partial_J \phi) + \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J g_{ij})} \bar{\delta}(\partial_J g_{ij}) \Bigg) \\ &\quad + \frac{\partial(\xi^i \mathcal{L}_F)}{\partial x^i} + \frac{\partial \mathcal{L}_F}{\partial(\partial\phi/\partial x^i)} \bar{\delta}\left(\frac{\partial \phi}{\partial x^i}\right) + \frac{\partial \mathcal{L}_F}{\partial g_{ij}} \bar{\delta} g_{ij} \Bigg] \\ &= 0. \end{aligned} \tag{86}$$

The invariance shown in Eq. (86) can also be confirmed using Eq. (84) and the following formula for the derivative of $\mathcal{L}_F[\partial\phi/\partial x^i, g_{ij}]$ with respect to $\bar{\delta} = -L_\xi$,

$$\bar{\delta}\mathcal{L}_F = -\frac{\partial(\xi^i \mathcal{L}_F)}{\partial x^i} = \frac{\partial \mathcal{L}_F}{\partial(\partial\phi/\partial x^i)} \bar{\delta}\left(\frac{\partial \phi}{\partial x^i}\right) + \frac{\partial \mathcal{L}_F}{\partial g_{ij}} \bar{\delta} g_{ij}. \tag{87}$$

It is summarized from Eqs. (83), (84), (86), and (87) that the invariance of the scalar constants of L_{GKa} and L_{GKF} under the infinitesimal spatial transformation can be verified using the chain rule formulas for the derivative operation $\bar{\delta} = -L_\xi$ on the scalar field L_{GYa} and the scalar field density $\mathcal{L}_F$. The invariance formulas shown in Eqs. (83) and (86) are used to derive the momentum balance equation for the single-species system in Sec. IV B and that for the whole system including all species and the field in Sec. IV C, respectively.

B. Derivation of the momentum balance for a single particle species

We now use the Euler–Lagrange equations for gyrocenter motion [Eqs. (56), (60), (62), and (63)] and perform partial integrals to rewrite (83) as

$$\begin{aligned}\bar{\delta}L_{GKa} &= \int_V d^3x \Bigg[\xi^i \int d^3v \left\{ \frac{\partial}{\partial t}\left(F_a \frac{\partial L_{GYa}}{\partial u^i_{ax}} \right) - \mathcal{K}_a \frac{\partial L_{GYa}}{\partial u^i_{ax}} \right\} \\ &\quad + \frac{\delta L_{GKa}}{\delta A_i} \bar{\delta} A_i + \frac{\delta L_{GKa}}{\delta \phi} \bar{\delta}\phi + \frac{\delta L_{GKa}}{\delta g_{ij}} \bar{\delta} g_{ij} \Bigg] + \text{B.T.} \\ &= 0, \end{aligned} \tag{88}$$

where, instead of using Eq. (30), the gyrocenter distribution function F_a is assumed to satisfy

$$\frac{\partial F_a}{\partial t} + \frac{\partial}{\partial x^j}(F_a u^j_{ax}) + \frac{\partial}{\partial v_\parallel}(F_a u_{av_\parallel}) + \frac{\partial}{\partial \mu}(F_a u_{a\mu}) + \frac{\partial}{\partial \vartheta}(F_a u_{a\vartheta}) = \mathcal{K}_a. \tag{89}$$

Here, $\mathcal{K}_a$ represents the rate of temporal change in F_a due to collisions and/or external sources for the species a. In the present work, we assume that

$$\sum_a e_a \int d^3v\, \mathcal{K}_a = 0 \tag{90}$$

is satisfied by $\mathcal{K}_a$. In fact, it is shown in Ref. 43 that Eq. (90) corresponds to the intrinsic ambipolarity of the classical particle fluxes when $\mathcal{K}_a$ represents the collision operator in the gyrocenter coordinates. The variational derivative of L_{GKa} with respect to A_i is given by

$$\begin{aligned}\frac{\delta L_{GKa}}{\delta A_i} &= \sum_J (-1)^{\#J} \partial_J \left(\frac{\partial \mathcal{L}_{GKa}}{\partial(\partial_J A_i)} \right) \\ &= \frac{\partial \mathcal{L}_{GKa}}{\partial A_i} - \frac{\partial}{\partial x^j}\left(\frac{\partial \mathcal{L}_{GKa}}{\partial(\partial A_i/\partial x^j)} \right) = \frac{e_a}{c} \Gamma^i_a, \end{aligned} \tag{91}$$

where the particle flux of species a is represented by

$$\begin{aligned}\Gamma^k_a \equiv \int d^3v\, F_a u^k_{ax} + \frac{c}{e_a} \epsilon^{kij} \frac{\partial}{\partial x^i} \Bigg(\int d^3v\, \frac{F_a}{\sqrt{g}} \\ \times \left[-\mu b_j + \frac{m_a v_\parallel}{B} \left\{ (u_{ax})_j - (u_{ax})_i b^i b_j \right\} - e_a \frac{\partial \Psi_a}{\partial B^j} \right] \Bigg). \end{aligned} \tag{92}$$

The variational derivatives $\delta L_{GKa}/\delta\phi$ and $\delta L_{GKa}/\delta g_{ij}$ are written as

$$\frac{\delta L_{GKa}}{\delta \phi} = \sum_J (-1)^{\#J} \partial_J \left(\frac{\partial \mathcal{L}_{GKa}}{\partial(\partial_J \phi)} \right) = -e_a N^{(p)}_a \tag{93}$$

and

$$\frac{\delta L_{GKa}}{\delta g_{ij}} = \sum_J (-1)^{\#J} \partial_J \left(\frac{\partial \mathcal{L}_{GKa}}{\partial(\partial_J g_{ij})} \right) = \frac{1}{2} P_a^{ij}, \tag{94}$$

respectively, where the particle density $N_a^{(p)}$ and the symmetric pressure tensor density P_a^{ij} of species a are defined by

$$e_a N_a^{(p)} \equiv e_a N_a^{(g)} - \nabla_i P_{Ga}^i \tag{95}$$

and

$$P_a^{ij} \equiv 2 \sum_J (-1)^{\#J} \int d^3v\, F_a \frac{\partial L_{GYa}}{\partial(\partial_J g_{ij})} = P_{\mathrm{CGL}a}^{ij} + \pi_{\wedge a}^{ij} + P_{\Psi a}^{ij}, \tag{96}$$

respectively, and $\#J = n$ represents the order of $J \equiv (j_1, j_2, \ldots, j_n)$.

On the right-hand side of Eq. (95), P_{Ga}^i represents the contribution of species a to the generalized polarization vector density P_G^i defined in Eq. (74) and it is written as

$$\begin{aligned} P_{Ga}^i &\equiv \frac{\delta L_{GKa}}{\delta (E_L)_i} \\ &= -\sum_{k=1}^{\infty} \nabla_{j_1} \cdots \nabla_{j_{2k-1}} Q_{0a}^{j_1 \cdots j_{2k-1} i} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m-1} \\ &\quad \times \nabla_{i_1} \cdots \nabla_{i_{m-1}} \left[\chi_a^{i i_1 \cdots i_{m-1}; j_1 \cdots j_n} \nabla_{j_1} \cdots \nabla_{j_{n-1}} (E_L)_{j_n} \right]. \end{aligned} \tag{97}$$

On the right-hand side of Eq. (96), $P_{\mathrm{CGL}a}^{ij}$ is given in the Chew–Goldberger–Low (CGL) form[41]

$$P_{\mathrm{CGL}a}^{ij} = \int d^3v\, F_a \left[m_a v_\parallel^2 b^i b^j + \mu B (g^{ij} - b^i b^j) \right], \tag{98}$$

and $\pi_{\wedge a}^{ij}$ is defined by

$$\pi_{\wedge a}^{ij} \equiv \int d^3v\, F m_a v_\parallel \left[b^i (u_x)_\perp^j + (u_x)_\perp^i b^j \right]. \tag{99}$$

Here, the perpendicular component of the gyrocenter velocity is represented by $(u_{ax})_\perp^i \equiv u_{ax}^i - u_{ax}^k b_k b^i$. We find that Eqs. (98) and (99) agree with those given by Eqs. (124) and (125) in Ref. 30 except for the effects of the electrostatic fluctuations included in F_a and $(u_{ax})_\perp^i$ on the right-hand side of Eqs. (98) and (99). We also note that Eq. (99) contains the product of the fluctuation parts of F_a and $(u_{ax})_\perp^i$, the ensemble average of which does not disappear but contributes to the turbulent momentum transport. In the neoclassical transport theory,[41,42] it is considered that the CGL pressure tensor shown in Eq. (98) contains the scalar (or isotropic) part, which represents background pressure, and the anisotropic part, the magnitude of which is smaller than the background pressure by the factor $\sim \rho/L$, where ρ and L represent the gyroradius and the equilibrium gradient scale length, respectively. The anisotropic part of the CGL pressure tensor causes the viscous force, which plays an essential role in the neoclassical transport processes when the distribution function deviates from the local Maxwellian under the influence of collisions.[41,42] The magnitude of $\pi_{\wedge a}^{ij}$ defined in Eq. (99) is regarded as $\sim (\rho/L)^2$.

The last term on the right-hand side of Eq. (96) is given by

$$P_{\Psi_a}^{ij} \equiv -2 \sum_J (-1)^{\#J} \int d^3v\, F_a e_a \frac{\partial \Psi_a}{\partial(\partial_J g_{ij})} = P_{E1a}^{ij} + P_{E2a}^{ij}, \tag{100}$$

where P_{E1a}^{ij} and P_{E2a}^{ij} are defined in Eqs. (C5) and (C8) of Appendix C, respectively. We note that the effects of the turbulent electrostatic potential are included in the definitions of $P_{\Psi_a}^{ij}$ explicitly as well as in $\pi_{\wedge a}^{ij}$ through the turbulent drift velocity part of $(u_{ax})_\perp^i$. Then, the nonlinear interaction of the turbulent potential and the fluctuation part of the gyrocenter distribution function included in $P_{\Psi_a}^{ij}$ and $\pi_{\wedge a}^{ij}$ causes the turbulent momentum transport. In Appendix D, the ensemble-averaged pressure tensor describing the turbulent momentum transport is given by the WKB representation.

Performing further partial integrals in Eq. (88) finally gives

$$\bar{\delta} L_{GKa} = \int_V d^3x\, \xi_j J_{GKa}^j + \mathrm{B.T.} = 0, \tag{101}$$

where

$$\begin{aligned} J_{GKa}^j &\equiv \frac{\partial}{\partial t} \left(m_a N_a^{(g)} V_{ag\parallel} b^j \right) - \int d^3v\, \mathcal{K}_a m_a v_\parallel b^j \\ &\quad + e_a \left(N_a^{(p)} g^{jk} \frac{\partial \phi}{\partial x^k} + \frac{1}{c} N_a^{(g)} \frac{\partial A^j}{\partial t} - \frac{1}{c} \frac{\epsilon^{jkl}}{\sqrt{g}} \Gamma_{ak} B_l \right) + \nabla_i P_a^{ij} \end{aligned} \tag{102}$$

and

$$N_a^{(g)} V_{ag\parallel} \equiv \int d^3v\, F_a v_\parallel. \tag{103}$$

In deriving Eq. (101)–(103), we have deformed Eq. (88) using Eqs. (89), (91), (93), (94), $\bar{\delta} A_i = -\xi^j (\partial_j A_i) - (\partial_i \xi^j) A_j$ [see Eq. (B8)], $\bar{\delta}\phi = -\xi^j (\partial_j \phi)$ [see Eq. (B3)], $\bar{\delta} g_{ij} = -\nabla_i \xi_j - \nabla_i \xi_i$ [see Eq. (B10)], and partial integration. In particular, the term $\nabla_i P_a^{ij}$ in Eq. (102) is derived from the term $(\delta L_{GKa}/\delta g_{ij}) \bar{\delta} g_{ij}$, which is rewritten as $(\delta L_{GKa}/\delta g_{ij}) \bar{\delta} g_{ij} = -\frac{1}{2} P_a^{ij} (\nabla_i \xi_j + \nabla_i \xi_i) = \xi_j (\nabla_i P_a^{ij}) - \nabla_i (\xi_j P_a^{ij})$, where $P_a^{ij} = P_a^{ji}$ is also used and the spatial integral of the last term $\nabla_i (\xi_j P_a^{ij})$ becomes one of the boundary terms. We now recall that an arbitrary infinitesimal vector field can be employed as ξ^i. Then, in order for Eq. (101) to hold for any ξ^i, we need to have $J_{GKa}^j = 0$, which gives the momentum balance equation as

$$\begin{aligned} &\frac{\partial}{\partial t} \left(m_a N_a^{(g)} V_{ag\parallel} b^j \right) - \int d^3v\, \mathcal{K}_a m_a v_\parallel b^j \\ &= e_a \left(-N_a^{(p)} g^{jk} \frac{\partial \phi}{\partial x^k} - \frac{1}{c} N_a^{(g)} \frac{\partial A^j}{\partial t} + \frac{1}{c} \frac{\epsilon^{jkl}}{\sqrt{g}} \Gamma_{ak} B_l \right) - \nabla_i P_a^{ij}. \end{aligned} \tag{104}$$

As seen in Eqs. (126) and (152) of Ref. 30, the relation between the symmetric and canonical pressure tensors is obtained from the other condition for the sum of the boundary terms (B.T.) in Eq. (101) to vanish although its complicated expression is not shown here. We see that the inertia term in the momentum balance equation, Eq. (104), contains only the parallel momentum component while the electric current $e_a \Gamma_a^k$ in the Lorentz force term consists of the guiding-center current and the magnetization current as shown in Eq. (92) [see also Eq. (114)]. For comparison with Eq. (104), the canonical momentum balance equation derived by the conventional method is shown in Eq. (E5) of Appendix E where the divergence of the asymmetric canonical pressure tensor appears. In addition, the energy balance equation for the single-species gyrokinetic system is given in Eq. (F2) of Appendix F.

C. Derivation of the momentum balance for the whole system

In the same way as in deriving Eq. (88), performing partial integrals in Eq. (86) and using Eqs. (89), we have

$$\bar{\delta}L_{GKF} = \int_V d^3x \left[\xi^i \sum_a \int d^3v \left\{ \frac{\partial}{\partial t}\left(F_a \frac{\partial L_{GYa}}{\partial u^i_{ax}} \right) - \mathcal{K}_a \frac{\partial L_{GYa}}{\partial u^i_{ax}} \right\} + \frac{\delta L_{GKF}}{\delta A_i}\bar{\delta}A_i + \frac{\delta L_{GKF}}{\delta \phi}\bar{\delta}\phi + \frac{\delta L_{GKF}}{\delta g_{ij}}\bar{\delta}g_{ij} \right] + \text{B.T.} \tag{105}$$

Now, substituting Eq. (91) into $\delta L_{GKF}/\delta A_i = \sum_a \delta L_{GKa}/\delta A_i$ and using the gyrokinetic Poisson's equation, $\delta L_{GKF}/\delta\phi = 0$, and Eq. (94), Eq. (105) is finally rewritten as

$$\bar{\delta}L_{GKF} = \int_V d^3x\, \xi_j J^j_{GKF} + \text{B.T.} = 0, \tag{106}$$

where

$$J^j_{GKF} \equiv \sum_a \left[\frac{\partial}{\partial t}\left(m_a N^{(g)}_a V_{ag\parallel} b^j \right) - \int d^3v\, \mathcal{K}_a m_a v_\parallel b^j \right] + \sum_a \frac{e_a}{c}\left(N^{(g)}_a \frac{\partial A^j}{\partial t} - \frac{\epsilon^{jkl}}{\sqrt{g}}\Gamma_{ak}B_l \right) + \nabla_i \Theta^{ij}. \tag{107}$$

Here, the symmetric pressure tensor Θ^{ij} is defined by

$$\Theta^{ij} \equiv 2\frac{\delta L_{GKF}}{\delta g_{ij}} \equiv 2\left[\sum_J (-1)^{\#J} \int d^3v\, F_a \frac{\partial L_{GYa}}{\partial(\partial_J g_{ij})} + \frac{\partial \mathcal{L}_F}{\partial g_{ij}} \right] = P^{ij}_{\text{CGL}} + \pi^{ij}_\wedge + P^{ij}_\Psi + \frac{\sqrt{g}}{4\pi}\left(\frac{(E_L)^k (E_L)_k}{2} g^{ij} - (E_L)^i (E_L)^j \right), \tag{108}$$

where the last group of terms including $(E_L)_i$ represents the Maxwell stress tensor due to the electrostatic field with the opposite sign. Taking the summation of Eqs. (98), (99), and (100) over species defines P^{ij}_{CGL}, $\pi^{ij}_\wedge$, and P^{ij}_Ψ on the right-hand side of Eq. (108) as

$$P^{ij}_{\text{CGL}} = \sum_a \int d^3v\, F_a \left[m_a v^2_\parallel b^i b^j + \mu B(g^{ij} - b^i b^j) \right], \tag{109}$$

$$\pi^{ij}_\wedge \equiv \sum_a \int d^3v\, F_a m_a v_\parallel \left[b^i (u_{ax})^j_\perp + (u_{ax})^i_\perp b^j \right], \tag{110}$$

and

$$P^{ij}_\Psi \equiv \sum_a P^{ij}_{\Psi_a} = \sum_a (P^{ij}_{E1a} + P^{ij}_{E2a}), \tag{111}$$

respectively. As mentioned after Eq. (100) as well as in Appendix F, the turbulent momentum transport caused by the nonlinear interaction of the turbulent potential and the fluctuation part of the gyrocenter distribution function is included in $\pi^{ij}_\wedge$ and P^{ij}_Ψ.

From Eqs. (106) and (107), we obtain $J^j_{DKF} = 0$, which represents the momentum balance equation for the whole system,

$$\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a m_a v_\parallel b^j \right) - \sum_a \int d^3v\, \mathcal{K}_a m_a v_\parallel b^j + \nabla_i \Theta^{ij} = \sum_a \frac{e_a}{c}\left(-N^{(g)}_a \frac{\partial A^j}{\partial t} + \frac{\epsilon^{jkl}}{\sqrt{g}}\Gamma_{ak}B_l \right). \tag{112}$$

For comparison with Eq. (112), the canonical momentum balance equation derived by the conventional method is shown in Eq. (E11) of Appendix E where the divergence of the asymmetric pressure tensor appears. The condition for the sum of the boundary terms (B.T.) in Eq. (106) to vanish results in a complicated expression, which is not shown here while it gives the relation between the symmetric pressure tensor Θ^{ij} and the asymmetric canonical pressure tensor in Eq. (E11). Here, using Eq. (92), the electric current density can be written as

$$J^k \equiv \sum_a e_a \Gamma^k_a = \sum_a e_a \int d^3v\, F_a u^k_{ax} + c\,\epsilon^{kij}\frac{\partial M_j}{\partial x^i}, \tag{113}$$

where the covariant components of the magnetization vector are defined by

$$M_j \equiv \sum_a \int d^3v\, \frac{F_a}{\sqrt{g}}\left[-\mu b_j + \frac{m_a v_\parallel}{B}\left\{ (u_{ax})_j - (u_{ax})_i b^i b_j \right\} - e_a \frac{\partial \Psi_a}{\partial B^j} \right]. \tag{114}$$

Using Eqs. (89) and (90), we have

$$\frac{\partial \rho^{(g)}}{\partial t} + \frac{\partial J^i}{\partial x^i} = 0, \tag{115}$$

where $\rho^{(g)}$ is defined in Eq. (46). Combining Eq. (72) and (115), we obtain

$$J^i_L + \frac{1}{4\pi}\frac{\partial\left(\sqrt{g}D^i_L\right)}{\partial t} = 0, \tag{116}$$

where the subscript L represents the longitudinal part. It should be noted that the polarization current $4\pi\partial P^i_G/\partial t$ is included not in J^i but in $\partial\left(\sqrt{g}D^i\right)/\partial t$.

It can be shown by performing further vector calculations that Eq. (112) is rewritten as

$$\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a m_a v_\parallel \mathbf{b} + \frac{1}{4\pi c}(\mathbf{D}_L \times \mathbf{B}) \right) + \nabla\cdot(\mathbf{P}_{\text{CGL}} + \boldsymbol{\pi}_\wedge + \mathbf{P}_\Psi) + \nabla\left(\frac{|\mathbf{E}_L|^2}{8\pi} + \frac{\mathbf{E}_T\cdot\mathbf{D}_L}{4\pi} \right) - \nabla\cdot\left(\frac{\mathbf{E}_L\mathbf{E}_L + \mathbf{D}_L\mathbf{E}_T + \mathbf{E}_T\mathbf{D}_L}{4\pi} \right) + \nabla\left(\frac{B^2}{8\pi} \right) - \nabla\cdot\left(\frac{\mathbf{BB}}{4\pi} \right) = \sum_a \int d^3v\, \mathcal{K}_a m_a v_\parallel \mathbf{b} + \left(\frac{\nabla\times\mathbf{B}}{4\pi} - \frac{\mathbf{J}_T}{c} \right)\times\mathbf{B}, \tag{117}$$

where the Cartesian spatial coordinates are used and three-dimensional vectors are represented in terms of boldface letters. The longitudinal part of the electric displacement vector defined in Eq. (73) is represented by $\mathbf{D}_L$. The longitudinal part of the electric field is written as $\mathbf{E}_L \equiv -\nabla\phi$, while $\mathbf{E}_T \equiv -c^{-1}\partial\mathbf{A}/\partial t$ gives the transverse part under the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$. We see that the time derivative of the momentum density $(\mathbf{D}_L \times \mathbf{B})/(4\pi c)$ due to the electromagnetic field and the spatial divergences of the pressure tensors produced by both the electric and magnetic fields (the opposite sign of the Maxwell stress tensor) appear in the momentum balance equation in Eq. (117) where the effects of the polarization $\mathbf{P}_G$ [see Eq. (74)] are included through $\mathbf{D}_L$.

Here, recall that $F_a(x,v,t)$ represents the gyrocenter distribution function of the gyrocenter (not particle) position coordinates $x \equiv (x^i)_{i=1,2,3}$, $v \equiv (v^i)_{i=1,2,3} \equiv (v_\parallel, \mu, \vartheta)$ and time t [see descriptions after Eq. (18)]. Then, from typical gyrokinetic codes using the gyrocenter position coordinates as independent variables for distribution functions, we can directly evaluate the term $\int d^3v\, F_a m_a v_\parallel \mathbf{b} \equiv \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty d\mu \int_0^{2\pi} d\vartheta\, F_a(x, v_\parallel, \mu, \vartheta) m_a v_\parallel \mathbf{b}(x,t)$ in Eq. (117), for which we do not need to specify the transformation from the gyrocenter position coordinates to the particle coordinates. As explained at the end of Sec. III C, the difference between the gyrocenter and particle positions is taken into account in Eq. (117) through the terms related to the polarization due to the effects of the finite gyroradius and the electrostatic fluctuation. Such polarization terms also appear in the energy balance equations as seen in Eqs. (F7) and (F8) in Appendix F.

When the polarization $\mathbf{P}_G$ is eliminated from Eq. (117), the terms including the electric field are given by $\partial[(\mathbf{E}_L \times \mathbf{B})/(4\pi c)]/\partial t$ and $\nabla[|\mathbf{E}_L|^2/(8\pi) + (\mathbf{E}_T \cdot \mathbf{E}_L)/(4\pi)] - \nabla \cdot [(\mathbf{E}_L\mathbf{E}_L + \mathbf{E}_L\mathbf{E}_T + \mathbf{E}_T\mathbf{E}_L)/(4\pi)]$, which are verified to be the same as given in the momentum conservation law for the Vlasov–Poisson–Ampère (or Vlasov–Darwin) system [see Eq. (32) of Ref. 39]. In the case using the quasineutrality condition and the self-consistent magnetic field given by $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$ ($\mathbf{J} = \mathbf{J}_T$ from $\nabla \cdot \mathbf{J} = 0$ due to the quasineutrality) as well as removing polarization effects and $\mathcal{K}_a$, we find that $(\mathbf{D}_L \times \mathbf{B})/(4\pi c)$ and the part of the pressure tensor caused by the electric field disappear from Eq. (117) and that Eq. (117) agrees with the momentum conservation law for the drift kinetic system shown in Eq. (151) of Ref. 30.

The energy balance equation for the whole system is derived in Appendix F [see Eqs. (F7) and (F8)] where, in the same way as seen in Eq. (117), we can confirm the consistency of the derived energy balance with the energy conservation laws obtained for the Vlasov–Poisson–Ampère (or Vlasov–Darwin) system[39] and the drift kinetic sytem.[30]

We note that, if the Lagrangian L_{GKF} for the gyrokinetic system defined by Eqs. (17) and (18) is modified to $L_{GKF*} \equiv L_{GKF} - \int_V d^3x \sqrt{g} B^2/(8\pi)$, the variational equation $\delta L_{GKF*}/\delta \mathbf{A} = 0$ yields $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}_T$, which makes Eq. (117) take the form of the total momentum conservation law in the case of $\sum_a \int d^3v\, \mathcal{K}_a m_a v_\parallel = 0$. Nevertheless, in the gyrokinetic turbulent system, this condition is not generally imposed on the given equilibrium magnetic field $\mathbf{B}$. It is because $\mathbf{B}$ is considered not to contain the fluctuation part while $\mathbf{J}_T$ can have fluctuations. However, when the background magnetic field satisfies spatial translation, rotation, or helical symmetry and the effect of $\mathcal{K}_a$ is neglected, the local conservation law of the canonical momentum in the direction of symmetry can be derived from $\bar{\delta} L_{GKF} = 0$ with Eq. (105) as shown in Appendix G.

V. CONCLUSIONS

In this paper, the governing equations of the gyrokinetic system with electrostatic turbulence are derived in the general spatial coordinates based on the Eulerian variational principle. The local momentum balance equation for each particle species and that for the whole system, which the gyrocenter distribution functions and the potential field satisfy, are obtained from the invariance of the Lagrangians of these systems under arbitrary spatial coordinate transformations.

It is shown that, when the background magnetic field satisfies the consistency condition that its rotation is given by the solenoidal part of the current density as in the Darwin model, the momentum and energy balance equations for the whole system are rewritten in the complete conservative forms where contributions of the turbulent electric field and the background magnetic field are clearly given in the expressions of the momentum and energy densities, the pressure tensor, and the energy flux. The effects of the collision and/or external source terms added into the gyrokinetic equation on the momentum and energy balance equations are clarified as well.

The symmetric pressure tensor is directly obtained by the variational derivative of the Lagrangian with respect to the metric tensor components and it is shown to contain the CGL part representing the neoclassical viscosity as well as the turbulent momentum transport part, the ensemble average of which is confirmed to agree with the previous result obtained from the gyrokinetic theory using the WKB representation.

The representation in terms of the general spatial coordinates is useful in treating complex toroidal plasmas in suitable coordinates such as the flux coordinates. The momentum and energy balance equations obtained here are applicable as a reference for verification of long-time global gyrokinetic simulations based on the Lagrangian and Hamiltonian formulations to study neoclassical and turbulent transport in plasmas with external sources. It may seem troublesome for global simulation codes to treat the finite gyroradius effect represented by the infinite series expansions appearing in Eq. (27) and other places. However, for those simulations in which the expansions are truncated or approximated by other simpler expressions such that the Lagrangian corresponding to the reduced model given by the truncation or approximation is clearly defined, the same technique as shown in this work can be applied to that Lagrangian to derive the local energy and momentum balance equations, which can be compared with those simulation results. Such applications of the present work to the global simulations based on the reduced gyrokinetic model are considered as future works. For comparison with local flux tube gyrokinetic simulations[53–57] treating the full finite gyroradius effect, useful information such as expressions of local turbulent momentum transport can be obtained from this work using the WKB approximation as shown in Appendix D. The extension of the present work to the case with magnetic microturbulence also remains as a future study.

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APPENDIX A: COVARIANT DERIVATIVES AND CHRISTOFFEL SYMBOLS

This appendix briefly describes definitions of covariant derivatives and Christoffel symbols, which are used in the main text of the present paper. In the general spatial coordinates $x = (x^i)_{i=1,2,3}$, the components $\nabla_i S$ ($i = 1, 2, 3$) of the covariant derivative of an arbitrary scalar field $S(x)$ is given by

$$\nabla_i S = \frac{\partial S}{\partial x^i}, \tag{A1}$$

while those of the covariant derivatives of arbitrary contravariant and covariant vector fields, V^j and W_j ($j = 1, 2, 3$), are written as

$$\nabla_i V^j = \frac{\partial V^j}{\partial x^i} + \Gamma^j_{ik} V^k \tag{A2}$$

and

$$\nabla_i W_j = \frac{\partial W_j}{\partial x^i} - \Gamma^k_{ij} W_k, \tag{A3}$$

respectively. Here, the Christoffel symbols Γ^k_{ij} $(i,j,k=1,2,3)$ are defined by[58]

$$\begin{aligned}\Gamma^k_{ij}(x) &\equiv g^{kl}(x)\Gamma_{l,ij}(x)\\ &\equiv \frac{1}{2}g^{kl}(x)\left[\frac{\partial g_{jl}(x)}{\partial x^i} + \frac{\partial g_{li}(x)}{\partial x^j} - \frac{\partial g_{ij}(x)}{\partial x^l}\right].\end{aligned} \tag{A4}$$

The covariant and contravariant components of the metric tensor are denoted by g_{ij} and g^{ij}, respectively, and they satisfy

$$g^{ik}g_{kj} = \delta^i_j, \tag{A5}$$

where δ^i_j represents the Kronecker delta defined by

$$\delta^i_j \equiv \begin{cases} 1 & (i=j)\\ 0 & (i\neq j).\end{cases} \tag{A6}$$

In general, the components of the covariant derivative of an arbitrary mixed tensor field $T^{j_1\cdots j_r}_{k_1\cdots k_s}$ of type (r,s) are defined by

$$\begin{aligned}\nabla_i T^{j_1\cdots j_r}_{k_1\cdots k_s} &= \frac{\partial T^{j_1\cdots j_r}_{k_1\cdots k_s}}{\partial x^i} + \Gamma^{j_1}_{il}T^{lj_2\cdots j_r}_{k_1\cdots k_s} + \cdots + \Gamma^{j_r}_{il}T^{j_1\cdots j_{r-1}l}_{k_1\cdots k_s}\\ &\quad -\Gamma^l_{ik_1}T^{j_1\cdots j_r}_{lk_2\cdots k_s} - \cdots - \Gamma^l_{ik_s}T^{j_1\cdots j_r}_{k_1\cdots k_{s-1}l}.\end{aligned} \tag{A7}$$

Here, it is also noted that the covariant derivatives of g_{jk}, g^{jk}, and $g \equiv \det(g_{ij})$ all vanish,

$$\nabla_i g_{jk} = 0, \nabla_i g^{jk} = 0, \quad \text{and} \quad \nabla_i g = 0. \tag{A8}$$

We also note that when the covariant derivatives act on arbitrary tensors, the commutative property,

$$\nabla_i\nabla_j = \nabla_j\nabla_i, \tag{A9}$$

holds for the case of the present paper where the considered real space is a flat one with no Riemann curvature.

APPENDIX B: VARIATIONS IN THE FUNCTIONAL FORMS OF VECTOR AND TENSOR FIELDS UNDER THE INFINITESIMAL TRANSFORMATION OF SPATIAL COORDINATES

It is shown in this appendix how to represent variations in the functional forms of vector and tensor fields under the infinitesimal spatial coordinate transformation. The infinitesimal transformation from $x = (x^i)_{i=1,2,3}$ to $x' = (x'^i)_{i=1,2,3}$ is given by Eq. (78) with the infinitesimal variation $\xi^i(x)$ in the spatial coordinate x^i. Here, $(\xi^i(x))_{i=1,2,3}$ can be regarded as components of a vector field $\xi(x)$.

We first consider a scalar field $S(x)$, which is invariant under the spatial coordinate transformation,

$$S'(x') = S(x), \tag{B1}$$

and define the variation $\bar{\delta}S$ in the functional form of a scalar field $S(x)$ under the infinitesimal spatial coordinate transformation by

$$\bar{\delta}S(x) \equiv S'(x) - S(x). \tag{B2}$$

Note that the spatial arguments of S' and S are the same as each other on the right-hand side of Eq. (B2), while they are different in Eq. (B1). Next, substituting $S'(x') \simeq S'(x) + \xi^i(x)\partial S'(x)/\partial x^i \simeq S'(x,t) + \xi^i(x)\partial S(x)/\partial x^i$ into Eq. (B1) and using Eq. (B2), we obtain

$$\bar{\delta}S(x) = -\xi^i(x)\frac{\partial S(x)}{\partial x^i} \equiv -(L_\xi S)(x), \tag{B3}$$

where L_ξ denotes the Lie derivative[52] with respect to the vector field ξ with the components (ξ^i).

Under the general spatial coordinate transformation, the components of a contravariant vector field $V^i(x)$ are transformed as

$$V'^i(x') = \frac{\partial x'^i}{\partial x^j}V^j(x). \tag{B4}$$

In the same way as in Eq. (B2), we define the variation $\bar{\delta}V^i(x)$ in the functional form of $V^i(x)$ under the infinitesimal spatial coordinate transformation by

$$\bar{\delta}V^i(x) \equiv V'^i(x) - V^i(x). \tag{B5}$$

Substituting the formulas $V'^i(x') \simeq V'^i(x,t) + \xi^j(x)\partial V^i(x)/\partial x^j$ and $\partial x'^i/\partial x^j \simeq \delta^i_j + \partial\xi^i(x)/\partial x^j$ into Eq. (B4) and using Eq. (B5), we obtain

$$\begin{aligned}\bar{\delta}V^i(x) &= -\xi^j(x)\frac{\partial V^i(x)}{\partial x^j} + \frac{\partial\xi^i(x)}{\partial x^j}V^j(x)\\ &\equiv -(L_\xi V^i)(x),\end{aligned} \tag{B6}$$

where we see that the Lie derivative L_ξ can be used again to represent $\bar{\delta}V^i(x)$. The components of a covariant vector field $W_i(x)$ are transformed as

$$W'_i(x') = \frac{\partial x^j}{\partial x'^i}W_j(x). \tag{B7}$$

Next, following the procedure similar to those used in deriving Eqs. (B3) and (B6), the variation in the functional form of $W_i(x)$ under the infinitesimal spatial coordinate transformation is derived as

$$\bar{\delta}W_i(x) = -(L_\xi W_i)(x) = -\xi^j\frac{\partial W_i}{\partial x^j} - \frac{\partial\xi^j}{\partial x^i}W_j. \tag{B8}$$

It is shown in the same way as shown above that the variation in the functional form of a mixed tensor field $T^{j_1\cdots j_r}_{k_1\cdots k_s}(x)$ is given by the opposite sign of its Lie derivative as

$$\begin{aligned}\bar{\delta}T^{j_1\cdots j_r}_{k_1\cdots k_s}(x) &= -(L_\xi T^{j_1\cdots j_r}_{k_1\cdots k_s})(x)\\ &= -\xi^i\frac{\partial T^{j_1\cdots j_r}_{k_1\cdots k_s}}{\partial x^i} + \frac{\partial\xi^{j_1}}{\partial x^l}T^{lj_2\cdots j_r}_{k_1\cdots k_s} + \cdots + \frac{\partial\xi^{j_r}}{\partial x^l}T^{j_1\cdots j_{r-1}l}_{k_1\cdots k_s}\\ &\quad -\frac{\partial\xi^l}{\partial x^{k_1}}T^{j_1\cdots j_r}_{lk_2\cdots k_s} - \cdots - \frac{\partial\xi^l}{\partial x^{k_s}}T^{j_1\cdots j_r}_{k_1\cdots k_{s-1}l}.\end{aligned} \tag{B9}$$

It is also shown that the variations in the functional forms of g_{ij} and g^{ij} under the infinitesimal spatial coordinate transformation are expressed as

$$\begin{aligned}\bar{\delta}g_{ij} &= -L_\xi g_{ij} = -\xi^k\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial \xi^k}{\partial x^i}g_{kj} - \frac{\partial \xi^k}{\partial x^j}g_{ik}\\ &= -\nabla_i\xi_j - \nabla_j\xi_i,\\ \bar{\delta}g^{ij} &= -L_\xi g^{ij} = -\xi^k\frac{\partial g^{ij}}{\partial x^k} + \frac{\partial \xi^i}{\partial x^k}g^{kj} + \frac{\partial \xi^j}{\partial x^k}g^{ik}\\ &= \nabla^i\xi^j + \nabla^j\xi^i. \end{aligned} \tag{B10}$$

We now apply the chain rule to the derivative operation $\bar{\delta} = -L_\xi$ on $\sqrt{g} = \sqrt{\det(g_{ij})}$ and use Eqs. (A8) and (B10) to obtain

$$\begin{aligned}\bar{\delta}\sqrt{g} &= -L_\xi\sqrt{g} = \frac{\bar{\delta}g}{2\sqrt{g}} = \frac{\sqrt{g}}{2}g^{ij}\bar{\delta}g_{ij}\\ &= -\sqrt{g}g^{ij}\nabla_i\xi_j = -\sqrt{g}\nabla_i\xi^i = -\frac{\partial\left(\sqrt{g}\xi^i\right)}{\partial x^i}. \end{aligned} \tag{B11}$$

We here note that an arbitrary scalar density $\mathcal{S}$ is transformed in the same manner as $\sqrt{g}$ under arbitrary spatial coordinate transformations,

$$\mathcal{S}'(x') = \left|\det\left(\frac{\partial x^i}{\partial x'^j}\right)\right|\mathcal{S}(x), \tag{B12}$$

and that $\mathcal{S}(x)/\sqrt{g}(x)$ is regarded as a scalar field. Then, the variation $\bar{\delta}\mathcal{S}(x)$ in the functional form of $\mathcal{S}(x)$ under the infinitesimal spatial coordinate transformation is given by

$$\begin{aligned}\bar{\delta}\mathcal{S}(x) &= -(L_\xi\mathcal{S})(x) = \left(\bar{\delta}\sqrt{g}\right)\frac{\mathcal{S}}{\sqrt{g}} + \sqrt{g}\,\bar{\delta}\left(\frac{\mathcal{S}}{\sqrt{g}}\right)\\ &= -\frac{\partial\left(\sqrt{g}\xi^i\right)}{\partial x^i}\frac{\mathcal{S}}{\sqrt{g}} - \sqrt{g}\xi^i\frac{\partial}{\partial x^i}\left(\frac{\mathcal{S}}{\sqrt{g}}\right) = -\frac{\partial(\xi^i\mathcal{S})}{\partial x^i}. \end{aligned} \tag{B13}$$

Since the distribution function $F(x, v, t)$ behaves as a scalar density under arbitrary spatial coordinate transformations, the variation $\bar{\delta}F$ in its spatial functional form due to the infinitesimal spatial coordinate transformation is written using Eq. (B13) as

$$\bar{\delta}F = -L_\xi F = -\frac{\partial}{\partial x^j}(F\xi^j). \tag{B14}$$

Note that u^i_{ax} are the contravariant components of the gyrocenter velocity vector field, while $u_{av_\parallel}$, $u_{a\mu}$, and $u_{a\vartheta}$, which represent the temporal change rates of the parallel velocity, magnetic moment, and gyrophase, respectively, behave as scalar fields under arbitrary spatial coordinate transformations. Then, we find from Eqs. (B6) and (B3) that the variations in the functional forms of u^i_{ax}, $u_{av_\parallel}$, $u_{a\mu}$, and $u_{a\vartheta}$ under the infinitesimal spatial coordinate transformation are given by

$$\begin{aligned}\bar{\delta}u^i_{ax} &= -L_\xi u^i_{ax} \equiv -\xi^j\frac{\partial u^i_{ax}}{\partial x^j} + u^j_{ax}\frac{\partial \xi^i}{\partial x^j},\\ \bar{\delta}u_{av_\parallel} &= -L_\xi u_{av_\parallel} \equiv -\xi^j\frac{\partial u_{av_\parallel}}{\partial x^j},\\ \bar{\delta}u_{a\mu} &= -L_\xi u_{a\mu} \equiv -\xi^j\frac{\partial u_{a\mu}}{\partial x^j},\\ \bar{\delta}u_{a\vartheta} &= -L_\xi u_{a\vartheta} \equiv -\xi^j\frac{\partial u_{a\vartheta}}{\partial x^j}. \end{aligned} \tag{B15}$$

It is also useful to know that even though the Christoffel symbols are not regarded as tensor components, their derivatives with respect to $\bar{\delta}$ can be defined by the variations in their functional forms under the infinitesimal spatial coordinate transformation and written as

$$\bar{\delta}\Gamma^k_{ij} = \frac{g^{kn}}{2}\left(\nabla_i\bar{\delta}g_{nj} + \nabla_j\bar{\delta}g_{ni} - \nabla_n\bar{\delta}g_{ij}\right). \tag{B16}$$

Equation (B16) can be derived from Eqs. (A4), (A7), (B10), and the commutative property,

$$\frac{\partial}{\partial x^i}\bar{\delta} = \bar{\delta}\frac{\partial}{\partial x^i}. \tag{B17}$$

APPENDIX C: VARIATIONAL DERIVATIVE WITH RESPECT TO METRIC TENSOR COMPONENTS

We here consider the parts of the Lagrangian including the polarization effects denoted by L_{E1a} and L_{E2a},

$$[L_{E1a}, L_{E2a}] \equiv \int_V d^3x\,[\mathcal{L}_{E1a}, \mathcal{L}_{E2a}], \tag{C1}$$

where the $\mathcal{L}_{E1a}$ and $\mathcal{L}_{E2a}$ defined in Eqs. (40) and (41) appear due to finite gyroradius for species a and they are given, respectively, in the linear and quadratic forms of the longitudinal electrostatic field and its spatial gradients. For the purpose of deriving the pressure tensor $P^{ij}_{\Psi a}$ due to electrostatic gyrokinetic turbulence, we need to evaluate the variational derivatives of $\mathcal{L}_{E1a}$ and $\mathcal{L}_{E2a}$ with respect to the metric tensor components g_{ij}. It should be noted here that partial derivatives with respect to g_{ij} need to be carefully performed because 3×3 metric tensor components g_{ij} are not completely independent of each other due to the constraint $g_{ij} = g_{ji}$. Here, for an arbitrary function f of g_{ij}, the notation $\partial f/\partial g_{ij}$ is defined such that the infinitesimal variations δg_{ij} in g_{ij} give rise to the variation $\delta f = (\partial f/\partial g_{ij})\delta g_{ij}$ in f where both δg_{ij} and $\partial f/\partial g_{ij}$ must be symmetric under exchange of the indices i and j.[38] For example, we have $\partial g_{kl}/\partial g_{ij} = \frac{1}{2}(\delta^i_k\delta^j_l + \delta^j_k\delta^i_l)$ according to the above-mentioned definition. In the same manner, derivatives with respective to $\partial g_{ij}/\partial x^k$ are defined taking into account the symmetry under exchange of the indices i and j.

The variation in L_{E1a} caused by the variation in the metric tensor with keeping the gyrocenter distribution function F_a fixed in L_{E1a} is written as

$$\begin{aligned}(\delta_g L_{E1a})_F &\equiv \left(\frac{d}{d\epsilon}L_{E1a}\left[g_{ij} + \epsilon\delta g_{ij}\right]\Big|_{\epsilon=0}\right)_F\\ &= \int_V d^3x\,(\delta_g\mathcal{L}_{E1a})_F\\ &= \int_V d^3x\sum_{k=1}^{\infty}\Big[(\delta_g Q^{j_1\cdots j_{2k}}_{0a})_F\nabla_{j_1}\cdots\nabla_{j_{2k-1}}(E_L)_{j_{2k}}\\ &\quad + Q^{j_1\cdots j_{2k}}_{0a}\delta_g(\nabla_{j_1}\cdots\nabla_{j_{2k-1}}(E_L)_{j_{2k}})\Big]\\ &= \int_V d^3x\left(\frac{\delta L_{E1a}}{\delta g_{ij}}\right)_F\delta g_{ij} + \text{B.T.}, \end{aligned} \tag{C2}$$

where integration by parts is repeatedly performed to finally derive $(\delta L_{E1a}/\delta g_{ij})_F$. Here, $(\cdots)_F$ implies that the gyrocenter function F_a is

fixed when taking the variation with respect to the metric tensor. From Eqs. (A7), we have

$$\delta_g(\nabla_{j_1}\cdots\nabla_{j_{n-1}}(E_L)_{j_n}) = \sum_{l=1}^{n-1}\nabla_{j_1}\cdots\nabla_{j_{l-1}}(\delta_g\nabla_{j_l})\nabla_{j_{l+1}}\cdots\nabla_{j_{n-1}}(E_L)_{j_n}, \tag{C3}$$

where

$$\begin{aligned}&(\delta_g\nabla_{j_l})\nabla_{j_{l+1}}\cdots\nabla_{j_{n-1}}(E_L)_{j_n}\\ &\equiv -(n-l)(\delta_g\Gamma^p_{j_lj_n})\nabla_{j_{l+1}}\cdots\nabla_{j_{n-1}}(E_L)_p\\ &= -(n-l)\frac{g^{pq}}{2}(\nabla_{j_l}\delta g_{j_nq}+\nabla_{j_n}\delta g_{j_lq}-\nabla_q\delta g_{j_lj_n})\\ &\quad\times\nabla_{j_{l+1}}\cdots\nabla_{j_{n-1}}(E_L)_p.\end{aligned} \tag{C4}$$

We note that, in Eq. (C4), $\delta_g\Gamma^p_{j_lj_n}$ is expressed in the same form as in Eq. (B16). Now, the pressure tensor P^{ij}_{E1a} is given by two times of $(\delta L_{E1a}/\delta g_{ij})_F$ as

$$\begin{aligned}P^{ij}_{E1a} &\equiv 2\left(\frac{\delta L_{E1a}}{\delta g_{ij}}\right)_F\\ &= \sum_{k=1}^{\infty}\left[2\left(\frac{\partial Q^{j_1\cdots j_{2k}}_{0a}}{\partial g_{ij}}\right)_F\nabla_{j_1}\cdots\nabla_{j_{2k-1}}(E_L)_{j_{2k}}\right.\\ &\quad+\sum_{l=1}^{2k-1}(-1)^{l-1}(2k-l)\left\{(\delta^i_{j_{2k}}g^{jp}+\delta^j_{j_{2k}}g^{ip})\nabla_{j_l}-\delta^i_{j_l}\delta^j_{j_{2k}}\nabla^p\right\}\\ &\quad\left.\times\left\{(\nabla_{j_1}\cdots\nabla_{j_{l-1}}Q^{j_1\cdots j_{2k}}_{0a})(\nabla_{j_{l+1}}\cdots\nabla_{j_{2k-1}}(E_L)_p)\right\}\right],\end{aligned} \tag{C5}$$

where Eqs. (C2)–(C4) are used and $(\partial Q^{j_1\cdots j_{2k}}_{0a}/\partial g_{ij})_F$ is given by using Eq. (42) and

$$\frac{\partial\alpha^{j_1\cdots j_{2k}}_a}{\partial g_{mn}} = k\left[\frac{h^{mn}}{2}\alpha^{j_1\cdots j_{2k}}_a+\frac{1}{(k!)^2}\left(\frac{\rho}{2}\right)^{2k}\times\sum_{\sigma\in\mathfrak{S}_{2k}}\frac{\partial h^{j_{\sigma(1)}j_{\sigma(2)}}}{\partial g_{mn}}h^{j_{\sigma(3)}j_{\sigma(4)}}\cdots h^{j_{\sigma(2k-1)}j_{\sigma(2k)}}\right]. \tag{C6}$$

Next, the variation in L_{E2a} caused by the variation in the metric tensor with keeping the gyrocenter distribution function F_a fixed is written as

$$\begin{aligned}(\delta_gL_{E2a})_F &= \int_V d^3x\,(\delta_g\mathcal{L}_{E2a})_F\\ &= \int_V d^3x\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{2}(\delta_g\chi^{i_1\cdots i_m;j_1\cdots j_n})_F\nabla_{i_1}\cdots\nabla_{i_{m-1}}(E_L)_{i_m}\right.\\ &\quad\left.\times\nabla_{j_1}\cdots\nabla_{j_{n-1}}(E_L)_{j_n}+Q^{j_1\cdots j_n}_{Ea}\delta_g(\nabla_{j_1}\cdots\nabla_{j_{n-1}}(E_L)_{j_n})\right]\\ &= \int_V d^3x\left(\frac{\delta L_{E2a}}{\delta g_{ij}}\right)_F\delta g_{ij}+\text{B.T.}\end{aligned} \tag{C7}$$

Then, using Eqs. (C3), (C4), and (C7), we obtain the pressure tensor P^{ij}_{E2a}, which is given by two times of $(\delta L_{E2a}/\delta g_{ij})_F$ as

$$\begin{aligned}P^{ij}_{E2a} &\equiv 2\left(\frac{\delta L_{E2a}}{\delta g_{ij}}\right)_F\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(\frac{\partial\chi^{i_1\cdots i_m;j_1\cdots j_n}_a}{\partial g_{ij}}\right)_F(\nabla_{i_1}\cdots\nabla_{i_{m-1}}(E_L)_{i_m})\right.\\ &\quad\times(\nabla_{j_1}\cdots\nabla_{j_{n-1}}(E_L)_{j_n})\\ &\quad+\sum_{l=1}^{n-1}(-1)^{l-1}(n-l)\left\{(\delta^i_{j_n}g^{jp}+\delta^j_{j_n}g^{ip})\nabla_{j_l}-\delta^i_{j_l}\delta^j_{j_n}\nabla^k\right\}\\ &\quad\left.\times\left\{(\nabla_{j_1}\cdots\nabla_{j_{l-1}}Q^{j_1\cdots j_n}_{Ea})(\nabla_{j_{l+1}}\cdots\nabla_{j_{n-1}}(E_L)_p)\right\}\right],\end{aligned} \tag{C8}$$

where $(\partial\chi^{i_1\cdots i_m;j_1\cdots j_n}_a/\partial g_{ij})_F$ is given by using Eqs. (15), (44), (C6), and

$$\frac{\partial}{\partial g_{ij}}\left(\frac{1}{B}\right) = -\frac{1}{B^2}\frac{\partial B}{\partial g_{ij}} = \frac{h^{ij}}{2B}. \tag{C9}$$

APPENDIX D: WKB REPRESENTATION

Here, we use the WKB (or ballooning) representation[49] for turbulent fluctuations, which have small wavelengths of the order of the gyroradius ρ in the directions perpendicular to the background magnetic field. Such rapid spatial variations are represented using the perpendicular wavenumber vector $\mathbf{k}_\perp$. We assume that $k_\perp\rho = \mathcal{O}(1)$ and $\rho/L \ll 1$, where L is the gradient scale length of equilibrium variables.

The gyrocenter distribution function F_a for species a is given by the sum of the zeroth and first-order parts in ρ/L as $F_a(x,v) = F_{a0}+F_{a1}+\hat{F}_{a1}$, where the zeroth-order part F_{a0} is the local Maxwellian equilibrium distribution function, and the first-order part representing the deviation from the local Maxwellian consists of the non-turbulent and turbulent functions denoted by F_{a1} and $\hat{F}_{a1}$, respectively. Then, the turbulent gyrocenter distribution function $\hat{F}_{a1}$ is expanded as $\hat{F}_{a1} = \sum_{\mathbf{k}_\perp}\hat{F}_{a1\mathbf{k}_\perp}\exp(i\mathbf{k}_\perp\cdot\mathbf{x})$, where $\mathbf{x}$ is the gyrocenter position vector and the $\mathbf{k}_\perp$-component $F_{a1\mathbf{k}_\perp}$ is given by the sum of the adiabatic and nonadiabatic parts as[59]

$$\hat{F}_{a1\mathbf{k}_\perp} = -F_{a0}\frac{e_a}{T_a}J_0(k_\perp\rho_a)\hat{\phi}_{\mathbf{k}_\perp}+\hat{h}_{a\mathbf{k}_\perp}. \tag{D1}$$

Here, $\hat{\phi}_{\mathbf{k}_\perp}$ and $\hat{h}_{a\mathbf{k}_\perp}$ are the $\mathbf{k}_\perp$-components of the turbulent electrostatic potential and the nonadiabatic part of the turbulent distribution function, respectively, J_0 is the zeroth-order Bessel function, and T_a is the background temperature of the species a.

We now follow the conventional assumption that $\langle\hat{\mathcal{F}}_{\mathbf{k}_\perp}\rangle_{\text{ens}} = 0$ and $\langle\hat{\mathcal{F}}\hat{\mathcal{F}}'\rangle_{\text{ens}} = \sum_{\mathbf{k}_\perp}\langle\hat{\mathcal{F}}^*_{\mathbf{k}_\perp}\hat{\mathcal{F}}'_{\mathbf{k}_\perp}\rangle_{\text{ens}}$ are satisfied by arbitrary real-valued turbulent fluctuations $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$, where $(\cdots)^*$ and $\langle\cdots\rangle_{\text{ens}}$ represent the complex conjugate and the ensemble average, respectively. Note that the ensemble-averaged quantities are smooth spatial functions with the gradient scale length L and that $\langle\hat{\mathcal{F}}^*_{\mathbf{k}_\perp}\hat{\mathcal{F}}'_{\mathbf{k}'_\perp}\rangle_{\text{ens}} = 0$ for $\mathbf{k}_\perp\neq\mathbf{k}'_\perp$. When taking the ensemble average of the momentum balance equation in Eq. (117), we find that the effects of the turbulent fluctuations on the momentum transport are included in $\langle\boldsymbol{\pi}_\wedge\rangle_{\text{ens}}$ and $\langle\mathbf{P}_\Psi\rangle_{\text{ens}}$ through the correlation between the nonadiabatic part of the turbulent distribution function

and the turbulent electrostatic potential, which are described by $\langle \hat{h}^*_{a\mathbf{k}_\perp} \hat{\phi}_{\mathbf{k}_\perp} \rangle_{\rm ens}$. Neglecting terms of higher orders in ρ/L, we find that the turbulent contribution $\langle \boldsymbol{\pi}_{\wedge\Psi} \rangle_{\rm ens}$ to $\langle \boldsymbol{\pi}_{\wedge} \rangle_{\rm ens}$ is given by

$$\langle \boldsymbol{\pi}_{\wedge\Psi} \rangle_{\rm ens} = \frac{c}{B} \sum_{\mathbf{k}_\perp} [\mathbf{b}(\mathbf{k}_\perp \times \mathbf{b}) + (\mathbf{k}_\perp \times \mathbf{b})\mathbf{b}] \times \sum_a \int d^3v\, m_a v_\parallel J_0(k_\perp \rho_a) {\rm Im} \left[\langle \hat{h}^*_{a\mathbf{k}_\perp} \hat{\phi}_{\mathbf{k}_\perp} \rangle_{\rm ens} \right], \tag{D2}$$

and $\langle \mathbf{P}_\Psi \rangle_{\rm ens}$ is written as

$$\langle \mathbf{P}_\Psi \rangle_{\rm ens} = \sum_{\mathbf{k}_\perp} \left[\frac{1}{2} (\mathbf{I} - \mathbf{bb}) - \frac{\mathbf{k}_\perp \mathbf{k}_\perp}{k_\perp^2} \right] \sum_a e_a \times \int d^3v\, k_\perp \rho_a J_1(k_\perp \rho_a) {\rm Re} \left[\langle \hat{h}^*_{a\mathbf{k}_\perp} \hat{\phi}_{\mathbf{k}_\perp} \rangle_{\rm ens} \right], \tag{D3}$$

where $\mathbf{I}$ denotes the unit tensor.

We now consider the axisymmetric toroidal system, in which the magnetic field is given by

$$\mathbf{B} = I \nabla\zeta + \nabla\zeta \times \nabla\chi, \tag{D4}$$

where ζ, I, and χ represent the toroidal angle, the covariant toroidal magnetic field component, and the poloidal flux function, respectively. Then, using $(\mathbf{k}_\perp \times \mathbf{b}) \cdot \nabla\chi = -BR^2 \nabla\zeta \cdot \mathbf{k}_\perp$, the radial transport of the toroidal angular momentum due to the turbulent electric field is obtained from Eqs. (D2) and (D3) as

$$\begin{aligned} &\nabla\chi \cdot \langle \boldsymbol{\pi}_{\wedge\Psi} + \mathbf{P}_\Psi \rangle_{\rm ens} \cdot R^2 \nabla\zeta \\ &= -\sum_a \sum_{\mathbf{k}_\perp} \int d^3v \left[\frac{cI}{B} m_a v_\parallel J_0(k_\perp \rho_a) {\rm Im} \left[\langle \hat{h}^*_{a\mathbf{k}_\perp} \hat{\phi}_{\mathbf{k}_\perp} \rangle_{\rm ens} \right] \right. \\ &\quad \times (\mathbf{k}_\perp \cdot R^2 \nabla\zeta) + e_a k_\perp \rho_a J_1(k_\perp \rho_a) {\rm Re} \left[\langle \hat{h}^*_{a\mathbf{k}_\perp} \hat{\phi}_{\mathbf{k}_\perp} \rangle_{\rm ens} \right] \\ &\quad \left. \times \frac{(\mathbf{k}_\perp \cdot \nabla\chi)(\mathbf{k}_\perp \cdot R^2 \nabla\zeta)}{k_\perp^2} \right]. \end{aligned} \tag{D5}$$

The flux-surface average of Eq. (D5) agrees with the electrostatic and low-flow ordering limit of the result given in Eq. (53) of Ref. 24 where the turbulent radial transport of the toroidal angular momentum double-averaged over the ensemble and the flux surface is presented for the general case allowing the turbulent magnetic field and the high-flow ordering.[28,29,60] It should be emphasized here that Eq. (D5) is not surface-averaged but it presents the spatially local expression. In the axisymmetric configuration with up-down symmetry, the flux-surface average of Eq. (D5) is shown to vanish in the case of the low-flow ordering[27,61] although it does not imply that the local value of Eq. (D5) itself vanishes as well.

APPENDIX E: ANOTHER DERIVATION OF THE MOMENTUM BALANCE

In Sec. IV, the momentum balance is derived using the invariance of the Lagrangian of the system under the infinitesimal spatial coordinate transformation induced by the vector field $\xi^i(x)$, which has an arbitrary functional form. For comparison with that derivation, another derivation of the momentum balance is given in this appendix in the way closer to the conventional derivation of the canonical momentum conservation law, which generally involves the asymmetric canonical pressure tensor.

Equation (83) for the invariance of the gyrokinetic Lagrangian L_{GKa} of species a can be rewritten without separating boundary terms from the spatial integral as

$$\bar{\delta} L_{GKa} = \int_V d^3x\, J_{GKa} = 0, \tag{E1}$$

where

$$\begin{aligned} J_{GKa} &= \int_V d^3x \int d^3v F_a \left(\xi^i \frac{\partial L_{GYa}}{\partial x^i} + \frac{\partial L_{GYa}}{\partial u^i_{ax}} \bar{\delta} u^i_{ax} \right. \\ &\quad \left. + \frac{\partial L_{GYa}}{\partial u^i_{a\vartheta}} \bar{\delta} u^i_{a\vartheta} + (\bar{\delta} L_{GYa})_u \right) \\ &= \xi^i \int d^3v \left[\frac{\partial}{\partial t} (F_a p_{ai}) - \mathcal{K}_a p_{ai} + \frac{\partial}{\partial x^j} (F_a u^j_{ax} p_{ai}) - F_a \left(\frac{\partial L_{GYa}}{\partial x^i} \right)_u \right]. \end{aligned} \tag{E2}$$

Here, $p_{ai} \equiv \partial L_{GYa} / \partial u^i_{ax}$ is the canonical momentum [see Eq. (57)] and

$$\begin{aligned} (\bar{\delta} L_{GYa})_u &\equiv (\bar{\delta} p_{aj}) u^j_{ax} - \bar{\delta} H_{GYa} \\ &= -\left(\xi^i \frac{\partial p_{aj}}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j} p_{ai} \right) u^j_{ax} + \xi^i \frac{\partial H_{GYa}}{\partial x^i} \\ &= -\xi^i \left(\frac{\partial L_{GYa}}{\partial x^i} \right)_u - \frac{\partial \xi^i}{\partial x^j} p_{ai} u^j_{ax} \end{aligned} \tag{E3}$$

represents the variation in the functional form of the single gyrocenter Lagrangian $L_{GY\,a}$ under the infinitesimal spatial coordinate transformation with (u^i_x, u_ϑ) kept fixed in L_{GYa}. The term $(\partial L_{GYa}/\partial x^i)_u$ in Eq. (E3) is written down as

$$\left(\frac{\partial L_{GYa}}{\partial x^i} \right)_u \equiv u^j_{ax} \frac{\partial p_{aj}}{\partial x^i} - \mu \frac{\partial B}{\partial x^i} - e_a \frac{\partial \Psi_a}{\partial x^i}. \tag{E4}$$

Equations (56), (89), (B15), and (E3) are used for deriving Eq. (E2). Since Eq. (E1) holds for an arbitrary spatial integral domain V, we find $J_{GKa} = 0$ for any ξ^i. Then, the canonical momentum balance equation for the gyrocenters of species a is derived from Eq. (E2) as

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\int d^3v\, F_a p_{ai} \right) + \frac{\partial}{\partial x^j} \left(\int d^3v\, F_a m_a u^j_{ax} p_{ai} \right) \\ &\quad = \int d^3v\, \mathcal{K}_a p_{ai} + \int d^3v\, F_a \left(\frac{\partial L_{GYa}}{\partial x^i} \right)_u. \end{aligned} \tag{E5}$$

Equation (E5) also can be derived directly from multiplying Eq. (89) by p_{ai}, taking its v-space integral, and using Eq. (56). It should be recalled that the general spatial coordinates $(x^i)_{i=1,2,3}$ are used here. For example, when $(x^i)_{i=1,2,3}$ represent the Cartesian coordinates (x, y, z), Eq. (E5) represents the linear canonical momentum balance. As another interesting example, we can treat the canonical angular momentum balance in toroidal plasmas such as tokamaks and stellarator/heliotron devices, for which it is convenient to use the cylindrical coordinates and/or the magnetic flux coordinates. In these coordinate systems, the toroidal angle component of Eq. (E5) represents the toroidal angular momentum balance. It is also noted that the momentum balance equations shown in Eqs. (104) and (E5) are equivalent to each other although

the transformation from Eq. (E5) to Eq. (104) requires complicated procedures involving an infinite number of times of partial integration.

We next consider the invariance of the Lagrangian L_{GKF} of the whole system under the infinitesimal spatial coordinate transformation shown in Eq. (86). In contrast to Eq. (105), we now rewrite Eq. (86) without separating boundary terms from the spatial integral as

$$\bar{\delta} L_{GKF} = \int_V d^3x\, J_{GKF} = 0, \tag{E6}$$

with

$$\begin{aligned} J_{GKF} &= \sum_a \int d^3v \left[\xi^i \left\{ \frac{\partial}{\partial t}(F_a p_{ai}) - \mathcal{K}_a p_{ai} \right\} + \frac{\partial}{\partial x^j}\left(\xi^i F_a u^j_{ax} p_{ai}\right) \right] \\ &\quad + J_{GKFA} + J_{GKFg} + J_{GKF\phi} \\ &= 0, \end{aligned} \tag{E7}$$

where Eqs. (56), (72), (89), and (B15) are used. Here, J_{GKFA} and J_{GKFg} originate from the terms including $\bar{\delta}A_j$ and $\bar{\delta}g_{ij}$ in Eq. (86), respectively, and they are written as

$$\begin{aligned} J_{GKFA} &\equiv \frac{1}{c} J^j \bar{\delta} A_j + \frac{\partial}{\partial x^j}\left(\epsilon^{jkl} M_l \bar{\delta} A_k\right) \\ &= -\frac{\sqrt{g}}{c}\epsilon_{ijk}\xi^i J^j B^k - \frac{1}{c} J^j \frac{\partial(\xi^i A_i)}{\partial x^j} \\ &\quad - \epsilon^{jkl}\frac{\partial}{\partial x^j}\left[M_l \left(\xi^i \frac{\partial A_k}{\partial x^i} + \frac{\partial \xi^i}{\partial x^k} A_i \right) \right] \\ &= -\frac{\sqrt{g}}{c}\epsilon_{ijk}\xi^i J^j B^k - \left(\frac{1}{c} J^j - \epsilon^{jkl}\frac{\partial M^l}{\partial x^k} \right) \frac{\partial(\xi^i A_i)}{\partial x^j} \\ &\quad - \epsilon^{jkl}\epsilon_{ikm}\frac{\partial}{\partial x^j}\left(\sqrt{g}\xi^i M_l B^m \right) \\ &= -\frac{\sqrt{g}}{c}\epsilon_{ijk}\xi^i J^j B^k - \left(\sum_a \frac{e_a}{c} \int d^3v\, F_a u^j_{ax} \right) \frac{\partial(\xi^i A_i)}{\partial x^j} \\ &\quad - \frac{\partial}{\partial x^i}\left(\sqrt{g}\xi^i M_j B^j \right) + \frac{\partial}{\partial x^j}\left(\sqrt{g}\xi^i M_i B^j \right) \end{aligned} \tag{E8}$$

and

$$J_{GKFg} \equiv \sum_a \int d^3v\, F_a \sum_J \frac{\partial L_{GYa}}{\partial(\partial_J g_{ij})} \bar{\delta}(\partial_J g_{ij}) + \frac{\partial \mathcal{L}_F}{\partial g_{ij}} \bar{\delta} g_{ij}. \tag{E9}$$

The last term $J_{GKF\phi}$ included in Eq. (E7) is given in terms of the turbulent electrostatic field as

$$\begin{aligned} J_{GKF\phi} &= \frac{\partial(\xi^i \mathcal{L}_F)}{\partial x^i} + \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^k \frac{\partial}{\partial x^{j_k}} \left[\frac{\partial^{n-k}(\xi^i \partial\phi/\partial x^i)}{\partial x^{j_{k+1}} \cdots \partial x^{j_n}} \right. \\ &\quad \left. \times \frac{\partial^{k-1}}{\partial x^{j_1} \cdots \partial x^{j_{k-1}}} \left(\frac{\partial \mathcal{L}_{GKF}}{\partial(\partial^n \phi/\partial x^{j_1} \cdots \partial x^{j_n})} \right) \right]. \end{aligned} \tag{E10}$$

Here, because of the symmetry with respect to permutations of the indices $j_1, \ldots, j_n$, the variables $\partial^n\phi/\partial x^{j_1}\cdots\partial x^{j_n}$ $[\equiv -\partial^n (E_L)_{j_n}/\partial x^{j_1}\cdots\partial x^{j_{n-1}}]$ are not regarded as completely independent variables and the meaning of the expression $\partial\mathcal{L}_{GKF}/\partial(\partial^n\phi/\partial x^{j_1}\cdots\partial x^{j_n})$ in Eq. (E10) needs to be clearly mentioned. It is defined such that the infinitesimal variations $\delta(\partial^n\phi/\partial x^{j_1}\cdots\partial x^{j_n})$ give rise to the variation $\delta\mathcal{L}_{GKF} = [\partial\mathcal{L}_{GKF}/\partial(\partial^n\phi/\partial x^{j_1}\cdots\partial x^{j_n})]\delta(\partial^n\phi/\partial x^{j_1}\cdots\partial x^{j_n})$, where $\partial\mathcal{L}_{GKF}/\partial(\partial^n\phi/\partial x^{j_1}\cdots\partial x^{j_n})$ must be symmetric with respect to arbitrary permutations of the indices $j_1, \ldots, j_n$.

We now suppose that ξ^i represents a Killing vector field or a vector field, which generates an isometric transformation so that $\bar{\delta} g_{ij} = -L_\xi g_{ij} = 0$, $\bar{\delta}(\partial_J g_{ij}) = \partial_J(\bar{\delta} g_{ij}) = 0$ and, accordingly, $J_{GKFg} = 0$. The isometry is generated by the infinitesimal linear translation and the infinitesimal rotation, and they are represented by the Killing vector fields, $\boldsymbol{\xi} = \epsilon\mathbf{n}$ and $\boldsymbol{\xi} = \epsilon\mathbf{n} \times \mathbf{r}$, respectively, where ϵ is an infinitesimal constant, $\mathbf{n}$ a constant unit vector, and $\mathbf{r}$ the spatial position vector. Here and hereafter, the Cartesian coordinate system is used for $(x^i)_{i=1,2,3}$ and three-dimensional vectors are represented in terms of boldface letters. For example, the canonical momentum of the gyrocenter of species a is denoted by $\mathbf{p}_a \equiv \partial L_{GYa}/\partial \mathbf{u}_{ax} = m_a v_\parallel \mathbf{b} + (e_a/c)\mathbf{A}$.

Recall again that J_{GKF} defined in Eq. (E7) vanishes because Eq. (E6) holds for any V. Then, substituting $\boldsymbol{\xi} = \epsilon\mathbf{n}$ into Eq. (E7) and noting that $J_{GKF} = 0$ is satisfied for any direction vector $\mathbf{n}$, we obtain the linear canonical momentum balance equation for the whole system,

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a \mathbf{p}_a \right) + \nabla \cdot \left(\sum_a \int d^3v\, F_a \mathbf{u}_{ax} \mathbf{p}_a \right) \\ &\quad + \nabla \cdot \left[-(\mathbf{B}\cdot\mathbf{M})\mathbf{I} + \mathbf{B}\mathbf{M} + \frac{|\mathbf{E}_L|^2}{8\pi}\mathbf{I} - \frac{\mathbf{E}_L\mathbf{E}_L}{4\pi} + \boldsymbol{\Pi}_\Psi \right] \\ &= \left(\sum_a \frac{e_a}{c} \int d^3v\, F_a \mathbf{u}_{ax} \right) \cdot \nabla \mathbf{A} + \frac{1}{c}\mathbf{J} \times \mathbf{B} + \sum_a \int d^3v\, \mathcal{K}_a \mathbf{p}_a, \end{aligned} \tag{E11}$$

where $\mathbf{M}$ is the magnetization vector defined in Eq. (114) and $\boldsymbol{\Pi}_\Psi$ is defined using Eq. (45) as

$$\begin{aligned} (\boldsymbol{\Pi}_\Psi)^{ij} &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^k \left[\frac{\partial^{n-k}(E_L)^j}{\partial x^{j_k}\cdots\partial x^{j_{n-1}}} \right. \\ &\quad \left. \times \frac{\partial^{k-1}}{\partial x^{j_1}\cdots\partial x^{j_{k-1}}} \left(\frac{\partial \mathcal{L}_\Psi}{\partial(\partial^{n-1}(E_L)_i/\partial x^{j_1}\cdots\partial x^{j_{n-1}})} \right) \right]. \end{aligned} \tag{E12}$$

Like $\mathbf{P}_\Psi$ given in Eq. (111), $\boldsymbol{\Pi}_\Psi$ contains the effect of the turbulent electrostatic field on the momentum transport although $\mathbf{P}_\Psi$ and $\boldsymbol{\Pi}_\Psi$ are symmetric and asymmetric tensors, respectively. Using Eqs. (90) and (115), Eq. (E11) also can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a m_a v_\parallel \mathbf{b} \right) + \nabla \cdot \left(\sum_a \int d^3v\, F_a m_a v_\parallel \mathbf{u}_{ax} \mathbf{b} \right) \\ &\quad + \nabla \cdot \left[-(\mathbf{B}\cdot\mathbf{M})\mathbf{I} + \mathbf{B}\mathbf{M} + \frac{|\mathbf{E}_L|^2}{8\pi}\mathbf{I} - \frac{\mathbf{E}_L\mathbf{E}_L}{4\pi} + \boldsymbol{\Pi}_\Psi \right] \\ &= -\left(\sum_a e_a \int d^3v\, F_a \right) \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{J} \times \mathbf{B} + \sum_a \int d^3v\, \mathcal{K}_a m_a v_\parallel \mathbf{b}. \end{aligned} \tag{E13}$$

The momentum balance equation shown in Eq. (E13) can be transformed into Eq. (112) although again it is so complicated involving infinite number of times of partial integration.

Substituting $\boldsymbol{\xi} = \epsilon \mathbf{n} \times \mathbf{r}$ into Eq. (E7) and using $J_{GKF} = 0$ for any direction vector $\mathbf{n}$, the angular momentum balance equation for the whole system is derived as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\mathbf{r} \times \sum_a \int d^3v\, F_a m_a v_\parallel \mathbf{b}\right) + \nabla \cdot \left(\sum_a \int d^3v\, F_a m_a v_\parallel \mathbf{u}_{ax} \mathbf{r} \times \mathbf{b}\right) \\
&\quad + \nabla \cdot \left[\left\{(\mathbf{B}\cdot\mathbf{M})\mathbf{I} - \mathbf{B}\mathbf{M} - \frac{|\mathbf{E}_L|^2}{8\pi}\mathbf{I} + \frac{\mathbf{E}_L\mathbf{E}_L}{4\pi}\right\} \times \mathbf{r} + \mathbf{T}_\Psi\right] \\
&= \mathbf{r} \times \left[-\left(\sum_a e_a \int d^3v\, F_a\right)\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{J} \times \mathbf{B} \right. \\
&\quad \left. + \sum_a \int d^3v\, \mathcal{K}_a m_a v_\parallel \mathbf{b}\right], \qquad \text{(E14)}
\end{aligned}$$

where $\mathbf{T}_\Psi$ is the asymmetric tensor representing the transport of the angular momentum due to the electrostatic field and defined by

$$\begin{aligned}
(\mathbf{T}_\Psi)^{ij} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^k &\left[\frac{\partial^{n-k}(\mathbf{r}\times\mathbf{E}_L)^j}{\partial x^{j_k}\cdots\partial x^{j_{n-1}}}\right. \\
&\left.\times \frac{\partial^{k-1}}{\partial x^{j_1}\cdots\partial x^{j_{k-1}}}\left(\frac{\partial \mathcal{L}_\Psi}{\partial(\partial^{n-1}(E_L)_i/\partial x^{j_1}\cdots\partial x^{j_{n-1}})}\right)\right]. \qquad \text{(E15)}
\end{aligned}$$

APPENDIX F: ENERGY BALANCE IN THE GYROKINETIC SYSTEM

In this appendix, the energy balance in the electrostatic gyrokinetic turbulent system is derived. In contrast to the case of in Sec. IV where the momentum balance is derived, we here use only the Cartesian coordinate system and represent three-dimensional vectors in terms of boldface letters. Then, the metric tensor components are represented by the Kronecker delta, and they form the 3×3 unit matrix with determinant unity.

The partial time derivative of the gyrokinetic Lagrangian density $\mathcal{L}_{GKa}$ for particle species a is written as

$$\begin{aligned}
\frac{\partial \mathcal{L}_{GKa}}{\partial t} &= \frac{\partial}{\partial t}\left(\int d^3v\, F_a L_{GYa}\right) \\
&= \int d^3v \left[\frac{\partial F_a}{\partial t}L_{GYa} + F_a\left\{\frac{\partial L_{GYa}}{\partial \mathbf{u}_{ax}}\cdot\frac{\partial \mathbf{u}_{ax}}{\partial t}\right.\right. \\
&\quad \left.\left. + \frac{\partial L_{GYa}}{\partial u_{a\vartheta}}\frac{\partial u_{a\vartheta}}{\partial t} + \left(\frac{\partial L_{GYa}}{\partial t}\right)_u\right\}\right], \qquad \text{(F1)}
\end{aligned}$$

where $(\partial L_{GYa}/\partial t)_u$ denotes the time derivative of L_{GYa} with $(u^i_{ax}, u_{a\vartheta})$ kept fixed in L_{GYa}. Here, we consider the gyrocenter distribution function F_a satisfying Eq. (89), which includes the term $\mathcal{K}_a$ representing the rate of temporal change in F_a due to collisions and/or external sources for the species a. Substituting Eq. (89) into Eq. (F1) and using $\delta I_{GK}/\delta \mathbf{x}_{aE} = 0$ [Eq. (56)], $\delta I_{GK}/\delta v_{a\|E} = 0$ [Eq. (60)], $\delta I_{GK}/\delta \mu_{aE} = 0$ [Eq. (62)], and $\delta I_{GK}/\delta \vartheta_{aE} = 0$ [Eq. (63)], we obtain the energy balance equation,

$$\frac{\partial}{\partial t}\left(\int d^3v\, F_a \mathcal{E}_a\right) + \nabla \cdot \left(\int d^3v\, F_a \mathcal{E}_a \mathbf{u}_{ax}\right) = \int d^3v\, \left(F_a \dot{\mathcal{E}}_a + \mathcal{K}_a \mathcal{E}_a\right), \qquad \text{(F2)}$$

where the gyrocenter velocity $\mathbf{u}_{ax}$ is given by Eq. (64) and $\mathcal{E}_a$ represents the energy of the single particle (or the gyrocenter Hamiltonian H_{GYa}) defined by

$$\begin{aligned}
\mathcal{E}_a \equiv H_{GYa} &\equiv \frac{\partial L_{GYa}}{\partial \mathbf{u}_{ax}}\cdot\mathbf{u}_{ax} + \frac{\partial L_{GYa}}{\partial u_{a\vartheta}}u_{a\vartheta} - L_{GYa} \\
&= \frac{1}{2}m_a v_\parallel^2 + \mu B + e_a \Psi_a. \qquad \text{(F3)}
\end{aligned}$$

The rate of change in the particle's energy is given by

$$\dot{\mathcal{E}}_a \equiv -\left(\frac{\partial L_{GYa}}{\partial t}\right)_u = e\frac{\partial \Psi_a}{\partial t} + \mu\frac{\partial B}{\partial t} - \frac{e_a}{c}\mathbf{u}_{ax}\cdot\frac{\partial \mathbf{A}^*_a}{\partial t}. \qquad \text{(F4)}$$

The energy balance equation shown in Eq. (F2) agrees with Eq. (C7) in Ref. 30 except for the effects of the turbulent electrostatic potential included here. It can be seen in Eq. (F2) how the energy balance is modified when the collision (or source) term $\mathcal{K}_a$ is added into the gyrokinetic equation.

We now consider the energy balance in the extended system consisting of particles of all species and the self-consistent electrostatic field. The partial time derivative of the Lagrangian density $\mathcal{L}_{GKF}$ of this extended system is written as

$$\begin{aligned}
\frac{\partial \mathcal{L}_{GKF}}{\partial t} &= \frac{\partial}{\partial t}(\mathcal{L}_{GK} + \mathcal{L}_F) = \frac{\partial}{\partial t}\left(\sum_a \mathcal{L}_{GKa} + \mathcal{L}_F\right) \\
&= \frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a L_{GYa} + \mathcal{L}_F\right) \\
&= \sum_a \int d^3v \left\{\frac{\partial F_a}{\partial t}L_{GYa} + F_a\left(\frac{\partial L_{GYa}}{\partial \mathbf{u}_{ax}}\cdot\frac{\partial \mathbf{u}_{ax}}{\partial t} + \frac{\partial L_{GYa}}{\partial u_{a\vartheta}}\frac{\partial u_{a\vartheta}}{\partial t}\right)\right\} \\
&\quad + \sum_J \frac{\partial \mathcal{L}_{GKF}}{\partial(\partial_J\phi)}\frac{\partial(\partial_J\phi)}{\partial t} + \frac{\partial \mathcal{L}_{GKF}}{\partial \mathbf{A}}\frac{\partial \mathbf{A}}{\partial t} \\
&\quad + \frac{\partial \mathcal{L}_{GKF}}{\partial(\partial \mathbf{A}/\partial x^i)}\cdot\frac{\partial(\partial \mathbf{A}/\partial x^i)}{\partial t}\Big]. \qquad \text{(F5)}
\end{aligned}$$

In the same way as in deriving Eq. (F2), we use the Euler–Lagrange equations [Eqs. (56), (60), (62), and (63)] for $(\mathbf{u}_{ax}, u_{av_\parallel}, u_{a\mu}, u_{a\vartheta})$, Eq. (89) for F_a, and Eq. (72) for ϕ in order to rewrite Eq. (F5) as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a H_{GYa} - \mathcal{L}_F\right) \\
&\quad + \nabla \cdot \left(\sum_a \int d^3v\, F_a H_{GYa}\mathbf{u}_{ax}\right) + \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1} \\
&\quad \times \frac{\partial}{\partial x^{j_k}}\left\{\frac{\partial^{k-1}}{\partial x^{j_1}\cdots\partial x^{j_{k-1}}}\left(\frac{\partial \mathcal{L}_{GKF}}{\partial(\partial^n\phi/\partial x^{j_1}\cdots\partial x^{j_n})}\right)\right. \\
&\quad \left.\times \frac{\partial}{\partial t}\left(\frac{\partial^{n-k}\phi}{\partial x^{j_{k+1}}\cdots\partial x^{j_n}}\right)\right\} \\
&= \mathbf{J}\cdot\mathbf{E}_T + \nabla\cdot(c\mathbf{E}_T \times \mathbf{M}) + \sum_a \int d^3v\, \mathcal{K}_a \\
&\quad \times \left\{\frac{1}{2}m_a v_\parallel^2 + \mu B + e_a(\Psi_{E1a} + \Psi_{E2a})\right\}, \qquad \text{(F6)}
\end{aligned}$$

where $\mathbf{E}_T \equiv -c^{-1}\partial \mathbf{A}/\partial t$ represents the electric field induced by the temporal change in the background magnetic field and the

magnetization vector $\mathbf{M}$ is defined by Eq. (114). It should be recalled that the background magnetic field is allowed to temporally change in this paper so that the long time evolution of the system over the transport timescale can be treated. It is also noted that Eq. (90) is used in deriving Eq. (F6).

It can be shown after several analytical manipulations that Eq. (F6) is further rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\sum_a \int d^3v\, F_a\left(\frac{1}{2}m_a v_\parallel^2+\mu B\right)+\frac{|\mathbf{E}_L|^2}{8\pi}\right.\\
&\left.+\frac{\mathbf{E}_L\cdot\mathbf{P}_D}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\partial^n (E_L)_i}{\partial x^{j_1}\cdots\partial x^{j_n}}Q_E^{ij_1\cdots j_n}+\frac{|\mathbf{B}|^2}{8\pi}\right]\\
&+\nabla\cdot\left[\sum_a\int d^3v\, F_a\left\{\frac{1}{2}m_a v_\parallel^2+\mu B+e_a(\Psi_{E1a}+\Psi_{E2a})\right\}\mathbf{u}_{ax}\right.\\
&\left.+\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}-\frac{1}{4\pi}\frac{\partial\phi_D}{\partial t}\mathbf{E}_T+\phi\frac{\partial(\mathbf{P}_G)_T}{\partial t}\right]\\
&+\frac{\partial}{\partial x^i}\left[\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-k}\frac{\partial^k (E_L)_{j_n}}{\partial x^{j_{n-k}}\cdots\partial x^{j_{n-1}}}\right.\\
&\left.\times\frac{\partial^{n-k}}{\partial x^{j_1}\cdots\partial x^{j_{n-k-1}}}\left(\frac{\partial Q^{ij_1\cdots j_n}}{\partial t}\right)\right]\\
&=\left(\mathbf{J}_T-\frac{c}{4\pi}\nabla\times\mathbf{B}\right)\cdot\mathbf{E}+\sum_a\int d^3v\,\mathcal{K}_a\\
&\times\left\{\frac{1}{2}m_a v_\parallel^2+\mu B+e_a(\Psi_{E1a}+\Psi_{E2a})\right\}
\end{aligned}\tag{F7}$$

or

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\sum_a \int d^3v\, F_a\left(\frac{1}{2}m_a v_\parallel^2+\mu B\right)+\frac{|\mathbf{E}_L|^2}{8\pi}+\frac{\mathbf{E}_L\cdot\mathbf{P}_D}{2}\right.\\
&\left.+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\partial^n (E_L)_i}{\partial x^{j_1}\cdots\partial x^{j_n}}Q_E^{ij_1\cdots j_n}+\frac{\mathbf{E}_T\cdot\mathbf{D}_L}{4\pi}+\frac{|\mathbf{B}|^2}{8\pi}\right]\\
&+\nabla\cdot\left[\sum_a\int d^3v\, F_a\left\{\frac{1}{2}m_a v_\parallel^2+\mu B+e_a(\Psi_{E1a}+\Psi_{E2a})\right\}\mathbf{u}_{ax}\right.\\
&\left.+\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}+\frac{1}{4\pi}\phi_D\frac{\partial\mathbf{E}_T}{\partial t}+\phi\frac{\partial(\mathbf{P}_G)_T}{\partial t}\right]\\
&+\frac{\partial}{\partial x^i}\left[\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-k}\frac{\partial^k (E_L)_{j_n}}{\partial x^{j_{n-k}}\cdots\partial x^{j_{n-1}}}\right.\\
&\left.\times\frac{\partial^{n-k}}{\partial x^{j_1}\cdots\partial x^{j_{n-k-1}}}\left(\frac{\partial Q^{ij_1\cdots j_n}}{\partial t}\right)\right]\\
&=\left(\mathbf{J}_T-\frac{c}{4\pi}\nabla\times\mathbf{B}\right)\cdot\mathbf{E}+\sum_a\int d^3v\,\mathcal{K}_a\\
&\times\left\{\frac{1}{2}m_a v_\parallel^2+\mu B+e_a(\Psi_{E1a}+\Psi_{E2a})\right\},
\end{aligned}\tag{F8}$$

where the longitudinal (irrotational) part of the electric displacement vector [see Eq. (73)] is denoted by $\mathbf{D}_L\equiv(\mathbf{E}+4\pi\mathbf{P}_G)_L\equiv-\nabla\phi_D$ and the magnetic intensity field $\mathbf{H}$ is defined by $\mathbf{H}\equiv\mathbf{B}-4\pi\mathbf{M}$ with the magnetic induction field $\mathbf{B}$ and the magnetization vector field $\mathbf{M}$ [see Eq. (114)]. Equations (F7) and (F8) take the conservative form including no other terms than the time derivative of the total energy density and the divergence of the total energy flux when the right-hand sides vanish. The effect of the collision (or source) term on the total energy balance is shown by the integral term, which appears on the right-hand sides of Eqs. (F7) and (F8). This term vanishes when $\mathcal{K}_a$ represents the collision operator, which conserves the summation of the gyrocenter kinetic and polarization energies $\sum_a\int d^3v\{\frac{1}{2}m_a v_\parallel^2+\mu B+e_a(\Psi_{E1a}+\Psi_{E2a})\}$. In addition, the right-hand sides of Eqs. (F7) and (F8) contain the difference between the transverse current density $\mathbf{J}_T$ and $(c/4\pi)\nabla\times\mathbf{B}$, which vanishes if the self-consistency condition $\nabla\times\mathbf{B}=(4\pi/c)\mathbf{J}_T$ is imposed as is done in the Darwin model[62] and in our previous work by including the magnetic energy with the minus sign into the Lagrangian for the drift kinetic system with self-consistent fields.[30] We also find from Eqs. (F7) and (F8) that the terms including $\mathbf{E}_T$ appear on their left-hand sides in the way consistent with the case of the energy conservation law for the Vlasov–Poisson–Ampère (or Vlasov–Darwin) system shown in Eq. (22) of Ref. 39.

All the kinetic energy, the electric energy, and the magnetic energy with the polarization including the dipole and other multipole moment effects are described in Eqs. (F7) and (F8), where the energy flux contains the kinetic energy flow due to the gyrocenter motion, the Poynting vector, and the extra energy flux due to electrostatic turbulent fluctuations with wavelengths of the order of gyroradius.

APPENDIX G: THE LOCAL CONSERVATION LAW OF THE CANONICAL MOMENTUM IN THE DIRECTION OF SYMMETRY

It is shown in this appendix how the local conservation law of the canonical momentum in the direction of symmetry can be derived for the gyrokinetic system considered in the present paper. We start from Eq. (105) for which the gyrokinetic equation including the collision (or source) term $\mathcal{K}_a$ [see Eq. (89)] is used. Equation (105) is rewritten as

$$\begin{aligned}
\bar{\delta}L_{GKF}=\int_V d^3x\left[\xi^i\sum_a\int d^3v\left\{\frac{\partial}{\partial t}(F_a p_{ai})-\mathcal{K}_a p_{ai}\right\}\right.\\
\left.+\frac{1}{c}J^i\bar{\delta}A_i+\frac{\delta L_{GKF}}{\delta\phi}\bar{\delta}\phi+\frac{1}{2}\Theta^{ij}\bar{\delta}g_{ij}\right]+\text{B.T.},
\end{aligned}\tag{G1}$$

where we have used the canonical momentum of the single gyrocenter, the electric current density, and the symmetric pressure tensor defined by $p_{ai}\equiv\partial L_{GYa}/\partial u_{ax}^i$, $J^i\equiv\sum_a e_a\Gamma_a^i=c\sum_a\delta L_{GKa}/\delta A_i=c\,\delta L_{GKF}/\delta A_i$, and $\Theta^{ij}\equiv 2\,\delta L_{GKF}/\delta g_{ij}$, respectively, as shown in Eqs. (57), (91), (113), and (108). Next, we use $\delta L_{GKF}/\delta\phi=0$, which is equivalent to the gyrokinetic Poisson's equation [see Eq. (72)], and substitute $\bar{\delta}A_i=-\xi^j(\partial_j A_i)-(\partial_i\xi^j)A_j$ [Eq. (B8)], $\bar{\delta}g_{ij}=-\nabla_i\xi_j-\nabla_j\xi_i$ [Eq. (B10)] into Eq. (G1). Then, Eq. (G1) is rewritten after performing partial integrals as

$$\bar{\delta}L_{GKF} = \int_V d^3x \left[\xi^j \sum_a \int d^3v \left\{ \frac{\partial}{\partial t}(F_a p_{aj}) - \mathcal{K}_a p_{aj} \right\} + \xi^j \frac{1}{c} \left\{ \partial_i (J^i A_j) - J^i (\partial_j A_i) \right\} + \xi_j \nabla_i \Theta^{ij} \right] + \text{B.T.}, \quad \text{(G2)}$$

where we can also write the last term of the integrand as $\xi_j \nabla_i \Theta^{ij} = \xi^j \nabla_i \Theta^i_j$ by using $\xi_j \equiv g_{jk}\xi^k$, $\Theta^i_j \equiv g_{jk}\Theta^{ik}$, and Eq. (A8). We now recall the invariance of the Lagrangian L_{GKF} under an arbitrary spatial coordinate transformation, which implies that $\bar{\delta}L_{GKF} = 0$ for any ξ^j. Consequently, the canonical momentum balance equation is obtained from Eq. (G2) as

$$\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a p_{aj} \right) - \sum_a \int d^3v\, \mathcal{K}_a p_{aj} + \frac{1}{c}\left\{ \partial_i (J^i A_j) - J^i (\partial_j A_i) \right\} + \nabla_i \Theta^i_j = 0. \quad \text{(G3)}$$

It is emphasized here that Eq. (G3) is valid for arbitrary spatial coordinates $(x^i)_{i=1,2,3}$ in which the spatial position vector in the Cartesian coordinate system is represented as $\mathbf{r} = \mathbf{r}(x^1, x^2, x^3)$.

Hereafter, we assume that the background magnetic field has symmetry with respect to continuous isometric transformations (spatial translations, rotations, or screw motions generated by the combination of translations and rotations) in a certain direction. Owing to the use of general spatial coordinates, translation, rotation, and screw (or helical) symmetries can be treated in a unified manner. Under the symmetry assumption for the background magnetic field, there exists a certain coordinate system $(x^i)_{i=1,2,3}$ where all components of the magnetic field, the vector potential, and the metric tensor become independent of one coordinate for which x^3 is chosen here without loss of generality,

$$\partial_3 B^i = 0, \quad \partial_3 A_i = 0, \quad \text{and} \quad \partial_3 g_{ij} = 0. \quad \text{(G4)}$$

For example, such coordinate systems are given by Cartesian, cylindrical (or spherical), and helical coordinate systems for the cases of translation, rotation, and screw (or helical) symmetries, respectively. Then, from Eq. (G3) with $j = 3$, we have

$$\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a p_{a3} \right) + \frac{1}{c}\partial_i (J^i A_3) + \nabla_i \Theta^i_3 = \sum_a \int d^3v\, \mathcal{K}_a p_{a3}. \quad \text{(G5)}$$

Here, we define the basis vector $\mathbf{e} \equiv \partial\mathbf{r}/\partial x^3$ associated with the coordinate x^3. For translation symmetry, $\mathbf{e}$ represents the constant vector parallel to the direction of symmetry while, for rotation symmetry, we have $\mathbf{e} = \mathbf{n} \times \mathbf{r}$ where $\mathbf{n}$ is the direction vector of the rotation axis. For helical symmetry, $\mathbf{e}$ is given by the combination of the two types of the vectors described above. Then, the boldface notation $\mathbf{p}_a$ is used to represent the vector with the covariant components $(p_{aj})_{j=1,2,3}$, and the inner product of the vectors $\mathbf{p}_a$ and $\mathbf{e}$ is represented by $\mathbf{p}_a \cdot \mathbf{e} = p_{aj} e^j = p_{a3}$, where the jth contravariant component of $\mathbf{e} \equiv \partial\mathbf{r}/\partial x^3$ is given by $e^j = \partial x^j/\partial x^3 = \delta^j_3$. Thus, p_{a3} is hereafter treated as a scalar field produced by the inner product of the vectors. In the same way, using $\mathbf{A}$ to represent the vector with the covariant components $(A_j)_{j=1,2,3}$, we can write $A_3 \equiv \mathbf{A} \cdot \mathbf{e}$, which is treated as a scalar field produced by the inner product of $\mathbf{A}$ and $\mathbf{e}$.

It is noted that, with respect to the spatial coordinate transformation, the distribution function F_a is transformed as a scalar density field [see Eq. (B12)], which means $F_a/\sqrt{g}$ is regarded as a scalar field. Since F_a enters the definitions of J^i [Eq. (113)] and Θ^{ij} [Eq. (108)], J^i and Θ^{ij} are the components of the contravariant vector density field and the symmetric contravariant tensor density field, respectively. Correspondingly, $J^i/\sqrt{g}$ and $\Theta^{ij}/\sqrt{g}$ represent the vector field and the tensor field. In the Cartesian coordinates where $\sqrt{g} = 1$, the scalar, vector, and tensor density fields have the same components as the corresponding scalar, vector, and tensor fields so that we do not need to distinguish these density fields from the corresponding fields. Using the notation $\mathbf{J}$ to represent the vector field with the contravariant components $(J^i/\sqrt{g})_{i=1,2,3}$, we can regard $J^i A_3/\sqrt{g}$ as the ith contravariant component of the vector field $\mathbf{J}(\mathbf{A} \cdot \mathbf{e})$. We also use Eqs. (A2), (A8), and the formula $\Gamma^i_{ij} = (\partial_j \sqrt{g})/\sqrt{g}$ to write

$$\begin{aligned} \partial_i (J^i A_3) &= \sqrt{g}\left\{ \partial_i \left[(J^i/\sqrt{g}) A_3 \right] + \Gamma^i_{ik} (J^i/\sqrt{g}) A_3 \right\} \\ &= \sqrt{g} \nabla \cdot [\mathbf{J}(\mathbf{A} \cdot \mathbf{e})]. \end{aligned} \quad \text{(G6)}$$

From Eq. (A4) and $\partial_3 g_{ij} = 0$ in Eq. (G4), we find that $\Gamma_{i,j3} = \frac{1}{2}(\partial_j g_{i3} - \partial_i g_{j3}) = -\Gamma_{j,i3}$, which is combined with $\Theta^{ij} = \Theta^{ji}$ to obtain $\Gamma^j_{i3}\Theta^i_j = \Gamma_{j,i3}\Theta^{ij} = 0$. We now use the notation $\boldsymbol{\Theta}$ to represent the tensor field with the contravariant components $\Theta^{ij}/\sqrt{g}$. Also, the tensor-vector contraction $\boldsymbol{\Theta} \cdot \mathbf{e}$ is used to represent the vector field, the ith component of which is written as $\left(\Theta^{ij}/\sqrt{g}\right) e_j = \left(\Theta^i_j/\sqrt{g}\right) e^j = \Theta^i_3/\sqrt{g}$. Then, using Eq. (A7) and (A8), we obtain

$$\begin{aligned} \nabla_i \Theta^i_3 &= \sqrt{g} \nabla_i \left(\Theta^i_3/\sqrt{g} \right) \\ &= \sqrt{g}\left[\partial_i \left(\Theta^i_3/\sqrt{g} \right) + \Gamma^i_{ij} \left(\Theta^j_3/\sqrt{g} \right) + \Gamma^j_{i3} \left(\Theta^i_j/\sqrt{g} \right) \right] \\ &= \sqrt{g}\left[\partial_j \left(\Theta^j_3/\sqrt{g} \right) + \Gamma^i_{ij} \left(\Theta^j_3/\sqrt{g} \right) \right] \\ &= \sqrt{g} \nabla \cdot (\boldsymbol{\Theta} \cdot \mathbf{e}) = \partial_i \Theta^i_3, \end{aligned} \quad \text{(G7)}$$

where $\Gamma^i_{ij} = (\partial_j \sqrt{g})/\sqrt{g}$ and $\Gamma^j_{i3}\Theta^i_j = 0$ are used.

Using Eq. (G5)–(G7), the local canonical momentum balance equation in the direction of symmetry is written as

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a p_{a3} \right) + \nabla_i \left(\frac{1}{c} J^i A_3 + \Theta^i_3 \right) \\ &= \frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a p_{a3} \right) + \partial_i \left(\frac{1}{c} J^i A_3 + \Theta^i_3 \right) \\ &= \sum_a \int d^3v\, \mathcal{K}_a p_{a3}, \end{aligned} \quad \text{(G8)}$$

in the spatial coordinates, where x^3 is the coordinate in the direction of symmetry. Equation (G8) is also written using the conventional vector and tensor notations in the Cartesian coordinates (with $\sqrt{g} = 1$) as

$$\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a (\mathbf{p}_a \cdot \mathbf{e}) \right) + \nabla \cdot \left[\frac{1}{c} \mathbf{J}(\mathbf{A} \cdot \mathbf{e}) + \boldsymbol{\Theta} \cdot \mathbf{e} \right] = \sum_a \int d^3v\, \mathcal{K}_a (\mathbf{p}_a \cdot \mathbf{e}). \quad \text{(G9)}$$

Finally, we clearly see that Eqs. (G8) and (G9) represent the local conservation law of the canonical momentum conjugate to the coordinate in the direction of symmetry when the collision (or source) term $\mathcal{K}_a$ satisfies $\sum_a \int d^3v\, \mathcal{K}_a\,(\mathbf{p}_a \cdot \mathbf{e}) = 0$.

Let us compare Eqs. (G8) and (G9) with the result derived from the conventional method explained in Appendix E. In deriving Eqs. (G8) and (G9), after the invariance of the Lagrangian under arbitrary spatial coordinate transformations is used to obtain the local momentum balance equation, Eq. (G1), in the general spatial coordinates, Eq. (G1) is represented in the special coordinates, one of which is the coordinate in the direction of symmetry. On the other hand, in Appendix E, the invariance of the Lagrangian under not arbitrary but only translational and rotational coordinate transformations are used to derive the local linear and angular momentum equations in Eqs. (E11) and (E14). In the case where the background magnetic field has translation symmetry along the constant direction vector $\mathbf{e}$, the inner product of Eq. (E11) and $\mathbf{e}$ results in the local canonical momentum conservation law,

$$\frac{\partial}{\partial t}\left(\sum_a \int d^3v\, F_a\,(\mathbf{p}_a \cdot \mathbf{e})\right) + \nabla \cdot \left[\frac{1}{c}\mathbf{J}\,(\mathbf{A}\cdot\mathbf{e}) + \mathbf{\Pi}_c \cdot \mathbf{e}\right] = \sum_a \int d^3v\, \mathcal{K}_a\,(\mathbf{p}_a \cdot \mathbf{e}), \tag{G10}$$

with the asymmetric pressure tensor $\mathbf{\Pi}_c$ given by

$$\mathbf{\Pi}_c \equiv \sum_a \int d^3v\, F_a \mathbf{u}_{ax}\mathbf{u}_{ax} - (\mathbf{B}\cdot\mathbf{M})\mathbf{I} + \mathbf{BM} + \frac{|\mathbf{E}_L|^2}{8\pi}\mathbf{I} - \frac{\mathbf{E}_L\mathbf{E}_L}{4\pi} + \mathbf{\Pi}_\Psi, \tag{G11}$$

where Eqs. (57), (113), and $\mathbf{e}\cdot\nabla\mathbf{A} = 0$ are used, and the term including $\mathcal{K}_a$ is still retained for comparison with Eq. (G9). When the background field has rotational symmetry around the axis, which is parallel to the direction vector $\mathbf{n}$ and passes through the origin of the position vector $\mathbf{r}$, $\mathbf{A}$ satisfies $(\mathbf{n}\times\mathbf{r})\cdot\nabla\mathbf{A} = \mathbf{n}\times\mathbf{A}$. Then, we can use the inner product of Eq. (E14) and $\mathbf{n}$ to derive the local canonical angular momentum conservation law, which has the same form as shown in Eqs. (G10) and (G11) where $\mathbf{e}$ is regarded as given by $\mathbf{e} = \mathbf{n}\times\mathbf{r}$. In the same way, it can be shown that, for the case of helical symmetry, the local conservation law of the canonical momentum in the direction of symmetry is written again by Eqs. (G10) and (G11), where we put $\mathbf{e} = \mathbf{n} + \alpha\mathbf{n}\times\mathbf{r}$ with a constant α to represent the direction of helical symmetry. Now, it is clear that the difference between Eqs. (G9) and (G10) is expressed by that between the symmetric and asymmetric pressure tensors denoted as $\mathbf{\Theta}$ and $\mathbf{\Pi}_c$, respectively. It is not easy to find out the well-known CGL tensor $\mathbf{P}_{CGL}$ [Eq. (109)] in $\mathbf{\Pi}_c$ while $\mathbf{P}_{CGL}$ naturally appears in $\mathbf{\Theta}$ [Eq. (108)]. Incidentally, the procedure of the Belinfante–Rosenfeld type is known for derivation of the symmetric pressure tensor from the asymmetric canonical pressure tensor[39,40] although the symmetric tensor is more directly derived by the present method using the variational derivative of the Lagrangian with the metric tensor.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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