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# Explicit approximations to estimate the perturbative diffusivity in the presence of convectivity and damping. III. Cylindrical approximations for heat waves traveling inwards 

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In this paper, a number of new explicit approximations are introduced to estimate the perturbative diffusivity $(\chi)$, convectivity $(V)$, and damping $(\tau)$ in cylindrical geometry. For this purpose, the harmonic components of heat waves induced by localized deposition of modulated power are used. The approximations are based on the heat equation in cylindrical geometry using the symmetry (Neumann) boundary condition at the plasma center. This means that the approximations derived here should be used only to estimate transport coefficients between the plasma center and the off-axis perturbative source. If the effect of cylindrical geometry is small, it is also possible to use semi-infinite domain approximations presented in Part I and Part II of this series. A number of new approximations are derived in this part, Part III, based upon continued fractions of the modified Bessel function of the first kind and the confluent hypergeometric function of the first kind. These approximations together with the approximations based on semi-infinite domains are compared for heat waves traveling towards the center. The relative error for the different derived approximations is presented for different values of the frequency, transport coefficients, and dimensionless radius. Moreover, it is shown how combinations of different explicit formulas can be used to estimate the transport coefficients over a large parameter range for cases without convection and damping, cases with damping only, and cases with convection and damping. The relative error between the approximation and its underlying model is below $2 \%$ for the case, where only diffusivity and damping are considered. If also convectivity is considered, the diffusivity can be estimated well in a large region, but there is also a large region in which no suitable approximation is found. This paper is the third part (Part III) of a series of three papers. In Part I, the semi-infinite slab approximations have been treated. In Part II, cylindrical approximations are treated for heat waves traveling towards the plasma edge assuming a semi-infinite domain.
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## I. INTRODUCTION

In this paper, Part III of a series of three papers, approximate solutions are obtained of the heat equation in a cylindrical domain with symmetry (Neumann) boundary condition and the unknown diffusivity $\chi$, convectivity $V$, and damping $\tau$. These unknown transport coefficients can be estimated using the experimental data of heat pulse propagation. These approximations should be used when heat waves travel towards the center of the plasma, which is the case when the plasma is perturbated using an off-axis heating source. For a general introduction of the series of three papers, the reader is referred to Ref. 1.

This paper is structured as follows. Section II gives an overview of the relevant assumptions and models used for perturbative transport analysis. Then, in Sec. III, continued
fractions are used to find approximations for $\chi, V$, and $\tau$. Section IV gives an overview and comparison of possible explicit approximations that can be used to estimate $\chi, V$, and $\tau$ for heat waves traveling towards the center. In Sec. V, the main results are summarized and discussed for Part III and, in Sec. VI, a general conclusion for the set of papers is given.

## II. MODELING OF THERMAL TRANSPORT

This section briefly reviews the relevant Partial Differential Equation describing transport in fusion reactors and its solution in the Laplace domain. This solution is necessary to derive explicit approximations for the transport coefficients, which is the subject of Sec. III. For a more
extensive discussion on the heat equation, the reader is referred to Refs. 1 and 2.

## A. Perturbative transport analysis

Linearized thermal transport inside a fusion reactor is often modeled as a one-dimensional radial transport in cylindrical geometry due to the magnetic confined plasma topology ${ }^{1,3,4}$

$$
\begin{align*}
\frac{3}{2} \frac{\partial}{\partial t}(n T)= & \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(n \rho \chi(\rho) \frac{\partial T}{\partial \rho}+n \rho V(\rho) T\right) \\
& -\frac{3}{2} n \tau_{\text {inv }}(\rho) T+P_{\text {mod }} \tag{1}
\end{align*}
$$

where $\chi$ is the diffusivity, $V$ is the convectivity, $\tau_{i n v}$ is the damping ( $\tau_{i n v}=1 / \tau$ ), $T$ is the electron temperature, $n$ is the density, $\rho$ is the radius, and $P_{\text {mod }}$ is a perturbative heat source. Analytical solutions based on (1) can be derived using a number of standard assumptions. These assumptions are: ${ }^{2,3,5}$ constant transport coefficients with respect to time and $\rho$ (homogenous or uniform); no transients due to initial conditions; on the considered domains $P_{\text {mod }}=0$; and density $n$ is assumed constant with respect to $\rho$ and time.

Under these assumptions, the analytical solution of (1) in the Laplace domain can be expressed in terms of confluent hypergeometric functions $\Phi$ and $\Psi^{2,6-8}$

$$
\begin{align*}
\Theta(\rho, s)= & e^{\lambda_{1} \rho} D_{1}(s) \Psi\left(\frac{\lambda_{2}}{\lambda_{2}-\lambda_{1}}, 1,\left(\lambda_{2}-\lambda_{1}\right) \rho\right) \\
& +e^{\lambda_{1} \rho} D_{2}(s) \Phi\left(\frac{\lambda_{2}}{\lambda_{2}-\lambda_{1}}, 1,\left(\lambda_{2}-\lambda_{1}\right) \rho\right) \tag{2}
\end{align*}
$$

where $s$ is the Laplace variable

$$
\begin{equation*}
\lambda_{1,2}=-\frac{V}{2 \chi} \mp \sqrt{\left(\frac{V}{2 \chi}\right)^{2}+\frac{3}{2} \frac{s+\tau_{i n v}}{\chi}} \tag{3}
\end{equation*}
$$

and $\Theta$ is the Laplace transformed temperature $\Theta=\mathcal{L}(T)$. The boundary constants are denoted by $D_{1}(s)$ and $D_{2}(s)$. If $V=0$, (2) can be simplified in terms of modified Bessel functions $I_{\nu}$ and $K_{\nu}$ of order $\nu=0$ resulting in Refs. 6 and 9

$$
\begin{equation*}
\Theta(\rho, s)=\frac{1}{\sqrt{\pi}} D_{1}(s) K_{0}(z \rho)+D_{2}(s) I_{0}(z \rho) \tag{4}
\end{equation*}
$$

with

$$
\begin{equation*}
z=\sqrt{\frac{3}{2} \frac{s+\tau_{i n v}}{\chi}} \tag{5}
\end{equation*}
$$

The unknown boundary constants $D_{1}(s)$ and $D_{2}(s)$ need to be determined or partly eliminated such that analytical solutions can be determined, which are used in Sec. III to calculate $\chi$ explicitly.

## B. Logarithmic temperature derivative and transfer function

In Refs. 1 and 2, models have been derived to determine $\chi$ for heat waves traveling outwards in a cylindrical semi-infinite
domain. The semi-infinite domain is, in principle, unnatural as there is a true finite boundary (in the form of a plasma end or wall). However, this assumption is necessary to find approximations for $\chi, V$, and $\tau_{i n v}$. On the other hand, if heat waves travel towards the center in a cylindrical geometry, the natural boundary condition in a cylindrical system is a symmetry (Neumann) boundary condition at $\rho=0$, i.e.,

$$
\begin{equation*}
\frac{\partial \Theta}{\partial \rho}(\rho=0)=0 \tag{6}
\end{equation*}
$$

Note that the choice of a symmetry boundary condition makes the domain finite. This can be unrealistic in the context of drift-wave turbulence as this results in a finite box size. The drift wave turbulence has an inverse cascade, which will make wave intensities pile up at the longest wavelengths possible in the system (determined by the box size) unless there is an efficient damping of long wavelengths. This will be due to zonal flows. This damping of drift waves by zonal flows can be seen as a predator-prey system. In steady state, there could be a balance between zonal flows and drift waves such that drift waves with the longest wavelengths (box size) are completely damped out. In this case, absorbing (no reflections) boundary for long wavelengths can be used as is chosen here. However, it takes some time for the equilibrium state in the predator-prey system to develop. This may mean that the equilibrium of the predator-prey system may not have time to develop in transient transport and, thus, reflections of waves occur, i.e., the absorbing boundary for long wavelengths as chosen here may not be valid for transients. This could distort the relation between steady state and transient transport and, in some cases, make the method of perturbative transport analysis less useful. This case is not considered here.

An advantage of assuming a (6) is that $D_{1}(s)=0$. This is difficult to prove analytically, but in case $V=0$, for this boundary condition, $D_{1}(s)=0$ in (4). ${ }^{10}$ This indicates that also $D_{1}(s)=0$ in (2) when $V \neq 0$. This has been numerically verified in (2) by comparing it to finite difference simulations using the boundary condition given in (6). This shows that the error between the analytic and numerical simulations is small and that the error is decreasing with increasing density of the discretization grid. Hence, it is concluded that $D_{1}(s)=0$ for a Neumann boundary condition in (2), i.e.,

$$
\begin{equation*}
\Theta(\rho, s)=D_{2}(s) e^{\lambda_{1} \rho} \Phi\left(\frac{\lambda_{2}}{\lambda_{2}-\lambda_{1}}, 1,\left(\lambda_{2}-\lambda_{1}\right) \rho\right) \tag{7}
\end{equation*}
$$

Basically, there are two possibilities to handle the unknown $D_{2}(s)$ when $s=i \omega$. One possibility is to use the logarithmic temperature derivative $(\partial \Theta / \partial \rho) / \Theta$ to eliminate $D_{2}(s)$, where $\Theta=A \exp (i \phi)$ resulting in (see Ref. 7 for the definition of the derivative of $\Phi$ )

$$
\begin{equation*}
\frac{\Theta^{\prime}}{\Theta}=\lambda_{1}+\lambda_{2} \frac{\Phi\left(a+1,2,\left(\lambda_{2}-\lambda_{1}\right) \rho\right)}{\Phi\left(a, 1,\left(\lambda_{2}-\lambda_{1}\right) \rho\right)} \tag{8}
\end{equation*}
$$

with the left hand side

$$
\begin{equation*}
\frac{\Theta^{\prime}}{\Theta}=\frac{A^{\prime}}{A}+i \phi^{\prime} \tag{9}
\end{equation*}
$$

Note that the derivatives are defined in terms of $\rho$ and not in terms of distance to the source. Hence, the derivatives are defined positively for heat waves traveling towards the center. If $V=0$, this simplifies to

$$
\begin{equation*}
\frac{\Theta^{\prime}}{\Theta}=z \frac{I_{1}(z \rho)}{I_{0}(z \rho)} \tag{10}
\end{equation*}
$$

where $z$ is defined according to (5). This last relationship is well known in the literature. ${ }^{5,10}$ In the logarithmic temperature derivative representation, it is necessary to approximate spatial derivatives $A^{\prime} / A$ and $\phi^{\prime}$ from the measured $A$ and $\phi$. This can be avoided by using the transfer function representation, where $D_{2}(s)$ is fixed by assuming a second boundary condition.

The most logical choice for a second boundary condition is $\Theta(\rho, s)=\Theta\left(\rho_{1}, s\right)$, which is natural as $\Theta\left(\rho_{1}, s\right)$ is measured. The transfer function using (7), then, becomes

$$
\begin{equation*}
\frac{\Theta\left(\rho_{2}, s\right)}{\Theta\left(\rho_{1}, s\right)}=e^{\lambda_{1} \Delta \rho} \frac{\Phi\left(a, 1,\left(\lambda_{2}-\lambda_{1}\right) \rho_{2}\right)}{\Phi\left(a, 1,\left(\lambda_{2}-\lambda_{1}\right) \rho_{1}\right)} \tag{11}
\end{equation*}
$$

where the solution at a second measurement point $\rho_{1}>\rho_{2}$ is used as resulting temperature $\Theta\left(\rho_{2}\right)$. This description is expressed directly in terms of the measured Fourier coefficients $(\Theta=A \exp (i \phi))$. However, it is not straightforward to derive explicit relationships for $\chi, V$, and $\tau_{i n v}$ using this relationship.

## III. DERIVATION OF EXPLICIT APPROXIMATIONS

In this section, continued fractions are used to find approximations for the transport coefficients in cylindrical geometry using (8) and (10). In Sec. II, the logarithmic temperature derivative is introduced, which is described by the ratio of modified Bessel functions of the first kind or confluent hypergeometric functions of the first kind. From the literature, it is well known that these ratios of transcendental functions can be approximated by truncation of their continued fraction representation. ${ }^{2,11,12}$ Based on this concept, a number of new approximations are derived, which are summarized in three tables in Sec. IV, and their derivations can be found in Appendix. Here, only the three most important approximations are introduced.

## A. Diffusivity and damping only

The continued fraction for a ratio of Bessel functions of the first kind is used to find approximations for $\chi$ under influence of damping by assuming that $V=0$. Therefore, the logarithmic temperature derivative introduced in (10) is used. The following continued $S$-fraction of the ratio of Bessel functions can be found in Ref. 12

$$
\begin{equation*}
\frac{I_{1}(z \rho)}{I_{0}(z \rho)}=\frac{a_{1}}{1+\frac{a_{2}}{1+\frac{a_{3}}{1+\ldots}}} \tag{12}
\end{equation*}
$$

where $a_{k+1}=(z \rho)^{2} /(4 k(k+1))$ and $a_{1}=z \rho / 2$. If this continued fraction is truncated taking only the first term $a_{1}$ into
account, then the logarithmic temperature derivative in (10) is approximated by

$$
\begin{equation*}
\frac{\Theta^{\prime}}{\Theta}=z \frac{z \rho}{2} \tag{13}
\end{equation*}
$$

This can be solved in terms of $\chi$ and $\tau_{i n v}$ using (5), resulting in

$$
\begin{equation*}
\chi_{I s \phi}=\frac{3}{4} \frac{\omega}{\phi^{\prime}} \rho \quad \text { and } \quad \tau_{I s \phi}=\frac{\omega}{\phi^{\prime}} \frac{A^{\prime}}{A} \tag{14}
\end{equation*}
$$

This relationship can also be found based on the asymptotic expansions given in Ref. 13.

Continued fractions can also be used to find more accurate approximations by using more terms in the continued fraction before truncation. In this case, the best approximation is found by truncating at $a_{4}$ in (12), which can be written in terms of a second order polynomial in $z^{2}$

$$
\begin{equation*}
0=c_{2} z^{4}+c_{1} z^{2}+c_{0} \tag{15}
\end{equation*}
$$

with coefficients

$$
\begin{align*}
& c_{2}=12 \rho^{3}-\frac{\Theta^{\prime}}{\Theta} \rho^{4}, c_{1}=192 \rho-72 \frac{\Theta^{\prime}}{\Theta} \rho^{2} \\
& \quad \text { and } c_{0}=-384 \frac{\Theta^{\prime}}{\Theta} \tag{16}
\end{align*}
$$

where $\Theta^{\prime} / \Theta$ is given by (9). The second order polynomial yields two solutions in terms of $z^{2}$. However, generally only one solution can be used to determine $\chi$, because $z^{2}$ lies in the first quadrant of the complex plane $\left(\chi>0, \tau_{i n v}>0\right.$, and $s=i \omega$ with $\omega>0$ ) and the other two are often outside this domain. However, using a truncation of (12) at location $a_{5}$ results in more solutions within this domain. Hence, it is no longer straightforward to select the correct solution. Therefore, truncations higher than $a_{4}$ will not be considered here. For the truncation using $a_{4}$ as the last term, it has been numerically determined that the useful zero is given by

$$
\begin{equation*}
z^{2}=\frac{-c_{1}+\sqrt{c_{1}^{2}-4 c_{0} c_{2}}}{c_{2}} \tag{17}
\end{equation*}
$$

which covers the largest region of interest. The solution for the diffusivity $\chi$ and the damping $\tau_{i n v}$ is found by substituting (17) into

$$
\begin{equation*}
\chi=\frac{3}{2} \frac{\omega}{\Im\left(z^{2}\right)} \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau_{i n v}=\omega \frac{\Re\left(z^{2}\right)}{\Im\left(z^{2}\right)} \tag{19}
\end{equation*}
$$

The continued fraction in (12) can also be used to find two other approximations belonging to $a_{2}$ and $a_{3}$, which are named as $\chi_{I s 2}$ and $\chi_{I s 3}$, respectively. These can found in Table II in Sec. IV. In the Appendix, another continued fraction for the ratio of Bessel functions of the first kind is presented, which is also used to find explicit approximations for $\chi$ and $\tau_{i n v}$.

In this subsection, the convectivity is assumed zero such that the continued fractions for Bessel functions can be considered. In Subsection III B, non-zero $V$ is considered. Therefore, the continued fraction for the ratio of confluent hypergeometric functions of the first kind is used to find approximations for $\chi$.

## B. Diffusivity, convectivity, and damping

The logarithmic amplitude derivative $A^{\prime} / A$ and phase derivative $\phi^{\prime}$ are given in (8) as a function of $\chi, V$, and $\tau_{i n v}$. However, only two quantities are known, i.e., $A^{\prime} / A$ and $\phi^{\prime}$, whereas on the right hand side, three unknowns are given. Therefore, a third quantity needs to be introduced to calculate the transport coefficients, which can be done by introducing a second harmonic, i.e., $A^{\prime}\left(\omega_{2}\right) / A\left(\omega_{2}\right)$ or $\phi^{\prime}\left(\omega_{2}\right)$. In addition, the expression in (8) needs to be approximated using a continued fraction. In this case, the continued $C$-fraction of the ratio of confluent hypergeometric functions of the first kind, given in Ref. 12, is used

$$
\begin{equation*}
\frac{\Phi(a+1, b+1, z)}{\Phi(a, b, z)}=\frac{1}{1-\frac{\frac{b-a}{(b+0)(b+1)} z}{1+\frac{\left.\frac{a+1}{(b+1)(b+2)} z\right|_{\mathrm{I}}}{\frac{b-a+1}{(b+2)(b+3)} z}}} \tag{20}
\end{equation*}
$$

This continued fraction needs to be truncated and substituted into (8) to find a proper approximation for $\chi$. Here, it is chosen to truncate (20) at locations I and II, because, in these special cases, there are no square roots in the resulting approximation of the logarithmic temperature derivative in (8). Hence, it is easier to derive explicit approximations for $\chi, V$, and $\tau_{i n v}$. In the Appendix, the truncations at locations I and II are derived for various combinations of amplitude and phase. In this section, only the truncation at location II is given using two amplitudes and one phase, i.e., $A^{\prime}\left(\omega_{1}\right) / A\left(\omega_{1}\right), \phi^{\prime}\left(\omega_{1}\right)$, and $A_{2}^{\prime}(\omega) / A\left(\omega_{2}\right)$, because, in a numerical comparison, this gave the best result. This does not necessarily mean that, in practice, it also gives the best result. For instance, calibration errors will influence this approximation more than the one based on two phases, because the sensitivity of the amplitude to calibration errors is larger.

Although it is now possible to calculate explicit solutions for $\chi, V$, and $\tau_{i n v}$, the calculations are too complicated to do by hand. Therefore, Mathematica ${ }^{\odot}$ was used to derive approximations for $\chi, V$, and $\tau_{i n v}$ based on the truncation in (20). Truncating at location II results in a third order polynomial such that there are three solutions. However, only one is different from $\chi=0$, which is given by

$$
\begin{equation*}
\chi_{\Phi 4 a}=\frac{3}{2} \frac{6859 \rho^{3} \omega_{1}^{2} \phi_{1}^{\prime} d A}{8\left(o_{3}+27436 \omega_{1}^{3} d A^{3}+45 o_{1}^{3}\right)} \tag{21}
\end{equation*}
$$

where

$$
\begin{aligned}
& d A=\frac{A_{1}^{\prime}}{A_{1}}-\frac{A_{2}^{\prime}}{A_{2}}, \\
& o_{1}=\omega_{1}\left(d A\left(\frac{A_{1}^{\prime}}{A_{1}} \rho-4\right)+\rho\left(\phi_{1}^{\prime}\right)^{2}\right)-\rho \omega_{2} \phi_{1}^{\prime} \phi_{2}^{\prime}, \\
& o_{2}=\omega_{1}^{2}\left(d A^{2}+\left(\phi_{1}^{\prime}\right)^{2}\right)-2 \omega_{1} \omega_{2} \phi_{1}^{\prime} \phi_{2}^{\prime}+\omega_{2}^{2}\left(\phi_{2}^{\prime}\right)^{2},
\end{aligned}
$$

and

$$
o_{3}=1311 o_{1}^{2} \omega_{1} d A+10108 o_{1} \omega_{1}^{2} d A^{2}
$$

The corresponding $V$ and $\tau_{i n v}$ are given by

$$
\begin{equation*}
V_{\Phi 4 a}=-\chi_{\Phi 4 a} \frac{30 o_{1}}{38 \rho \omega_{1} d A} \tag{22}
\end{equation*}
$$

and (subscripts $\Phi 4 a$ have been omitted on the right hand side)

$$
\begin{align*}
\tau_{\Phi 4 a}= & \frac{3}{2} \chi \frac{-15 \rho^{2} \frac{\omega}{\chi} \frac{A_{1}^{\prime}}{A_{1}}-6\left(\frac{V}{\chi}\right)^{2} \rho^{2} \phi^{\prime}-19 \rho^{2} \omega \frac{V}{\chi^{2}}}{15 \rho^{2} \phi^{\prime}} \\
& +\frac{3}{2} \chi \frac{48\left(\frac{V}{\chi}\right) \rho \phi^{\prime}+60 \rho \frac{\omega}{\chi}-120 \phi^{\prime}}{15 \rho^{2} \phi^{\prime}} \tag{23}
\end{align*}
$$

These solutions are complicated, but are the only explicit approximations found for the combined problem of estimating $\chi$ under convectivity and damping. The other approximations are given in the Appendix. All the approximations are summarized and compared in Sec. IV.

## IV. INWARD SOLUTIONS

In this section, the different approximations to determine $\chi, V$, and, $\tau_{i n v}$ are summarized and compared for heat waves traveling towards the center. The approximations are based on the underlying models (8) and (10), which are used to calculate $A^{\prime} / A$ and $\phi^{\prime}$ for a large number of combinations of the transport coefficients. In addition, the semi-infinite approximations derived in Refs. 1 and 2 can also be used. The comparison is based on five parameters ( $\rho, \omega, \chi, V$, and $\tau_{\text {inv }}$ ) and is presented in terms of normalized transport coefficients, i.e., $\bar{\chi}=\chi / \omega, \bar{V}=V / \omega$, and $\bar{\tau}_{i n v}=\tau_{i n v} / \omega$. In case, two harmonics are necessary, $A^{\prime} / A\left(\omega_{1}\right), A^{\prime} / A\left(\omega_{2}\right), \phi^{\prime}\left(\omega_{1}\right)$, and $\phi^{\prime}\left(\omega_{2}\right)$ are calculated using $\omega_{1}=\omega$ and $\omega_{2}=2 \omega$. This corresponds to the first and second harmonics.

## A. Overview of possible explicit approximations

In Table I, all derived approximations from Sec. III and the Appendix to estimate $\chi$ are summarized. Some infinite domain solutions from Refs. 1 and 2 are also included in Table I. The reason is that if cylindrical effects are small, i.e., the ratio of $\omega \rho / \chi$ is large; the infinite domain approximations

TABLE I. Overview of the approximations for $\chi$ for heat waves traveling towards the center in a cylindrical geometry, where a symmetry boundary condition is assumed. From left to right, the columns denote: the approximation of $\chi$ either explicit or in terms of $z$ in which case Table II gives the relationship for $z$; the equation numbers for $\chi, V$, and $\tau_{i n v}$ refer either to Sec. III or to the Appendix or the reference in which they are derived. The short-hand notations $\phi^{\prime}\left(\omega_{1}\right)=\phi_{1}^{\prime}$ and $\frac{A^{\prime}\left(\omega_{1}\right)}{A\left(\omega_{1}\right)}=\frac{A_{1}^{\prime}}{A_{1}}$ are used, which also means two harmonics are necessary.

| $\chi$ | Equation for $\chi$ | $V$ |  |
| :--- | :--- | :--- | :--- |
| $\chi_{z}$ | $\frac{3}{2} \frac{\omega}{\Im\left(z^{2}\right)}$ | $($ see Table II for $z)$ | 0 |
| inv | $(19)$ |  |  |

Approximations based on symmetry boundary condition

| $\chi_{\text {Is }}$ | $\frac{3}{4} \frac{\omega}{\phi^{\prime}} \rho$ | 0 | (14) |
| :---: | :---: | :---: | :---: |
| $\chi_{l s 2 \phi}$ | $\frac{3}{2} \omega \rho \frac{2+\sqrt{4-\rho^{2}\left(\phi^{\prime}\right)^{2}}}{8 \phi^{\prime}}$ | 0 | 0 |
| $\chi_{I S 2 A}$ | $\omega \frac{3 \rho^{3} \sqrt{4-A^{\prime} / A \rho}}{16 \rho^{2} \sqrt{A^{\prime} / A}}$ | 0 | 0 |
| $\chi_{\text {Ф2V }}$ | $\frac{9 \rho \omega}{4 \phi^{\prime}\left(\frac{A^{\prime}}{A} \rho+3\right)}$ | (B6) | 0 |
| $\chi_{\text {Ф4V }}$ | $\frac{3}{2} \frac{6859 \rho^{3} \omega \phi^{\prime}}{l_{2}+\left(6 \frac{A^{\prime}}{A} \rho\left(15 \frac{A^{\prime}}{A} \rho+32\right)+680\right) \cdot\left(2 \frac{A^{\prime}}{A} \rho-l_{1}+30\right)},$ | (B14) | 0 |
| with $l_{1}=\sqrt{4\left(\frac{A^{\prime}}{A} \rho+15\right)^{2}-285 \rho^{2}\left(\phi^{\prime}\right)^{2}} \quad$ and $\quad l_{2}=114 \rho^{2}\left(\phi^{\prime}\right)^{2}\left(32 \frac{A^{\prime}}{A} \rho+l_{1}+62\right)$ |  |  |  |
| $\chi_{\Phi 4 a}$ | $\frac{3}{2} \frac{6859 \rho^{3} \omega_{1}^{2} \phi_{1}^{\prime} d A o_{2}}{8\left(o_{3}+27436 \omega_{1}^{3} d A^{3}+45 o_{1}^{3}\right)},$ | (22) | (23) |
| with $o_{1}=\omega_{1}\left(d A\left(\frac{A_{1}^{\prime}}{A_{1}} \rho-4\right)+\rho\left(\phi_{1}^{\prime}\right)^{2}\right)-\rho \omega_{2} \phi_{1}^{\prime} \phi_{2}^{\prime}, \quad o_{2}=\left(\omega_{1}^{2}\left(d A^{2}+\left(\phi_{1}^{\prime}\right)^{2}\right)-2 \omega_{1} \omega_{2} \phi_{1}^{\prime} \phi_{2}^{\prime}+\omega_{2}^{2}\left(\phi_{2}^{\prime}\right)^{2}\right), \quad o_{3}=1311 o_{1}^{2} \omega_{1} d A+10108 o_{1} \omega_{1}^{2} d A^{2}$ |  |  |  |
| $\chi_{\Phi 4 b}$ | $\frac{3}{2} \frac{6859 d \omega \rho^{3} \omega_{1} \phi_{1}^{\prime}\left(\omega_{2}^{2}\left((d A)^{2}+\left(\phi_{1}^{\prime}\right)^{2}\right)+\omega_{1} \phi_{2}^{\prime}\left(d \omega-\omega_{2} \phi_{1}^{\prime}\right)\right)}{8(19 d \omega+o)\left(1444 d \omega^{2}+456 d \omega o+45 o^{2}\right)}$ | (B18) | (B19) |
|  | with $d \omega=\omega_{1} \phi_{2}^{\prime}-\omega_{2} \phi_{1}^{\prime}, d A=\frac{A_{1}^{\prime}}{A_{1}}-\frac{A_{2}^{\prime}}{A_{2}}, \quad$ and $\quad o=\frac{A_{1}^{\prime}}{A_{1}} \rho \omega_{1} \phi_{2}^{\prime}-\frac{A_{2}^{\prime}}{A_{2}} \rho \omega_{2} \phi_{1}^{\prime}-4 d \omega$ |  |  |

Approximations based on semi-infinite domain
$\chi_{c}$

$$
\frac{3}{4} \frac{\omega}{\phi^{\prime}\left(\frac{A^{\prime}}{A}+\frac{1}{2 \rho}\right)} \text { from Ref. } 5
$$

see Ref. 2

$$
\frac{3}{4} \sqrt{\frac{\left(\omega_{1} \phi_{\omega_{2}}^{\prime}\right)^{2}-\left(\omega_{2} \phi_{\omega_{1}}^{\prime}\right)^{2}}{\phi_{\omega_{1}}^{\prime 2} \phi_{\omega_{2}}^{\prime 2}\left(\phi_{\omega_{1}}^{\prime 2}-\phi_{\omega_{2}}^{\prime 2}\right)}} \text { from Ref. } 1
$$

give a good approximation again. However, in case, polynomials in $z$ are used based on infinite domains, different solutions need to be selected. The reason is that $A^{\prime} / A$ and $\phi^{\prime}$ are negative for heat waves traveling towards the wall and are positive for heat waves traveling towards the center.

To keep the table compact, Table I only states $\chi$ explicitly. The corresponding equation numbers of $V$ and $\tau_{i n v}$ are given instead.

In Table II, the polynomials expressed in terms of $z$ using $\Theta^{\prime} / \Theta=A^{\prime} / A+i \phi^{\prime}$ and $\rho$ to directly calculate $\chi$ and $\tau_{i n v}$ are given. In Table III, the approximations in terms of polynomials in $z$ are given for approximations based on infinite domains.

The useful solutions of the polynomials in $z$ are different from those used for the analysis of heat waves traveling towards the wall, because the sign of $A^{\prime} / A$ and $\phi^{\prime}$ is opposite. Therefore, to distinguish between the solutions for outward
heat waves and inward heat waves, the superscript inw (inward) is added. The useful solutions have been selected by comparing the three possibilities numerically. The other outward solutions given in Ref. 2 can also still be used with the only exceptions of $\chi_{A E K}$ and $\chi_{A E \Phi}$ as they approximate $\chi$ in a strong cylindrical geometry for the outward case, which is very different from the inward case.

The approximations in Tables I-III are compared in the rest of this section.

## B. Selection of interesting approximations

The comparison of the approximations when only the diffusivity $\chi$ is present, i.e., $V=0$ and $\tau_{i n v}=0$, is made based on a large number of possibilities of $\rho$ and the combined parameter $\bar{\chi}=\chi / \omega$. Therefore, (8) and (10) are used to

TABLE II. Overview of approximations for $\chi$ in terms of $z$ for heat waves traveling towards the center in a cylindrical geometry, where a symmetry boundary condition is assumed. This table denotes the coefficients to calculate $z$ using $\Theta^{\prime} / \Theta=A^{\prime} / A+i \phi^{\prime}$ and $\rho$, which is used to calculate $\chi=\frac{3}{2} \omega / \Im\left(z^{2}\right)$ and $\tau_{\text {inv }}=\omega \Re\left(z^{2}\right) / \Im\left(z^{2}\right)$; the equation numbers refer either to Sec. III or to the Appendix.

| $\chi$ | Equation for $z$ | Equation |
| :---: | :---: | :---: |
| Quadratic polynominal in $z$ | $z=\left(-b_{1}+\sqrt{b_{1}^{2}-4 b_{0} b_{2}}\right) / b_{2}$ |  |
| $\chi_{I t 1}$ | $b_{2}=\rho, \quad b_{1}=-\frac{\Theta^{\prime}}{\Theta} \rho, \quad b_{0}=-2 \frac{\Theta^{\prime}}{\Theta}$ | (A2) |
| Cubic polynominal in $z$ | $z=\frac{1}{a_{3}}\left(-\frac{a_{2}}{3}-\frac{\sqrt[3]{2} p_{0}}{3 p_{1}}+\frac{p_{1}}{3 \sqrt[3]{2}}\right), \quad p_{2}=-27 a_{0} a_{3}^{2}+9 a_{1} a_{2} a_{3}-2 a_{2}^{3}, \quad p_{1}=\sqrt[3]{p_{2}+\sqrt{4 p_{0}^{3}+p_{2}^{2}}}, \quad \mathrm{p}_{0}=3 \mathrm{a}_{1} \mathrm{a}_{3}-\mathrm{a}_{2}^{2}$ |  |
| $\chi_{\text {It } 3}$ | $a_{3}=2 \rho^{2}, \quad a_{2}=\left(3 \rho-2 \rho^{2} \frac{\Theta^{\prime}}{\Theta}\right), \quad a_{1}=-4 \rho \frac{\Theta^{\prime}}{\Theta}, \quad a_{0}=-6 \frac{\Theta^{\prime}}{\Theta}$ | (A3) |
| Quadratic polynominals in $z^{2}$ | $z^{2}=\left(-c_{1}+\sqrt{c_{1}^{2}-4 c_{0} c_{2}}\right) / c_{2}$ |  |
| $\chi_{I s 2}$ $\chi_{I s 3}$ $\chi_{I s 4}$ | $\begin{gathered} c_{2}=\frac{\Theta^{\prime}}{\Theta} \rho^{2}-4 \rho, \quad c_{1}=0, \quad c_{0}=8 \frac{\Theta^{\prime}}{\Theta} \\ c_{2}=\rho^{3}, \quad c_{1}=24 \rho-8 \rho^{2} \frac{\Theta^{\prime}}{\Theta}, \quad c_{0}=-48 \frac{\Theta^{\prime}}{\Theta} \\ c_{2}=12 \rho^{3}-\frac{\Theta^{\prime}}{\Theta} \rho^{4}, \quad c_{1}=192 \rho-72 \frac{\Theta^{\prime}}{\Theta} \rho^{2}, \quad c_{0}=-384 \frac{\Theta^{\prime}}{\Theta} \end{gathered}$ | (12) at $a_{2}$ <br> (12) at $a_{3}$ <br> (16) |

generate $A^{\prime} / A$ and $\phi^{\prime}$. The most interesting and best approximations are shown in Fig. 1 in terms of the relative error with respect to the true diffusivity $\chi$.

The use of infinite domain approximations for heat waves traveling towards the center gives a good approximation, if the ratio $\rho \omega / \chi$ is large. In that case, $\chi_{c}$ has the largest region with a good accuracy, but the highest accuracy is generally given by $\chi_{K j 3}$. In that case, the approximations based on cylindrical geometry for heat waves traveling towards the center give good approximations for $\chi$. Hence, $\chi_{I t 3}$ almost approximates the entire presented region well, albeit with a slightly less accuracy than $\chi_{I s 4}$. Also, $\chi_{\Phi 4 V}$ performs well, although it was mainly derived to perform well under convectivity. $\chi_{I s \phi}$ is also shown as it is the most simple cylindrical approximation found. Unfortunately, its region of applicability is much smaller than the other approximations.

In summary, $\chi_{I t 3}$ has the largest region of applicability. Its relative error is only in a small region larger (maximally $\varepsilon_{\text {rel }} \approx 30 \%$ ). In this region, different approximations are necessary, for instance $\chi_{K j 3}$ or $\chi_{c}$ and $\chi_{I s 4}$.

## C. Diffusivity and damping only

It is not possible to use one approximation to approximate $\chi$ well for all combinations of $\chi, \omega, \rho$, and $\tau_{i n v}$. However, it turns out that by combining two approximations to estimate $\chi$ almost the entire presented region of interest for heat waves traveling inwards can be covered. This is shown in Fig. 2, where the maximum relative error over the entire presented region is below $2 \%$. Hence, it is always possible to get an accurate result for the presented combination of $\chi, \omega, \tau_{i n v}$, and $\rho$.

TABLE III. Overview of approximations for $\chi$ in terms of $z$ for heat waves traveling towards the edge in a cylindrical geometry, where an infinite domain is assumed. This table denotes the coefficients to calculate $z$ using $\Theta^{\prime} / \Theta=A^{\prime} / A+i \phi^{\prime}$ and $\rho$, which is used to calculate $\chi=\frac{3}{2} \omega / \Im\left(z^{2}\right)$ and $\tau_{\text {inv }}=\omega \Re\left(z^{2}\right) / \Im\left(z^{2}\right)$; The coefficients are taken from Ref. 2.

| $\chi_{z}^{\text {inw }}$ | Equation for z |
| :---: | :---: |
| Quadratic polynominal in $z$ | $z=\left(-b_{1}-\sqrt{b_{1}^{2}-4 b_{0} b_{2}}\right) / b_{2}$ |
| $\chi_{K c 2}^{i n w}$ $\chi_{K j 2}^{i n v}$ | $\begin{gathered} b_{2}=4 \rho, \quad b_{1}=3+4 \frac{\Theta^{\prime}}{\Theta} \rho, \quad b_{0}=\frac{\Theta^{\prime}}{\Theta} \\ b_{2}=8 \rho^{2}, \quad b_{1}=8 \rho^{2} \frac{\Theta^{\prime}}{\Theta}+4 \rho, \quad b_{0}=8 \frac{\Theta^{\prime}}{\Theta} \rho-3 \end{gathered}$ |
| Cubic polynominal in $z$ | $z=\frac{1}{a_{3}}\left(-\frac{a_{2}}{3}+\frac{1-i \sqrt{3}}{3 \cdot \sqrt[3]{4}} \frac{p_{0}}{p_{1}}-\frac{1+i \sqrt{3}}{6 \cdot \sqrt[3]{2}} p_{1}\right), \quad p_{2}=-27 a_{0} a_{3}^{2}+9 a_{1} a_{2} a_{3}-2 a_{2}^{3}, \quad p_{1}=\sqrt[3]{p_{2}+\sqrt{4 p_{0}^{3}+p_{2}^{2}}}, \quad \mathrm{p}_{0}=3 \mathrm{a}_{1} \mathrm{a}_{3}-\mathrm{a}_{2}^{2}$ |
| $\chi_{K j 3}^{i n \omega}$ | $a_{3}=16 \rho^{3}, \quad a_{2}=16 \frac{\Theta^{\prime}}{\Theta} \rho^{3}+56 \rho^{2}, \quad a_{1}=48 \frac{\Theta^{\prime}}{\Theta} \rho^{2}+45 \rho, \quad a_{0}=23 \frac{\Theta^{\prime}}{\Theta} \rho+7.5$ |
| $\chi_{K c 5}^{i n w}$ | $a_{3}=16 \rho^{2}, \quad a_{2}=\left(36 \rho+16 \rho^{2} \frac{\Theta^{\prime}}{\Theta}\right), \quad a_{1}=\left(15+28 \rho \frac{\Theta^{\prime}}{\Theta}\right), \quad a_{0}=3 \frac{\Theta^{\prime}}{\Theta}$ |



FIG. 1. Comparison between the different relative errors of the $\chi$ estimates for a large range of $\bar{\chi}=\chi / \omega$ and $\rho$. The relative error is defined as $\varepsilon_{r e l}=100 \times \frac{\left|\chi-\chi_{\text {est }}\right|}{\chi}[\%]$, where $\chi_{\text {est }}$ is $\chi_{\phi}$ from Table I, $\chi_{c}$ from Table I, $\chi_{K j 3}^{i n w}$ from Table III using (18), and $\chi_{K c 5}^{i n w}$ from Table III using (18). These approximations are based on infinite domains (first row). The true cylindrical models (second row) are estimated by $\chi_{I s \phi}$ from (14), $\chi_{I s 4}$ from (16), $\chi_{I t 3}$ from (A3), and $\chi_{\Phi 4 V}$ from (B10). This comparison is based on a cylindrical geometry using a symmetry boundary condition with $\chi$ and $V=\tau_{i n v}=0$, where the heat waves travel inwards. The darkest blue represents $\varepsilon_{\text {rel }}<1 \%$ and the darkest red represents all $\varepsilon_{\text {rel }}>150 \%$.

Both $\chi_{K j 3}^{i n w}$ and $\chi_{I s 4}$ have been chosen because they give the most accurate approximations in their regions of applicability and they are complementary. The white line shows the approximate boundary of the regions of applicability of $\chi_{K j 3}^{i n w}$ and $\chi_{I s 4}$. At this boundary, the error is largest.

## D. Diffusivity and convectivity with $\tau_{i n v}=0$ and $\tau_{i n v}=2$

For the inward case, multiple approximations are available to estimate $V$. It is not easy to choose a suitable approximation before the measurements have been analyzed, because the approximations all depend on different harmonic informations. For instance, $\chi_{\phi}$ uses only the phases of two harmonics, but $\chi_{\Phi 4 a}$ uses two phases and one amplitude. On the other hand, when $\tau_{i n v}=0$, then, $\chi_{\Phi 4 V}$ can be used, which
uses only one harmonic. Therefore, it is not possible to point out the best approximation. However, the regions of applicability of the approximation are again clearly defined. $\chi_{\phi}$, which originates from slab-geometry, is best at approximating $\chi$ for large $\omega \rho / \chi$. On the other hand, the approximations based on the symmetry boundary conditions estimate $\chi$ as well for small $\omega \rho / \chi$.

From a numerical point of view, $\chi_{\Phi 4 V a}$ performed best, but it is comparable to the other cylindrical approximations. Therefore, it is chosen to combine $\chi_{\Phi 4 V a}$ and $\chi_{\phi}$ separated by the white line, which is shown in Figs. 3 and 4 for $\tau_{i n v}=0$ and $\tau_{i n v}=2$.

Both figures show similar regions, where $\chi$ can be estimated well and where not. The large error close to the boundary is caused by the limited region of approximation,


FIG. 2. The relative error of the $\chi$ estimates for the combination of $\chi_{I s 4}$ and $\chi_{K i 3}^{i n w}$ is presented for different $\bar{\chi}=\chi / \omega$ and $\bar{\tau}_{\text {inv }}=\tau_{\text {inv }} / \omega$ at a number of spatial locations $\rho$. The relative error is defined as $\quad \varepsilon_{\text {rel }}=100 \times \frac{\left|\chi-\chi_{\text {est }}\right|}{\chi}[\%]$. This figure combines the approximations $\chi_{I s 4}$ and $\chi_{K j 3}^{i n w}$, which are separated by the boundary represented by the white line. This figure is based on a cylindrical geometry using a symmetry boundary condition with $V=0$, where the heat waves travel inwards.


FIG. 3. The relative error of the $\chi$ estimates for the combination of $\chi_{\Phi 4 V a}$ and $\chi_{\phi}$ is presented for different $\bar{\chi}=\chi / \omega$ and $\bar{V}=V / \omega$ at a number of spatial locations $\rho$. The relative error is defined as $\quad \varepsilon_{\text {rel }}=100 \times \frac{\left|\chi-\gamma_{\text {est }}\right|}{\chi} \quad[\%]$. This figure combines the approximations $\chi_{\Phi 4 V a}$ and $\chi_{\phi}$, which are separated by the boundary represented by the white line. This figure is based on a cylindrical geometry using a symmetry boundary condition with $\tau_{i n v}=0$, where the heat waves travel inwards. The darkest blue represents $\varepsilon_{\text {rel }}<1 \%$ and the darkest red represents all $\varepsilon_{\text {rel }}>150 \%$.
which is similar to the previous figures. There is no suitable approximation, which handles the regions with large errors.

## V. SUMMARY AND DISCUSSION

In this paper, the problem of determining the thermal diffusion coefficient from electron temperature measurements during power modulation experiments has been revisited. A large number of new approximations have been introduced to estimate $\chi$ directly from $A^{\prime} / A$ and $\phi^{\prime}$ for different combinations of $\chi, V$, and $\tau_{i n v}$ for heat waves traveling towards the center. This corresponds to the case of off-axis heating. The approximations are based on a symmetry boundary condition and are derived on the basis of cylindrical geometry using standard assumptions.

The quality of the approximations is presented in several figures. In case, only $\chi$ and $\tau_{i n v}$ are considered $(V=0)$, the relative error of the $\chi$ estimate for the region of interest is smaller than $2 \%$. These errors are achievable by combining $\chi_{K j 3}^{i n w}$ and $\chi_{I s 4}$. In case also $V$ is considered, the new
approximations show a significant region in which $\chi$ can be estimated well, but also regions in which no suitable approximation exists. Combining $\chi_{\Phi 4 V a}$ and $\chi_{\phi}$ cover a large region, where $\chi$ can be well estimated.

## VI. GENERAL CONCLUSION

In this set of papers (Parts I, II, and III), the problem of determining the thermal diffusion coefficient from electron temperature measurements during power modulation experiments has been revisited. A large number of new approximations have been introduced to estimate $\chi$ directly from $A^{\prime} / A$ to $\phi^{\prime}$ for different combinations of $\chi, V$, and $\tau_{i n v}$. The approximations are based on infinite domains and Neumann boundary conditions and are derived on the basis of slab or cylindrical geometry using common assumptions. These approximations, including the well known approximations from the literature, have been compared in Part III for heat waves traveling towards the center (inward) and in Part I and


FIG. 4. The relative error of the $\chi$ estimates for the combination of $\chi_{\Phi 4 V a}$ and $\chi_{\phi}$ is presented for different $\bar{\chi}=\chi / \omega$ and $\bar{V}=V / \omega$ at a number of spatial locations $\rho$. The relative error is defined as $\quad \varepsilon_{\text {rel }}=100 \times \frac{\left|\chi-\chi_{\text {est }}\right|}{\chi} \quad[\%]$. This figure combines the approximations $\chi_{\Phi 4 V a}$ and $\chi_{\phi}$, which are separated by the boundary represented by the white line. This figure is based on a cylindrical geometry using a symmetry boundary condition with $\bar{\tau}_{i n v}=2$, where the heat waves travel inwards. The darkest blue represents $\varepsilon_{r e l}<1 \%$ and the darkest red represents all $\varepsilon_{\text {rel }}>150 \%$.

Part II towards the edge (outwards). The approximations are derived based on $A^{\prime} / A$ and $\phi^{\prime}$. The quality of the approximations is presented in several figures. In case, only $\chi$ and $\tau_{i n v}$ are considered $(V=0)$, the relative error of the $\chi$ estimate for the region of interest is, in general, $\varepsilon_{\text {rel }}<1 \%$. However, in a small region, the errors are larger with a maximum relative error for heat waves traveling towards the edge $\varepsilon_{r e l}<20 \%$ and for heat waves traveling towards the center $\varepsilon_{r e l}<2 \%$. These errors are achievable by combining $\chi_{K j 3}$ and $\chi_{A E \Psi}$ in the outward case $\chi_{K j 3}^{i n w}$ and $\chi_{I s 4}$ in the inward case, respectively. In case also $V$ is considered, the new approximations show a significant region in which $\chi$ can be estimated well, but also regions in which no suitable approximation exists. Combining $\chi_{\Phi 4 V a}$ and $\chi_{\phi}$ for the inward case and $\chi_{A E \Psi}$ and $\chi_{\phi}$ for the outward case cover a large region, where $\chi$ can be well estimated.

However, there are also a number of important issues when using the approximations presented in these parts and the literature. The combination of assuming the transport coefficients independent of $\rho$ and the use of infinite domains or symmetry boundary conditions necessary to arrive at explicit approximations will result in errors, if the profile is not constant, which is shown in Part I Sec. VII B. However, as is discussed in the same section, the variation of the profiles in space is small; it is still possible to determine the spatial varying transport coefficients. However, with increasing, variation of the transport coefficients, different boundary conditions, and decreasing modulation frequency, the estimate will become more erroneous. In Part I, it has been shown that these errors influence the estimation of the convectivity and the damping more significantly, making the estimated $V$ and $\tau_{i n v}$ often erroneous. On the other hand, as has been shown in Part II, it is important to still estimate $V$ and $\tau_{i n v}$ as they are necessary to select the proper approximation and to select the correct estimates of $\chi$ in the presence of $V$ and $\tau_{i n v}$. A second important issue is the determination of $A^{\prime} / A$ and $\phi^{\prime}$ from $\phi$ to $A$. To investigate this relationship, the notion of transfer functions has been introduced, which makes the relationship between $A^{\prime} / A$ and $A$, and, $\phi^{\prime}$ and $\phi$ explicit. It showed that the relationship in slab-geometry is straightforward (Part I), but in cylindrical geometry is more complicated (Part II). It also shows that the dependency of the transport coefficients on $\rho$ is contained in $A^{\prime}$ and $\phi^{\prime}$ and as such the transport coefficients differ depending on how $A^{\prime}$ and $\phi^{\prime}$ are calculated. Therefore, it is always important to clearly state how $A^{\prime}$ and $\phi^{\prime}$ are calculated from $\phi$ to $A$ to arrive at comparable results. However, for cylindrical domains, there is not a clear recipe to calculate $A^{\prime}$ and $\phi^{\prime}$ from $\phi$ to $A$. Hence, in this paper, we do not wish to comment on this issue, and instead for the analysis, the true $A^{\prime} / A$ and $\phi^{\prime}$ are used. A third problem, only touched upon briefly, is the effect of noise, which is not taken into account by the methods proposed in this set of papers. However, if two harmonics are used (in the presence of $V$ ), it is important that both harmonics do not contain too much noise, which can be achieved by using a non-symmetric duty cycle.

The above discussed issues are partly related to the use of explicit approximations. These problems can, in principle, be avoided by using implicit methods, as these implicit
methods allow the use of more realistic boundary conditions, the direct estimation of $\chi$ from $A$ and $\phi$ using the transfer function representation, and the inclusion of noise in the estimation process. However, such implicit methods will come at the price of more complex optimization problems and require a number of different concepts as presented in this set of papers. Nevertheless, in the case implicit methods are used finding good starting values are also important, for which the approximations presented in this set of papers can be used.

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## APPENDIX

The Appendix consists of three sections in which approximations are derived based on the continued $T$-fraction of modified Bessel functions of the first kind. They lead to approximations for $\chi$ and $\tau_{i n v}$ in terms of $z$. In addition, the other approximations for $\chi, V$, and $\tau_{i n v}$ based on the continued fraction given in (20) are presented. Finally, also an approximation is given for a case, where only $\chi$ is present.

## APPENDIX A: CONTINUED T-FRACTION OF THE RATIO OF BESSEL FUNCTIONS OF THE FIRST KIND

The following continued $T$-fraction of $I_{1}(z \rho) / I_{0}(z \rho)$ is based on Ref. 12 and is useful for approximating (10). The continued $T$-fraction is given by

$$
\begin{equation*}
\frac{I_{1}(z \rho)}{I_{0}(z \rho)}=\frac{z \rho}{2+\left.z \rho\right|_{\text {III }}+\frac{-3 z \rho}{3+\left.2 z \rho\right|_{\text {IV }}+\frac{-5 z \rho}{4+2 z \rho+\cdots}}} \tag{A1}
\end{equation*}
$$

which needs to be substituted into (10) to find explicit solutions. This continued fraction is truncated at locations III and IV.
(a) Truncating (A1) at location III results in the polynomial

$$
\begin{equation*}
0=\rho z^{2}-\frac{\Theta^{\prime}}{\Theta} \rho z-2 \frac{\Theta^{\prime}}{\Theta} \tag{A2}
\end{equation*}
$$

(b) Truncating (A1) at location IV results in the following third order polynomial

$$
\begin{equation*}
0=2 \rho^{2} z^{3}+\left(3 \rho-2 \rho^{2} \frac{\Theta^{\prime}}{\Theta}\right) z^{2}-4 \rho \frac{\Theta^{\prime}}{\Theta} z-6 \frac{\Theta^{\prime}}{\Theta} \tag{A3}
\end{equation*}
$$

Again, continued fractions with more terms result in fourth order or higher order polynomials.

## APPENDIX B: CONTINUED C-FRACTION OF CONFLUENT HYPERGEOMETRIC FUNCTION OF THE FIRST KIND

The continued $C$-fraction for $\Phi(a+1, b+1, z) /$ $\Phi(a, b, z)$ given in (20) is used to derive several approximations.
(a) Truncating (20) at location I and substituting it into (8) results in the following logarithmic temperature derivative

$$
\begin{equation*}
\frac{\Theta^{\prime}}{\Theta}=\lambda_{1}+\lambda_{2} \frac{1}{1-\frac{\frac{\lambda_{2}-\lambda_{1}-\lambda_{2}}{2} \rho}{1+\frac{\lambda_{2}+\left(\lambda_{2}-\lambda_{1}\right)}{6} \rho}} . \tag{B1}
\end{equation*}
$$

This can be further simplified by partly substituting $\lambda_{1}$ and $\lambda_{2}$

$$
\begin{equation*}
\frac{\Theta^{\prime}}{\Theta}=\frac{2 \lambda_{1}^{2} \rho+\lambda_{1} \lambda_{2} \rho+2 \lambda_{2}^{2} \rho-6 \frac{V}{\chi}}{6-2 \frac{V}{\chi} \rho} \tag{B2}
\end{equation*}
$$

where $2 \lambda_{1}^{2}+\lambda_{1} \lambda_{2}+2 \lambda_{2}^{2}=2\left(\frac{V}{\chi}\right)^{2}+\frac{9}{2} \frac{\tau_{i v v}+\omega i}{\chi}$ such that

$$
\begin{align*}
& \left(6-2 \frac{V}{\chi} \rho\right) \frac{A^{\prime}}{A}+\left(6-2 \frac{V}{\chi} \rho\right) i \phi^{\prime} \\
& \quad=\left(2\left(\frac{V}{\chi}\right)^{2}+\frac{9}{2} \frac{\tau_{i n v}+\omega i}{\chi}\right) \rho-6 \frac{V}{\chi} \tag{B3}
\end{align*}
$$

By splitting (B3) in its real and imaginary parts, i.e.,

$$
\begin{equation*}
\left(6-2 \frac{V}{\chi} \rho\right) \phi^{\prime}=\frac{9}{2} \frac{\omega}{\chi} \rho \tag{B4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(6-2 \frac{V}{\chi} \rho\right) \frac{A^{\prime}}{A}=2\left(\frac{V}{\chi}\right)^{2} \rho+\frac{9}{2} \frac{\tau_{\text {inv }}}{\chi} \rho-6 \frac{V}{\chi} \tag{B5}
\end{equation*}
$$

$\chi$ can be calculated. The imaginary part for $V=0$ yields an approximation for $\chi$, i.e.,

$$
\begin{equation*}
\chi_{I s \phi}=\frac{3}{4} \frac{\omega}{\phi^{\prime}} \rho \quad \text { and } \quad \tau_{I s \phi}=\frac{\omega}{\phi^{\prime}} \frac{A^{\prime}}{A} \tag{B6}
\end{equation*}
$$

which is also found using asymptotic expansions and in (14) using a continued fraction based on Bessel functions. If $\tau_{i n v}$ is assumed to be zero, then solving (B4) and (B5) together gives

$$
\begin{equation*}
\chi_{\Phi 2 V}=\frac{9 \rho \omega}{4 \phi^{\prime}\left(\frac{A^{\prime}}{A} \rho+3\right)} \tag{B7}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{\Phi 2 V}=-\frac{9 \frac{A^{\prime}}{A} \rho \omega}{4 \phi^{\prime}\left(\frac{A^{\prime}}{A} \rho+3\right)} \tag{B8}
\end{equation*}
$$

The mixed case of $\chi, V$, and $\tau_{i n v}$ cannot be solved, due to the system of equations even if one harmonic is added.
(b) Truncating (20) at location II and substituting it into (8) results in following logarithmic temperature derivative by substituting $\lambda_{1}$ and $\lambda_{2}$

$$
\begin{align*}
\frac{\Theta^{\prime}}{\Theta}= & \frac{4 \rho\left(4 V^{2}+5 \chi\left(\tau_{i n v}+i \omega\right)\right)}{\left(\rho^{2}\left(2 V^{2}+5 \chi\left(\tau_{i n v}+i \omega\right)\right)-16 \rho V \chi+40 \chi^{2}\right)} \\
& -\frac{\rho^{2} V\left(6 V^{2}+19 \chi\left(\tau_{i n v}+i \omega\right)\right)+120 V \chi^{2}}{3 \chi\left(\rho^{2}\left(2 V^{2}+5 \chi\left(\tau_{i n v}+i \omega\right)\right)-16 \rho V \chi+40 \chi^{2}\right)} . \tag{B9}
\end{align*}
$$

The complexity of (B9) makes it difficult to calculate approximations by hand. Therefore, Mathematica ${ }^{\odot}$ has been used to calculate the approximations for $\chi, V$, and $\tau_{i n v}$.
(b1) If $\tau_{i n v}=0$, (B9) results in

$$
\begin{equation*}
\chi_{\Phi 4 V}=\frac{3}{2} \frac{6859 \rho^{3} \omega \phi^{\prime}}{l_{2}-l_{3}\left(-2 \frac{A^{\prime}}{A} \rho \pm l_{1}+30\right)} \tag{B10}
\end{equation*}
$$

with

$$
\begin{gather*}
l_{1}=\sqrt{4\left(\frac{A^{\prime}}{A} \rho+15\right)^{2}-285 \rho^{2}\left(\phi^{\prime}\right)^{2}}  \tag{B11}\\
l_{2}=114 \rho^{2}\left(\phi^{\prime}\right)^{2}\left(32 \frac{A^{\prime}}{A} \rho \pm l_{1}+62\right) \tag{B12}
\end{gather*}
$$

and

$$
\begin{equation*}
l_{3}=6 \frac{A^{\prime}}{A} \rho\left(15 \frac{A^{\prime}}{A} \rho+32\right)+680 \tag{B13}
\end{equation*}
$$

There are two solutions possible, the second option ( $-\mathrm{in} \pm$ ) gives a solution in a region with poor approximations and is disregarded. Hence, + solution is used. The convectivity $V$ is given by

$$
\begin{equation*}
V_{\Phi 4 V}=\chi_{\Phi 4 V} \frac{-l_{1}-17 \frac{A^{\prime}}{A} \rho+30}{19 \rho} \tag{B14}
\end{equation*}
$$

(b2) If $A_{1}^{\prime} / A_{1}, \phi_{1}^{\prime}$, and $\phi_{2}^{\prime}$ are used, only one solution is found

$$
\begin{align*}
& \chi_{\Phi 4 b}=\frac{3}{2} \frac{6859 d \omega \rho^{3} \omega_{1} \phi_{1}^{\prime}}{8\left(19 d \omega+o_{2}\right)} \\
& \frac{\omega_{2}^{2}\left((d A)^{2}+\left(\phi_{1}^{\prime}\right)^{2}\right)+\omega_{1} \phi_{2}^{\prime}\left(d \omega-\omega_{2} \phi_{1}^{\prime}\right)}{\left(1444 d \omega^{2}+456 d \omega o_{2}+45 o_{2}^{2}\right)} \tag{B15}
\end{align*}
$$

with

$$
\begin{equation*}
d \omega=\omega_{1} \phi_{2}^{\prime}-\omega_{2} \phi_{1}^{\prime} \tag{B16}
\end{equation*}
$$

and

$$
\begin{equation*}
o_{2}=\frac{A_{1}^{\prime}}{A_{1}} \rho \omega_{1} \phi_{2}^{\prime}-\frac{A_{2}^{\prime}}{A_{2}} \rho \omega_{2} \phi_{1}^{\prime}-4 d \omega . \tag{B17}
\end{equation*}
$$

The corresponding $V$ and $\tau_{i n v}$ are given by

$$
\begin{equation*}
V_{\Phi 4 b}=-\chi_{\Phi 4 b} \frac{30 o_{2}}{38 \rho d \omega} \tag{B18}
\end{equation*}
$$

and (subscripts $\Psi 4 b$ have been omitted)

$$
\begin{align*}
\tau_{\Phi 4 b}= & \frac{3}{2} \chi \frac{-15 \frac{A_{1}^{\prime}}{A_{1}} \rho^{2} \omega \frac{1}{\chi}-24\left(\frac{V}{2 \chi}\right)^{2} \rho^{2} \phi^{\prime}-120 \phi^{\prime}}{15 \rho^{2} \phi^{\prime}} \\
& +\frac{3}{2} \chi \frac{-38\left(\frac{V}{2 \chi^{2}}\right) \rho^{2} \omega+96\left(\frac{V}{2 \chi}\right) \rho \phi^{\prime}+60 \rho \omega \frac{1}{\chi}}{15 \rho^{2} \phi^{\prime}} \tag{B19}
\end{align*}
$$

This equation is exactly the same as (23).
(b3) The approximation using the same truncation as (B15) and $A_{1}^{\prime} / A_{1}, A_{2}^{\prime} / A_{2}$ and $\phi_{1}^{\prime}$, is given in (21) in the main text. Also, a $T$-fraction is given in Refs. 11 and 12, but it showed less accurate results than the continued $C$-fraction.

## APPENDIX C: APPROXIMATIONS FOR X ONLY ( $V=T_{I N V}=0$ )

If $\tau_{i n v}=0$ and $V=0$, the truncation of (12) with the last term $a_{2}$ is given by

$$
\begin{equation*}
0=\left(\left(\frac{A^{\prime}}{A}+i \phi^{\prime}\right) \rho^{2}-4 \rho\right) \frac{3}{2} \frac{\omega i}{\chi}+8\left(\frac{A^{\prime}}{A}+i \phi^{\prime}\right) \tag{C1}
\end{equation*}
$$

The real part is given by

$$
\begin{equation*}
0=-\phi^{\prime} \rho^{2} \frac{3}{2} \frac{\omega}{\chi}+8 \frac{A^{\prime}}{A} \tag{C2}
\end{equation*}
$$

and the imaginary part is given by

$$
\begin{equation*}
0=\frac{A^{\prime}}{A} \rho^{2} \frac{3}{2} \frac{\omega i}{\chi}-4 \rho \frac{3}{2} \frac{\omega i}{\chi}+8 i \phi^{\prime} \tag{C3}
\end{equation*}
$$

Solving for $A^{\prime} / A$ and $\phi^{\prime}$ and rewriting in terms of $\chi$ yields

$$
\begin{equation*}
\chi_{I S 2 A}=\omega \frac{3 \rho^{3} \sqrt{4-A^{\prime} / A \rho}}{16 \rho^{2} \sqrt{A^{\prime} / A}} \tag{C4}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi_{I s 2 \phi}=3 \omega \rho \frac{2+\sqrt{4-\rho^{2}\left(\phi^{\prime}\right)^{2}}}{16 \phi^{\prime}} \tag{C5}
\end{equation*}
$$

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