Nested invariant tori foliating a vector field and its curl: toward MHD equilibria and steady Euler flows in toroidal domains without continuous Euclidean isometries

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Abstract

This paper studies the problem of finding a three-dimensional solenoidal vector field such that both the vector field and its curl are tangential to a given family of toroidal surfaces. We show that this question can be translated into the problem of determining a periodic solution with periodic derivatives of a two-dimensional linear elliptic second-order partial differential equation on each toroidal surface, and prove the existence of smooth solutions. Examples of smooth solutions foliated by toroidal surfaces that are not invariant under continuous Euclidean isometries are also constructed explicitly, and they are identified as equilibria of anisotropic magnetohydrodynamics. The problem examined here represents a weaker version of a fundamental mathematical problem that arises in the context of magnetohydrodynamics and fluid mechanics concerning the existence of regular equilibrium magnetic fields and steady Euler flows in bounded domains without continuous Euclidean isometries. The existence of such configurations represents a key theoretical issue for the design of the confining magnetic field in nuclear fusion reactors known as stellarators.

1 Introduction

This paper is concerned with the equation

$$[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla \Psi = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{w} = 0 \quad \text{in} \ \ \Omega. \tag{1}$$

Here, the unknown $\boldsymbol{w}(\boldsymbol{x})$ is a three-dimensional vector field with Cartesian components w_i, $i = 1, 2, 3$, defined in a smooth toroidal domain $\Omega \subset \mathbb{R}^3$ foliated by nested toroidal surfaces corresponding to level sets of a smooth function $\Psi(\boldsymbol{x})$ such that the bounding surface is given by $\partial\Omega = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \Psi = \Psi_0 \in \mathbb{R}\right\}$.

Equation (1) has the following physical meaning: given a set of nested toroidal surfaces Ψ in the domain Ω, can one always find a magnetic field $\boldsymbol{B} = \boldsymbol{w}$ and an electric current $\boldsymbol{J} = \nabla \times \boldsymbol{w}$ that are tangent to the level sets of Ψ? In a more general interpretation of (1), both the function Ψ and the shape of the toroidal volume Ω can be treated as an unknown variables as well.

In the context of plasma physics, the problem posed by equation (1) represents a relaxation of a more difficult equation, namely the magnetohydrodynamic equilibrium equation

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla \Psi, \quad \nabla \cdot \boldsymbol{w} = 0 \quad \text{in} \ \ \Omega, \tag{2}$$

where $P = \Psi$ plays the role of the pressure field. This equation describes an equilibrium state where the Lorentz force is exactly balanced by the pressure force, and its solution is crucial for the design of confining magnetic fields in nuclear fusion reactors [1]. Notice that any solution of (2) is also a solution of (1). Equation (2) also occurs in fluid mechanics, where it describes a steady Euler flow with velocity field $\boldsymbol{v} = \boldsymbol{w}$ and mechanical pressure $P = -\Psi - \frac{1}{2}\boldsymbol{v}^2$ [2].

The challenge posed by equation (2) is exemplified by the unavailability of a general theory concerning the existence of solutions [3], although steady progress has been made since the original inception of the problem. As a consequence, it is not known whether smooth steady fluid flows or equilibrium magnetic fields (2) foliated by a smooth pressure field Ψ exist in a bounded domain Ω of arbitrary shape such that the bounding surface $\partial\Omega$ corresponds to a level set of Ψ. The intrinsic mathematical difficulty behind equation (2) can be understood in terms of characteristic surfaces. Indeed, if considered as a system of nonlinear first order partial differential equations for the unknowns $\boldsymbol{w}, \Psi$, the characteristic surfaces S of equation (2) are determined by the characteristic equation [4]

$$(\nabla S)^2 (\boldsymbol{w} \cdot \nabla S)^2 = 0. \tag{3}$$

Hence, equation (2) exhibits a mixed behavior, being twice elliptic and twice hyperbolic, with the nontrivial characteristic surfaces $(\boldsymbol{w} \cdot \nabla S)^2 = 0$ associated with hyperbolicity depending on the unknown $\boldsymbol{w}$. These features are the reason why finding solutions of (2) with a function Ψ whose level sets are not invariant under some continuous Euclidean isometry is a hard mathematical problem (as discussed below, continuous Euclidean isometries enable the reduction of (2) to an elliptic equation).

Weak solutions of (2) have been proposed where the function Ψ has a stepped profile [5, 6]. In this approach, the toroidal domain Ω is partitioned in a set of nested toroidal subdomains $\Omega_i \subset \Omega$, $i = 1, ..., N$, where the function $\Psi = \psi_i \in \mathbb{R}$ is constant, while the total pressure $\Psi + \boldsymbol{w}^2/2$ is continuous across the boundary separating adjacent subdomains. This construction has the advantage that in each subdomain Ω_i equation (2) reduces to the eigenvalue problem for the curl operator, a system of linear first order partial differential equations for which strong solutions are available [7]. However, this comes at the price of reduced regularity of solutions, which fall in the class $L^2(\Omega)$. An alternative approach to investigate existence and regularity of solutions of system (2) is represented by so-called inviscid regularizations of the Euler and magnetohydrodynamics equations, an idea that has been introduced in the context of regularization schemes for fluid models. In particular, the Voigt regularization of the three-dimensional incompressible Navier-Stokes equations (the Navier-Stokes-Voigt equations) is a globally well posed system [8, 9, 10] that shares the same steady state solutions and Reynolds-averaged equations as the Navier-Stokes equations. This latter fact suggests that regularized and original models share similar statistics, even though the dynamics of individual solutions is different [11, 12, 13]. With the aid of such Voigt approximation scheme of the time-dependent viscous non-resistive incompressible magnetohydrodynamics equations in the limit $t \to \infty$ in a setting where Ψ is not required to be constant on $\partial\Omega$, nontrivial strong solutions of (2) in the class $H^1(\Omega)$ have been reported in [14]. For completeness, we also mention that Voigt regularization can be applied to the study of finite-time blow-up of the three-dimensional incompressible Euler equations [15].

It is well known that equation (2) is greatly simplified whenever the vector field $\boldsymbol{w}$ and the function Ψ are invariant under a continuous Euclidean isometry, i.e. a continuous transformation of three-dimensional space that preserves the Euclidean distance $ds^2 = dx^2 + dy^2 + dz^2$ between points. Such transformations are characterized by a vector field $\boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$, with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, representing the general solution of the equation $\mathfrak{L}_{\boldsymbol{\eta}} ds^2 = 0$, where $\mathfrak{L}$ denotes the Lie-derivative, and physically correspond to combinations of translations (generated by $\boldsymbol{a}$) and rotations (generated by $\boldsymbol{b}$). In the context of plasma physics, invariance under a continuous Euclidean isometry is usually referred to as a symmetry of the system. In formulae, the vector field $\boldsymbol{w}$ and the function Ψ are invariant under a continuous Euclidean isometry whenever constant vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ with $\boldsymbol{a}^2 + \boldsymbol{b}^2 \neq \boldsymbol{0}$ exist such that

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}} \boldsymbol{w} = \boldsymbol{0}, \quad \mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}} \Psi = 0. \tag{4}$$

When condition (4) holds, equation (2) can be reduced to the Grad-Shafranov equation [16, 17], a nonlinear second order elliptic partial differential equation for the unknown Ψ, which is assumed to satisfy Dirichlet boundary conditions on $\partial\Omega$. Regular solutions of the Grad-Shafranov equation can be obtained in accordance with the theory of second order elliptic partial differential equations, thus providing regular (symmetric) solutions of (2). Notice that in this setting the symmetry of Ψ implies the symmetry of the bounding surface $\partial\Omega$, which corresponds to a level set of Ψ.

Unfortunately, the presence of a continuous Euclidean isometry (4) is a special requirement that does not apply in several situations of practical interest. In particular, the confining magnetic field in nuclear fusion reactors known as stellarators sacrifices axial symmetry in favor of a pronounced field line twist that aims at minimizing plasma losses at the vessel boundary $\partial\Omega$ caused by cross-field dynamics of charged particles [18]. In this context, it is therefore necessary to understand the existence of solutions of (2) that are not endowed with continuous Euclidean symmetries. For completeness, it should be emphasized that even if such 'asymmetric' solutions exist, they would not necessarily work as confining magnetic fields, because other requirements, such as quasisymmetry [19] and a small electric current, must be enforced on $\boldsymbol{w}$.

The nontrivial geometrical constraints on $\boldsymbol{w}$ and Ψ required for the existence of solutions of equation (2) in toroidal domains without continuous Euclidean symmetries have also raised the possibility that regular solutions of this kind may not exist: according to the Grad conjecture [20], only 'configurations of great geometrical symmetry' would produce well behaved equilibria. In modern plasma physics, this idea is usually understood as equation (4) being a necessary condition for the existence of regular solutions of (2). Although Grad's conjecture remains unsolved, Arnold's structure theorem [21] provides a topological characterization of the field lines of any analytic solution of (2) such that $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$ are not everywhere collinear. In particular, when equation (2) is considered in a connected analytic bounded domain Ω together with tangential boundary conditions $\boldsymbol{w}\cdot\boldsymbol{n}=0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$, any contour of Ψ that does not intersect the boundary $\partial\Omega$ and such that $\nabla\Psi\neq\boldsymbol{0}$ is a two-dimensional torus. This result is also the reason why toroidal volumes such that Ψ is constant on the boundary are considered in the study of equation (2). The fact that a simpler topology where level sets of Ψ are spherical is not consistent with (2) can also be understood through the hairy ball theorem [22], which precludes the existence of a continuous non-vanishing vector field always tangent to a 2-sphere.

Considering the challenge posed by equation (2) described above, here we examine the simplified problem of equation (1). Observe that while in (2) the magnitude of the component of $(\nabla\times\boldsymbol{w})\times\boldsymbol{w}$ along $\nabla\Psi$ is exactly $|\nabla\Psi|$, no such requirement appears in (1). As it will be shown later, under suitable assumptions this simplifies the mathematical difficulty by a 'half', since the governing equations can be reduced from two to one. Notice that studying equation (1) may provide useful information concerning the nature of the space of solutions of equation (2). Indeed, any conditions preventing the existence of solutions of (1) would also apply to (2). Furthermore, if regular solutions of (1) could be obtained, it would be possible to identify the geometrical obstruction preventing such solutions from solving (2) as well.

The strategy we adopt to examine equation (1) is to reduce the equation by a Clebsch representation [23, 24] of the vector field $\boldsymbol{w}$ through a pair of Clebsch potentials (Ψ,Θ) that reflect the foliated $(\nabla\Psi\cdot\boldsymbol{w}=0)$ and solenoidal $(\nabla\cdot\boldsymbol{w}=0)$ nature of the candidate solution $\boldsymbol{w}=\nabla\Psi\times\nabla\Theta$. This approach has the advantage that the topology of the foliation associated with the (given) function Ψ can be enforced a priori, leaving the analysis of the existence of solutions as an independent issue for the Clebsch potential Θ. In particular, we prove the following:

***Theorem* 1.** *Let $\Omega\subset\mathbb{R}^3$ denote a bounded domain. Assume that the bounding surface $\partial\Omega$ is a hollow torus corresponding to two distinct level sets of a smooth function $\Psi\in C^\infty(\Omega)$, with $\nabla\Psi\neq\boldsymbol{0}$ in Ω, and that level sets of Ψ foliate Ω with nested toroidal surfaces endowed with angle coordinates μ,ν with smooth gradients $\nabla\mu,\nabla\nu\in C^\infty(\Omega)$. Then, the system of partial differential equations*

$$[(\nabla\times\boldsymbol{w})\times\boldsymbol{w}]\times\nabla\Psi=\boldsymbol{0},\quad \nabla\cdot\boldsymbol{w}=0\quad\text{in}\ \ \Omega, \tag{5}$$

admits a nontrivial solution $\boldsymbol{w}\in C^\infty(\Omega)$ such that $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$ are not everywhere collinear.

The proof of theorem 1 follows by observing that, upon introducing the Clebsch representation $\boldsymbol{w}=\nabla\Psi\times\nabla\Theta$ of the solution, equation (5) reduces to a single linear elliptic second-order partial differential equation on each toroidal surface $\Psi=$ constant for the unknown Θ in a periodic domain. Regular periodic solutions of these equations can be obtained by elliptic theory. A global solution Θ can then be constructed by smoothly joining solutions corresponding to different toroidal surfaces, thus providing a smooth solution $\boldsymbol{w}$ of (5) in a hollow toroidal volume Ω.

We also state here a proposition that gives a straightforward way to look for explicit solutions of (1). This proposition will be discussed in detail in section 6.

Proposition 1. *Let $\Omega \subset \mathbb{R}^3$ be a toroidal volume with boundary $\partial\Omega$ foliated by toroidal surfaces corresponding to level sets of a function $\Psi \in C^1(\bar{\Omega})$. Let $\boldsymbol{\xi} \in L^2_H(\Omega)$ be a harmonic vector field in Ω, with*

$$L^2_H(\Omega) = \left\{\boldsymbol{\xi} \in L^2(\Omega); \nabla \times \boldsymbol{\xi} = \boldsymbol{0}, \nabla \cdot \boldsymbol{\xi} = 0, \boldsymbol{\xi} \cdot \boldsymbol{n} = 0\right\}, \tag{6}$$

where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. Further assume that $\boldsymbol{\xi}$ is foliated by Ψ, that is

$$\boldsymbol{\xi} \cdot \nabla\Psi = 0 \quad \text{in} \quad \Omega. \tag{7}$$

Then, the vector field $\boldsymbol{w} \in H^1_{\sigma\sigma}(\Omega)$ defined as

$$\boldsymbol{w} = f(\Psi)\,\boldsymbol{\xi}, \tag{8}$$

where f is any $C^1(\bar{\Omega})$ function of Ψ, is a nontrivial solution of (1) in Ω such that

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = -\frac{1}{2}\frac{\partial f^2}{\partial\Psi}\left|\boldsymbol{\xi}\right|^2 \nabla\Psi, \qquad \nabla \cdot \boldsymbol{w} = 0. \tag{9}$$

Here,

$$H^1_{\sigma\sigma}(\Omega) = \left\{\boldsymbol{w} \in L^2_\sigma(\Omega); \nabla \times \boldsymbol{w} \in L^2_\sigma(\Omega)\right\}, \tag{10}$$

$$L^2_\sigma(\Omega) = \left\{\boldsymbol{w} \in L^2(\Omega); \nabla \cdot \boldsymbol{w} = 0, \boldsymbol{w} \cdot \boldsymbol{n} = 0\right\}. \tag{11}$$

The present paper is organized as follows. In section 2, basic aspects of equation (1) are discussed, the reduction of the equation through Clebsch potentials to a three-dimensional linear second-order degenerate elliptic partial differential equation is presented, and a corresponding variational formulation is derived. In section 3, the reduced equation is reformulated as a family of two-dimensional linear elliptic second-order partial differential equations, each correponding to a given toroidal surface. Theorem 1 is proven in section 4, while an example of numerical solution is obtained in section 5. Smooth solutions of equation (1) with nested toroidal surfaces and without continuous Euclidean isometries are then constructed explicitly in section 6. It is further shown that such solutions can be ragarded as a equilibrium magnetic fields within the framework of anisotropic magnetohydrodynamics. Section 7 presents some considerations on the application of the theory to the study of equation (2). Concluding remarks are given in section 8.

2 General properties of the equation

The aim of this section is to discuss some basic properties of equation (1), as well as a variational formulation of the problem in terms of Clebsch potentials. For the purpose of this section, we shall assume that all involved quantities can be differentiated as many times as necessary.

First, it should be noted that the main difficulty in system (1) is not the requirement $[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla\Psi = \boldsymbol{0}$ that both $\boldsymbol{w}$ and $\nabla \times \boldsymbol{w}$ lie on the same surface $\Psi =$ constant per se, but rather its combination with the solenoidal condition $\nabla \cdot \boldsymbol{w} = 0$. Indeed, if one drops the equation $\nabla \cdot \boldsymbol{w} = 0$, any vector field of the form

$$\boldsymbol{w} = f(\Psi, \alpha)\,\nabla\alpha + g(\Psi, \beta)\,\nabla\beta, \tag{12}$$

where α and β are functions with the property that $\nabla\Psi \cdot \nabla\alpha = \nabla\Psi \cdot \nabla\beta = 0$, is a nontrivial solution of $[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla\Psi = \boldsymbol{0}$ where the proportionality coefficient λ between $(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}$ and $\nabla\Psi$ is given by

$$\lambda = -\frac{1}{2}\frac{\partial f^2}{\partial\Psi}\left|\nabla\alpha\right|^2 - \frac{1}{2}\frac{\partial g^2}{\partial\Psi}\left|\nabla\beta\right|^2 - \frac{\partial(fg)}{\partial\Psi}\nabla\alpha \cdot \nabla\beta. \tag{13}$$

Such configurations can be constructed explicitly. For example, introduce cylindrical coordinates (r, φ, z) and consider a family of axially symmetric toroidal surfaces with circular cross section and major radius $r_0 > 0$ corresponding to level sets of the function

$$\Psi_0 = \frac{1}{2}\left[(r - r_0)^2 + z^2\right]. \tag{14}$$

The axial symmetry of Ψ_0 can be broken by introducing a small displacement,

$$\Psi_\epsilon = \Psi_0 + \frac{1}{2}\epsilon \sin(m\varphi), \qquad m \in \mathbb{Z}, \qquad m \neq 0, \tag{15}$$

where $\epsilon > 0$ is a small real constant. In particular, observe that for a sufficiently small ϵ level sets of (15) generate toroidal surfaces. Furthermore, due to the dependence on the toroidal angle φ, these surfaces are not invariant under continuous Euclidean isometries, i.e. some appropriate combination of translations and rotations. Indeed, recalling that $\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is the generator of continuous Euclidean isometries, one sees that the only solution of the equation

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = (\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}) \cdot \nabla\Psi_\epsilon = 0, \tag{16}$$

is $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$. Let us verify this fact explicitly. We have

$$\begin{aligned}\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = &(a_x + b_y z - b_z y)\left[\left(1 - \frac{r_0}{r}\right)x - \frac{1}{2}\epsilon m \cos(m\varphi)\frac{y}{r^2}\right] \\ &+ (a_y + b_z x - b_x z)\left[\left(1 - \frac{r_0}{r}\right)y + \frac{1}{2}\epsilon m \cos(m\varphi)\frac{x}{r^2}\right] + z(a_z + b_x y - b_y x),\end{aligned} \tag{17}$$

where a_x, a_y, a_z, b_x, b_y, and b_z are the Cartesian components of $\boldsymbol{a}$ and $\boldsymbol{b}$. Next, evaluating the expression above along the positive x-axis where $\varphi = y = z = 0$ and $x = r$ gives

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = \frac{\epsilon m}{2}b_z - a_x r_0 + a_x x + a_y \frac{\epsilon m}{2x}. \tag{18}$$

Since x is not constant, this quantity vanishes only if $a_x = a_y = b_z = 0$. Now, the surviving terms in equation (17) are those involving a_z, b_x, and b_y. On the toroidal section $\varphi = 0$, equation (17) therefore becomes

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = -b_y r_0 z - b_x z\frac{\epsilon m}{2r} + z a_z. \tag{19}$$

It follows that the expression above vanishes if $b_x = 0$ and $a_z = b_y r_0$. Finally, consider the toroidal section $\varphi = \pi/2$. Here,

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = -b_y z\frac{\epsilon m}{2r}\cos\left(\frac{m\pi}{2}\right) + z b_y r_0. \tag{20}$$

This quantity vanishes for arbitrary r and z only if $b_y = a_z/r_0 = 0$. Hence, $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$, which implies that level sets of (15) are not invariant under continuous Euclidean isometries.

Now define the toroidal domain Ω as the volume enclosed by a contour of the function Ψ_ϵ in equation (15), and consider the vector field

$$\boldsymbol{w} = f(\Psi_\epsilon)\nabla\alpha, \tag{21}$$

where f is some function of Ψ_ϵ and $\alpha = \arctan[z/(r - r_0)]$. Since $\nabla\Psi_\epsilon \cdot \nabla\alpha = 0$, it readily follows that

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = -\frac{1}{2}\frac{\partial f^2}{\partial\Psi_\epsilon}|\nabla\alpha|^2\nabla\Psi_\epsilon = -\frac{1}{2\left[2\Psi_\epsilon - \epsilon\sin(m\varphi)\right]}\frac{\partial f^2}{\partial\Psi_\epsilon}\nabla\Psi_\epsilon, \tag{22}$$

although

$$\nabla \cdot \boldsymbol{w} = f(\Psi_\epsilon)\Delta\alpha = -\frac{z f(\Psi_\epsilon)}{r\left[2\Psi_\epsilon - \epsilon\sin(m\varphi)\right]}, \tag{23}$$

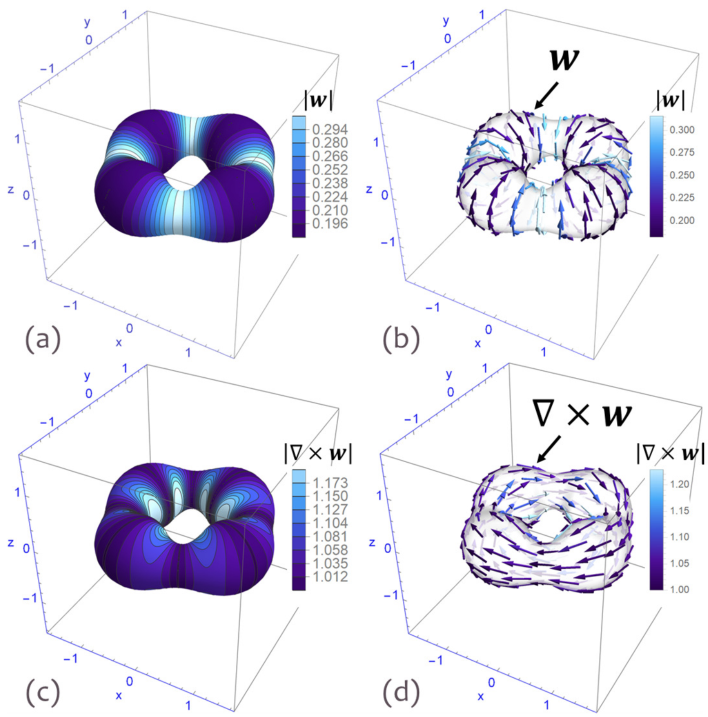

Figure 1: (a) Contour plot of $|\boldsymbol{w}|$ over the level set $\Psi_\epsilon = 0.1$. (b) Vector plot of $\boldsymbol{w}$ over the level set $\Psi_\epsilon = 0.1$. (c) Contour plot of $|\nabla \times \boldsymbol{w}|$ over the level set $\Psi_\epsilon = 0.1$. (d) Vector plot of $\nabla \times \boldsymbol{w}$ over the level set $\Psi_\epsilon = 0.1$. In (a), (b), (c), and (d) $\boldsymbol{w}$ is defined by equation (21), $f = \Psi_\epsilon$, and the values $r_0 = 1$, $\epsilon = 0.1$, and $m = 4$ are used.

which does not vanish in general. A plot of the vector field (21) and its modulus on a level set of Ψ_ϵ is given in figure 1. This example shows that if the space of solutions is not restricted to solenoidal vector fields, smooth vector fields obeying $[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla\Psi = \boldsymbol{0}$ in some bounded region can be obtained in a rather straightforward fashion.

We now return to the original problem (1). Observe that whenever $(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} \neq \boldsymbol{0}$ any solution of (1) must satisfy $\boldsymbol{w} \cdot \nabla\Psi = 0$, or

$$\boldsymbol{w} = \nabla\Psi \times \boldsymbol{q}, \tag{24}$$

for some vector field $\boldsymbol{q}(\boldsymbol{x})$. Since $\nabla \cdot \boldsymbol{w} = -\nabla\Psi \cdot \nabla \times \boldsymbol{q}$, a straightforward way to ensure that $\nabla \cdot \boldsymbol{w} = 0$ is to demand that $\nabla \times \boldsymbol{q} = \boldsymbol{0}$. For a given solenoidal vector field $\boldsymbol{w}$ in a small enough neighborhood the vector field $\boldsymbol{q}$ can always be found in the form $\boldsymbol{q} = \nabla\theta$, with θ a single-valued function, provided that $\boldsymbol{w}$ is sufficiently regular (Darboux theorem [25, 26]). This implies that locally solutions of (1) have the form $\boldsymbol{w} = \nabla\Psi \times \nabla\theta$. We shall therefore consider candidate global solutions of the type

$$\boldsymbol{w} = \nabla\Psi \times \nabla\Theta, \tag{25}$$

where Θ is allowed to be a multivalued (angle) variable. More precisely, we write $\nabla\Theta$ to denote an element of the kernel of the curl operator, $\nabla\Theta \in \mathrm{Ker}\,(\mathrm{curl})$. In the following, we shall refer to scalar quantities such as Ψ and Θ used to express a vector field as Clebsch potentials, and to the form (25) as a Clebsch representation of the solution $\boldsymbol{w}$ (see [23] for additional details on Clebsch representations and their completeness). It is now clear that finding a solution of (1) in the Clebsch form (25) is tantamount to determining two vector fields $\nabla\Theta$ and $\boldsymbol{p}$ such that

$$\nabla \times (\nabla\Psi \times \nabla\Theta) = \nabla\Psi \times \boldsymbol{p} \quad \text{in} \;\; \Omega. \tag{26}$$

Equation (26) is equivalent to demanding that $\nabla \times \boldsymbol{w}$ does not have any component in the direction of $\nabla\Psi$, i.e.

$$\nabla\Psi \cdot \nabla \times (\nabla\Psi \times \nabla\Theta) = 0 \quad \text{in} \;\; \Omega. \tag{27}$$

By standard vector identities, we thus arrive at

$$\nabla \cdot [\nabla \Psi \times (\nabla \Theta \times \nabla \Psi)] = 0 \quad \text{in} \ \ \Omega. \tag{28}$$

Hence, solutions of (1) with Clebsch representation (25) are solutions of (28). In the remainder of this paper we shall therefore concentrate our efforts on the study of equation (28) under appropriate boundary conditions for the variable Θ.

For a given Ψ, equation (28) is a second order degenerate elliptic partial differential equation for the unknown Θ. The equation is also linear, in contrast with the nonlinearity of system (1). To see this, observe that (28) can be written as

$$\begin{aligned} &\left|\nabla \Psi\right|^2 \Delta \Theta - \nabla \Psi \cdot (\nabla \Psi \cdot \nabla) \nabla \Theta - \nabla \Theta \cdot (\nabla \Psi \cdot \nabla) \nabla \Psi + \nabla \left|\nabla \Psi\right|^2 \cdot \nabla \Theta - (\nabla \Psi \cdot \nabla \Theta) \Delta \Psi \\ &= \sum_{i,j=1}^{3} \left|\nabla \Psi\right|^2 \left(\delta_{ij} - \frac{\Psi_i \Psi_j}{\left|\nabla \Psi\right|^2}\right) \Theta_{ij} + \sum_{i=1}^{3} \left(\frac{1}{2} \left|\nabla \Psi\right|_i^2 - \Psi_i \Delta \Psi\right) \Theta_i = 0 \quad \text{in} \ \ \Omega, \end{aligned} \tag{29}$$

where lower indices have been used as a shorthand notation for partial derivatives with respect to Cartesian coordinates $(x, y, z) = \left(x^1, x^2, x^3\right)$, e.g. $\Theta_1 = \partial \Theta / \partial x^1 = \partial \Theta / \partial x$. The coefficient matrix

$$\mathfrak{a}_{ij} = \left|\nabla \Psi\right|^2 \left(\delta_{ij} - \frac{\Psi_i \Psi_j}{\left|\nabla \Psi\right|^2}\right), \qquad i, j = 1, 2, 3, \tag{30}$$

is symmetric and positive semi-definite since

$$\mathfrak{a}_{ij} \xi^i \xi^j = \left|\nabla \Psi \times \boldsymbol{\xi}\right|^2 \geq 0, \qquad \boldsymbol{\xi} \in \mathbb{R}^3, \ \ \boldsymbol{x} \in \Omega, \tag{31}$$

and thus defines a degenerate elliptic differential operator. The degeneracy of the solution is evident from the fact that if Θ is a solution of (28), so is $\Theta + f(\Psi)$, with f a function of Ψ. In particular, it should be emphasized that the degeneracy is not expected to prevent the existence of solutions, but simply to affect their uniqueness.

Equation (28) also admits a variational formulation. Indeed, defining the magnetic energy (kinetic energy in the fluid analogy)

$$E_\Omega = \frac{1}{2} \int_\Omega \boldsymbol{w}^2 \, dV = \frac{1}{2} \int_\Omega \left|\nabla \Psi \times \nabla \Theta\right|^2 \, dV, \tag{32}$$

where dV is the volume element in $\mathbb{R}^3$, and assuming that variations $\delta\Theta$ vanish on the bounding surface $\partial\Omega$, one obtains

$$\delta E_\Omega = \int_\Omega \nabla \delta \Theta \cdot (\nabla \Psi \times \nabla \Theta) \times \nabla \Psi \, dV = -\int_\Omega \delta \Theta \nabla \cdot [\nabla \Psi \times (\nabla \Theta \times \nabla \Psi)] \, dV. \tag{33}$$

Hence, stationary points of the energy E_Ω correspond to solutions of (28).

3 Reformulation as an elliptic equation on a toroidal surface

As outlined in the introduction, we aim to remove the degeneracy of equation (28) by reducing it to a linear two-dimensional second-order elliptic partial differential equation over each toroidal surface $\Psi =$ constant. The degeneracy can be effectively removed, for example, by fixing the mean value $\langle\Theta\rangle$ of the unknown Θ over the surface. A unique solution of (28) can then obtained by patching two-dimensional solutions corresponding to different toroidal surfaces. In order to implement this construction, we introduce curvilinear coordinates $\left(x^1, x^2, x^3\right) = (\mu, \nu, \Psi)$ with $\mu, \nu \in [0, 2\pi)$ angle coordinates spanning the toroidal surfaces $\Psi =$ constant, ∂_i, $i = 1, 2, 3$, the corresponding tangent vectors, $J = \nabla\mu \cdot \nabla\nu \times \nabla\Psi$ the Jacobian determinant of the transformation, and $g_{ij} = \partial_i \cdot \partial_j$ the covariant metric tensor. Using these quantities, equation (28) restricted to the surface $\Sigma_{\Psi_0} = \{\boldsymbol{x} \in \Omega : \Psi(\boldsymbol{x}) = \Psi_0 \in \mathbb{R}\}$ takes the form

$$\frac{\partial}{\partial\mu}\left[J\left(g_{\nu\nu}\frac{\partial\Theta}{\partial\mu} - g_{\mu\nu}\frac{\partial\Theta}{\partial\nu}\right)\right] + \frac{\partial}{\partial\nu}\left[J\left(g_{\mu\mu}\frac{\partial\Theta}{\partial\nu} - g_{\mu\nu}\frac{\partial\Theta}{\partial\mu}\right)\right] = 0 \quad \text{in} \ \ \Sigma_{\Psi_0}. \tag{34}$$

Notice that in this notation $g_{11} = g_{\mu\mu}$, $g_{12} = g_{\mu\nu}$, and $g_{22} = g_{\nu\nu}$. Let us verify that the two-dimensional second order partial differential equation (34) is elliptic on each Σ_{Ψ_0}. First, observe that equation (34) can be written as

$$\begin{aligned} &g_{\nu\nu}\Theta_{\mu\mu} - 2g_{\mu\nu}\Theta_{\mu\nu} + g_{\mu\mu}\Theta_{\nu\nu} + \left[\frac{J_\mu}{J}g_{\nu\nu} + \frac{\partial g_{\nu\nu}}{\partial\mu} - \frac{J_\nu}{J}g_{\mu\nu} - \frac{\partial g_{\mu\nu}}{\partial\nu}\right]\Theta_\mu \\ &+ \left[\frac{J_\nu}{J}g_{\mu\mu} + \frac{\partial g_{\mu\mu}}{\partial\nu} - \frac{J_\mu}{J}g_{\mu\nu} - \frac{\partial g_{\mu\nu}}{\partial\mu}\right]\Theta_\nu = 0 \quad \text{in} \ \ \Sigma_{\Psi_0}. \end{aligned} \tag{35}$$

Equation (35) has the form

$$\sum_{i,j=1}^{2} a_{ij}\Theta_{ij} + \text{lower order terms} = 0, \tag{36}$$

where the coefficient matrix A with components a_{ij}, $i,j = 1,2$, is given by

$$A = \begin{bmatrix} g_{\nu\nu} & -g_{\mu\nu} \\ -g_{\mu\nu} & g_{\mu\mu} \end{bmatrix} \tag{37}$$

Evidently, $A = A^T$. Furthermore, the eigenvalues of A are given by

$$\lambda_\pm = \frac{\mathrm{Tr}A \pm \sqrt{(\mathrm{Tr}A)^2 - 4\det A}}{2}, \tag{38}$$

with

$$\mathrm{Tr}A = g_{\mu\mu} + g_{\nu\nu} > 0, \qquad \det A = g_{\nu\nu}g_{\mu\mu} - g_{\mu\nu}^2 = |\partial_\nu \times \partial_\mu|^2 > 0. \tag{39}$$

Both eigenvalues are real and positive with $\lambda_+ \geq \lambda_- > 0$ because

$$(\mathrm{TrA})^2 > (\mathrm{TrA})^2 - 4\det A = (g_{\mu\mu} - g_{\nu\nu})^2 + 4g_{\mu\nu}^2 \geq 0. \tag{40}$$

It therefore follows that equation (34) is strictly elliptic on each Σ_{Ψ_0}. Indeed, any vector $\boldsymbol{\xi} \in \mathbb{R}^2$ can be decomposed on the basis of normalized eigenvectors $(\boldsymbol{e}_+, \boldsymbol{e}_-)$ so that

$$a_{ij}\xi^i\xi^j \geq \lambda_- |\boldsymbol{\xi}|^2 \geq 0, \qquad \boldsymbol{\xi} \in \mathbb{R}^2, \qquad (\mu,\nu) \in [0, 2\pi), \qquad \Psi = \Psi_0. \tag{41}$$

Thanks to the strictly elliptic nature of the differential operator, once appropriate boundary conditions are enforced, solutions Θ of equation (34) exist and are unique (see e.g. [27]). Furthermore, as it will be shown later, solutions corresponding to different toroidal surfaces can be patched together to obtain a solution in the three-dimensional volume Ω. However, not all boundary conditions can be used to produce nontrivial vector fields $\boldsymbol{w}$ in Ω. For example, defining the doubly periodic domain $D = (0, 2\pi)^2$, Dirichlet boundary conditions $\Theta = 0$ on ∂D result in the trivial solution $\Theta = 0$, and thus $\boldsymbol{w} = \boldsymbol{0}$. Furthermore, even if a set of boundary conditions results in a nontrivial Θ, there is no guarantee that the corresponding vector field $\nabla\Theta$ is a periodic function of the variables μ and ν, a condition that is necessary for the continuity of the solution $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta$ in Ω. These difficulties can be avoided as follows. First, performing the change of variables

$$\Theta = \mu + \rho, \tag{42}$$

in equation (34) gives

$$\frac{\partial}{\partial\mu}\left[J\left(g_{\nu\nu}\frac{\partial\rho}{\partial\mu} - g_{\mu\nu}\frac{\partial\rho}{\partial\nu}\right)\right] + \frac{\partial}{\partial\nu}\left[J\left(g_{\mu\mu}\frac{\partial\rho}{\partial\nu} - g_{\mu\nu}\frac{\partial\rho}{\partial\mu}\right)\right] = \frac{\partial}{\partial\nu}(Jg_{\mu\nu}) - \frac{\partial}{\partial\mu}(Jg_{\nu\nu}) \quad \text{in} \ \ D. \tag{43}$$

Note that equation (43) is strictly elliptic because it shares the same coefficient matrix A with equation (34). Furthermore, Σ_{Ψ_0} has been replaced with D to emphasize that the problem is being considered within the domain of the angles μ and ν. Next, consider a periodic solution $\rho = \sum_{m,n} c_{mn}(\Psi_0)\, e^{\mathrm{i}(m\mu+n\nu)}$ such that the integral

$$\langle\rho\rangle = \int_D d\mu d\nu \rho = 0, \tag{44}$$

vanishes, i.e. such that $c_{00} = 0$ (this latter condition ensures that the solution ρ is unique on the toroidal surface Ψ_0). If such solution ρ could be found, the corresponding Θ would be nontrivial since its gradient $\nabla\Theta = \nabla\mu + \nabla\rho$ would be a periodic function of both μ and ν such that $\langle\Theta_\mu\rangle = \langle 1 + \rho_\mu\rangle = 4\pi^2$. Of course, other changes of variables, such as $\Theta = M(\Psi)\,\mu + N(\Psi)\,\nu + \rho$, could be used as well. In fact, by appropriate choice of the functions $M(\Psi)$ and $N(\Psi)$ one can control the rotational transform (the number of poloidal transits per toroidal transit of a field line on each level set of Ψ) of the solution $\boldsymbol{w}$.

It is now clear that the original problem (1) has been reduced to the existence of a periodic solution (with periodic derivatives) of equation (43) that depends in a regular fashion on the surface label Ψ. Although the coefficients appearing in equation (43) are periodic functions of μ and ν, enforcing a boundary condition such as $\rho(0,\nu,\Psi_0) = \rho(2\pi,\nu,\Psi_0) = \rho(\mu,0,\Psi_0) = \rho(\mu,2\pi,\Psi_0) = 0$ is not enough to ensure the periodicity of the partial derivatives ρ_μ, ρ_ν, ρ_Ψ, and so on. In other words, the corresponding solution will not generally correspond to a converging Fourier series. Therefore, the regularity of the solution $\boldsymbol{w} = \nabla\Psi\times\nabla\Theta$ will reflect the degree of periodicity of the solution ρ and its partial derivatives. In particular, denoting with $(\partial_\mu, \partial_\nu, \partial_\Psi)$ the tangent basis, observe that

$$\boldsymbol{w} = J\left(\frac{\partial\Theta}{\partial\mu}\partial_\nu - \frac{\partial\Theta}{\partial\nu}\partial_\mu\right), \tag{45a}$$

$$\begin{aligned}\nabla\times\boldsymbol{w} =& J\left\{\frac{\partial}{\partial\Psi}\left[J\left(g_{\mu\nu}\frac{\partial\Theta}{\partial\mu} - g_{\mu\mu}\frac{\partial\Theta}{\partial\nu}\right)\right] - \frac{\partial}{\partial\mu}\left[J\left(g_{\nu\Psi}\frac{\partial\Theta}{\partial\mu} - g_{\Psi\mu}\frac{\partial\Theta}{\partial\nu}\right)\right]\right\}\partial_\nu \\ &+ J\left\{\frac{\partial}{\partial\Psi}\left[J\left(g_{\mu\nu}\frac{\partial\Theta}{\partial\nu} - g_{\nu\nu}\frac{\partial\Theta}{\partial\mu}\right)\right] + \frac{\partial}{\partial\nu}\left[J\left(g_{\nu\Psi}\frac{\partial\Theta}{\partial\mu} - g_{\Psi\mu}\frac{\partial\Theta}{\partial\nu}\right)\right]\right\}\partial_\mu.\end{aligned} \tag{45b}$$

Hence, for $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$ to be continuous in Ω it is necessary that the partial derivatives $\Theta_\mu = 1 + \rho_\mu$, $\Theta_\nu = \rho_\nu$, $\Theta_{\mu\mu} = \rho_{\mu\mu}$, $\Theta_{\mu\nu} = \rho_{\mu\nu}$, and $\Theta_{\nu\nu} = \rho_{\nu\nu}$ are periodic functions of μ and ν. Conversely, if they fail to be periodic, $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$ will exhibit discontinuities on each toroidal surface in correspondence of the curves $\gamma_{\partial D} = \{\boldsymbol{x}\in\Omega : (\mu,\nu)\in\partial D, \Psi = \Psi_0\}$.

4 Proof of the main theorem

The purpose of this section is to prove theorem 1. As noted at the end of the previous section, in order to obtain a regular solution $\boldsymbol{w}$ of (1) in the domain Ω, any solution ρ of (43) must have periodic derivatives in the angles μ and ν. This implies that the standard theory for elliptic partial differential equations cannot be applied in a straightforward fashion because Dirichlet boundary conditions for ρ do not guarantee the periodicity of its partial derivatives. Since there are no requirements on the boundary values that the function $\Theta = \mu + \rho$ should take on ∂D, the idea is to construct a weak periodic solution of equation (43) in a two-dimensional lattice extending over $\mathbb{R}^2$ with unit cell D by introducing an appropriate Hilbert space $H^1_{\mathrm{per}}(D) \subset H^1(D)$ containing periodic functions. Then, interior regularity can be used to infer smoothness of weak solutions, and thus periodicity of their derivatives.

To carry out the program above, we begin by proving the following lemma:

Lemma 1. *Let $V = D\times U$ denote a doubly periodic three-dimensional domain spanned by coordinates $\mu,\nu\in[0,2\pi)$, $\Psi\in U$, with $D = (0,2\pi)^2$ and $U\subset\mathbb{R}$ a bounded open interval. Define $(x^1,x^2) = (\mu,\nu)$. Let $\alpha^{ij}\in C^\infty(\mathbb{R}^2\times U)$, $i,j = 1,2$, and $S\in C^\infty(\mathbb{R}^2\times U)$ be smooth functions which are periodic in D. Further assume that $\langle S\rangle = \int_D S\,d\mu d\nu = 0$, and that α^{ij} is strictly elliptic on each level set of Ψ, i.e.*

$$\alpha^{ij}\xi_i\xi_j \geq \lambda|\boldsymbol{\xi}|^2, \qquad \boldsymbol{\xi}\in\mathbb{R}^2, \qquad \mu,\nu\in[0,2\pi), \qquad \Psi\in U, \tag{46}$$

for some positive constant λ. Then, the boundary value problem

$$\frac{\partial}{\partial x^i}\left(\alpha^{ij}\frac{\partial \rho}{\partial x^j}\right)=S, \quad \langle\rho\rangle=\int_0^{2\pi}d\mu\int_0^{2\pi}d\nu\rho=0 \quad \text{in } V, \tag{47a}$$

$$\rho \ \text{ periodic in } D, \tag{47b}$$

admits a unique periodic solution $\rho\in C^{\infty}\left(\mathbb{R}^2\times U\right)$ with periodic derivatives of all orders. In particular, for fixed $\Psi\in U$ the function of two variables $\rho^{\Psi}(\mu,\nu)=\rho(\mu,\nu,\Psi)$ satisfies $\rho^{\Psi}\in C^{\infty}\left(\mathbb{R}^2\right)\cap H^1_{\mathrm{per}}(D)$. Here,

$$H^1_{\mathrm{per}}(D)=\left\{\rho^{\Psi}\in H^1(D)\,;\langle\rho^{\Psi}\rangle=0,\rho^{\Psi} \ \text{ periodic in } D\right\}. \tag{48}$$

Proof. First, observe that a function ρ is periodic in D provided that it takes the same values at opposite sides of the square,

$$\rho(0,\nu,\Psi)=\rho(2\pi,\nu,\Psi), \qquad \rho(\mu,0,\Psi)=\rho(\mu,2\pi,\Psi). \tag{49}$$

Considering a two-dimensional lattice with unit cell D, evidently a periodic solution satisfies the property

$$\rho(\mu,\nu,\Psi)=\rho(\mu+2\pi m,\nu+2\pi n,\Psi), \qquad \forall m,n\in\mathbb{Z}. \tag{50}$$

Hence, if derivatives of ρ exist, they are periodic functions as well. Next, notice that for each value of $\Psi\in U$ the strict ellipticity of α^{ij}, the regularity and periodicity of both α^{ij} and S, and the condition $\langle S\rangle=0$ guarantee that the boundary value problem

$$\frac{\partial}{\partial x^i}\left(\alpha^{ij}\frac{\partial \rho^{\Psi}}{\partial x^j}\right)=S, \quad \langle\rho^{\Psi}\rangle=0 \quad \text{in } D, \tag{51a}$$

$$\rho^{\Psi} \ \text{ periodic in } D, \tag{51b}$$

admits a unique solution $\rho^{\Psi}\in H^1_{\mathrm{per}}(D)$ (see for example [28]). Here, the notation $\rho^{\Psi}(\mu,\nu)=\rho(\mu,\nu,\Psi)$ stresses the fact that ρ is being considered a function of the angles (μ,ν) by fixing $\Psi\in U$. Let us briefly review the argument behind this result. Denote with $C^{\infty}_{\mathrm{per}}(D)=\left\{\rho^{\Psi}\in C^{\infty}\left(\mathbb{R}^2\right);\langle\rho^{\Psi}\rangle=0,\rho^{\Psi} \ \text{ periodic in } D\right\}$ the set of smooth functions periodic in D and with vanishing average. Note that $C^{\infty}_{\mathrm{per}}(D)=C^{\infty}\left(\mathbb{R}^2\right)\cap H^1_{\mathrm{per}}(D)$. Then, the Hilbert space $H^1_{\mathrm{per}}(D)$ can be identified with the completion of $C^{\infty}_{\mathrm{per}}(D)$ with respect to the H^1 norm. Now observe that the weak formulation of (51) is

$$\left(\rho^{\Psi},\psi\right)+\mathcal{F}_S[\psi]=\int_D\left(\alpha^{ij}\frac{\partial\psi}{\partial x^i}\frac{\partial\rho^{\Psi}}{\partial x^j}+S\psi\right)d\mu d\nu=0 \qquad \forall\psi\in H^1_{\mathrm{per}}(D). \tag{52}$$

Indeed, if $\rho^{\Psi}\in C^2_{\mathrm{per}}(D)$ with $C^2_{\mathrm{per}}(D)=\left\{\rho^{\Psi}\in C^2\left(\mathbb{R}^2\right);\langle\rho^{\Psi}\rangle=0,\rho^{\Psi} \ \text{ periodic in } D\right\}$, the partial derivatives $\partial\rho/\partial x^i$, $i=1,2$, are periodic, and therefore integration by parts shows that ρ^{Ψ} is a classical solution. We also remark that if ρ^{Ψ} is a weak solution in the sense of (52), it can also be tested against any $\psi_0\in H^1_0(D)$ since $\psi_0-\langle\psi_0\rangle/4\pi^2\in H^1_{\mathrm{per}}(D)$ by periodic extension of ψ_0 to $\mathbb{R}^2$. The converse is however not true, since a solution $\rho^{\Psi}_0\in H^1_0(D)$ of the standard Dirichlet boundary value problem cannot be tested against functions $\psi\in H^1_{\mathrm{per}}(D)$, i.e. $\left(\rho^{\Psi}_0,\psi\right)+\mathcal{F}_S[\psi]\neq 0$ in general.

Next, note that the inner product

$$\left(\rho^{\Psi},\psi\right)=\int_D\alpha^{ij}\frac{\partial\rho^{\Psi}}{\partial x^i}\frac{\partial\psi}{\partial x^j}\,d\mu d\nu, \tag{53}$$

defines a norm $\left|\left|\rho^{\Psi}\right|\right|_{H^1_{\mathrm{per}}(D)}=\left(\rho^{\Psi},\rho^{\Psi}\right)^{1/2}$ in $H^1_{\mathrm{per}}(D)$ due to the strict ellipticity of α^{ij},

$$\left(\rho^{\Psi},\rho^{\Psi}\right)\geq\lambda\left|\left|\nabla_{(\mu,\nu)}\rho^{\Psi}\right|\right|^2_{L^2(D)}\geq C\left|\left|\rho^{\Psi}\right|\right|^2_{H^1(D)}, \tag{54}$$

for some constant $C > 0$ and where in the last passage we used the Poincaré inequality [29], the fact that $\langle \rho^{\Psi} \rangle = 0$, and introduced the notation

$$
\begin{aligned}
&\left|\left|\rho^{\Psi}\right|\right|^2_{L^2(D)} = \int_D \left(\rho^{\Psi}\right)^2 \, d\mu d\nu, \\
&\left|\left|\nabla_{(\mu,\nu)}\rho^{\Psi}\right|\right|^2_{L^2(D)} = \int_D \left[\left(\rho^{\Psi}_{\mu}\right)^2 + \left(\rho^{\Psi}_{\nu}\right)^2\right] \, d\mu d\nu, \\
&\left|\left|\rho^{\Psi}\right|\right|^2_{H^1(D)} = \left|\left|\rho^{\Psi}\right|\right|^2_{L^2(D)} + \left|\left|\nabla_{(\mu,\nu)}\rho^{\Psi}\right|\right|^2_{L^2(D)}.
\end{aligned}
\tag{55}
$$

Hence, $H^1_{\text{per}}(D)$ is a Hilbert space with respect to the norm $||\cdot||_{H^1_{\text{per}}(D)}$. Finally, the linear functional

$$
\mathcal{F}_S[\psi] = \int_D S\psi \, d\mu d\nu \le C\,||\psi||_{H^1_{\text{per}}(D)}, \tag{56}
$$

is bounded, with $C > 0$ a constant. Hence, the Riesz representation theorem guarantees the existence of a unique element $\rho^{\Psi} \in H^1_{\text{per}}(D)$ such that $F_S[\psi] = -\left(\rho^{\Psi}, \psi\right)$, which thus provides a weak solution of (51).

The construction above applies even if the origin of the cell D is shifted by an arbitrary amount in $\mathbb{R}^2$. Let $D' \subset \mathbb{R}^2$ denote the shifted cell and $\rho'^{\Psi} \in H^1_{\text{per}}(D') = H^1_{\text{per}}(D)$ the corresponding solution. By interior regularity, any irregularity of the solution ρ'^{Ψ} that may occur on the boundary $\partial D'$ cannot affect the interior of the domain, and it can be shown that the regularity of α^{ij} and S is propagated to ρ'^{Ψ}. In particular, $\rho'^{\Psi} \in C^{\infty}(D')$ (see [30]). Since we may take $D' \cap D \neq \emptyset$ and $\rho^{\Psi} = \rho'^{\Psi}$ by uniqueness, this also implies the regularity of the derivatives of ρ^{Ψ} at the original cell boundary ∂D, and thus their periodicity. We conclude that $\rho^{\Psi} \in C^{\infty}_{\text{per}}(D)$ and that all partial derivatives of any order of the function ρ^{Ψ} are periodic functions in D.

We are now left with the task of showing that solutions of (51) corresponding to different values of Ψ define a smooth function in the variable Ψ. To see this, it is convenient to introduce the linear differential operators

$$
L = \frac{\partial}{\partial x^i}\left(\alpha^{ij}\frac{\partial}{\partial x^j}\right), \quad L_{\Psi} = \frac{\partial}{\partial x^i}\left(\alpha^{ij}_{\Psi}\frac{\partial}{\partial x^j}\right), \tag{57}
$$

where $L_{\Psi} = \partial L/\partial\Psi$ and we used the fact that α^{ij} is smooth in the variable Ψ to evaluate $\alpha^{ij}_{\Psi} = \partial\alpha^{ij}/\partial\Psi$. The first equation in (51) thus takes the form $L\rho^{\Psi} = S$. Furthermore, the linear operator L defines an invertible linear mapping from the function space $C^{\infty}_{\text{per}}(D)$ to itself (for $S \in C^{\infty}_{\text{per}}(D)$, $L\rho^{\Psi} = S$ admits a unique solution in $C^{\infty}_{\text{per}}(D)$). Denoting with L^{-1} the inverse, it follows that

$$
0 = \frac{\partial\left(LL^{-1}\right)}{\partial\Psi} = L_{\Psi}L^{-1} + LL^{-1}_{\Psi}, \tag{58}
$$

where $L^{-1}_{\Psi} = \partial L^{-1}/\partial\Psi$. Since $L^{-1}(0) = 0$, application of L^{-1} to the equation above gives

$$
L^{-1}_{\Psi} = -L^{-1}L_{\Psi}L^{-1}, \tag{59}
$$

so that the Ψ-derivative of the inverse operator L^{-1} is expressed in terms of the operators L^{-1} and L_{Ψ}. Higher order derivatives of the operator L^{-1} can be determined by differentiating (59) with respect to Ψ. We now consider ρ as a function of the three variables (μ, ν, Ψ). From $\rho = L^{-1}S$, and observing that the quantity $S_{\Psi} - L_{\Psi}\rho$ belongs to $C^{\infty}_{\text{per}}(D)$ when intended as a function of μ, ν, we thus conclude that

$$
\frac{\partial\rho}{\partial\Psi} = L^{-1}_{\Psi}S + L^{-1}S_{\Psi} = L^{-1}\left(S_{\Psi} - L_{\Psi}\rho\right), \qquad \left(\frac{\partial\rho}{\partial\Psi}\right)^{\Psi} \in C^{\infty}_{\text{per}}(D), \tag{60}
$$

where $\left(\partial\rho/\partial\Psi\right)^{\Psi}$ denotes the two variables function obtained by fixing Ψ in $\partial\rho/\partial\Psi$. Similarly, $\partial^2\rho/\partial\Psi^2$ and higher order partial derivatives can be evaluated by repeatedly differentiating $\rho = L^{-1}S$ with respect to Ψ. Hence, for each $\Psi \in U$ derivatives of ρ with respect to Ψ of all order exist and belong to $C^{\infty}_{\text{per}}(D)$. It follows that the function ρ is smooth in the variable Ψ, and therefore provides a unique solution $\rho \in C^{\infty}\left(\mathbb{R}^2 \times U\right)$ with period D of the original boundary value problem (47) such that $\rho^{\Psi} \in C^{\infty}\left(\mathbb{R}^2\right) \cap H^1_{\text{per}}(D)$. $\square$

We are now ready to prove theorem 1:

Proof. By hypothesis, the function Ψ is smooth and foliates the domain Ω with nested toroidal surfaces spanned by angle coordinates μ, ν. The smoothness of the derivatives of the curvilinear coordinate system $\left(x^1, x^2, x^3\right) = (\mu, \nu, \Psi)$ ensures that the components $g_{\mu\mu}$, $g_{\mu\nu}$, $g_{\nu\nu}$, $g_{\Psi\mu}$, $g_{\nu\Psi}$, $g_{\Psi\Psi}$ of the metric tensor and the Jacobian J are smooth functions in Ω. Indeed, they can be expressed in terms of derivatives of the coordinates. For example

$$g_{\mu\mu} = \frac{g^{\nu\nu}g^{\Psi\Psi} - \left(g^{\nu\Psi}\right)^2}{J^2} = \frac{|\nabla\nu|^2 |\nabla\Psi|^2 - (\nabla\nu\cdot\nabla\Psi)^2}{(\nabla\mu\cdot\nabla\nu\times\nabla\Psi)^2}. \tag{61}$$

Hence, the two-dimensional matrix A defined in (37) has smooth components a^{ij}, $i, j = 1, 2$, in Ω. Furthermore, as shown in the previous section the matrix A is symmetric and positive definite, and such that the corresponding differential operator $\partial_i\left(Ja^{ij}\partial_j\right)$ in (43) is strictly elliptic on each Ψ contour (notice that by hypothesis $J \geq J_m > 0$ for some positive constant J_m so that the strict ellipticity of a^{ij} implies the strict ellipticity of Ja^{ij}). Recalling that the composition of smooth functions is smooth, it is now clear that the hypothesis of lemma 1 are satisfied with $\alpha^{ij} = Ja^{ij}$ and source term S given by

$$S = \frac{\partial}{\partial\nu}\left(Jg_{\mu\nu}\right) - \frac{\partial}{\partial\mu}\left(Jg_{\nu\nu}\right). \tag{62}$$

In particular, observe that S is smooth. Let $\rho \in C^\infty\left(\mathbb{R}^2\times U\right)$ denote the periodic classical solution of equation (43) obtained in accordance with lemma 1. Evidently, $\rho \in C^\infty(\Omega)$ as well. Setting $\Theta = \mu + \rho$ and recalling equation (45), it follows that the vector field

$$\boldsymbol{w} = \nabla\Psi\times\nabla\Theta = J\left(\Theta_\mu\partial_\nu - \Theta_\nu\partial_\mu\right) = J\partial_\nu + \nabla\Psi\times\nabla\rho, \tag{63}$$

is a solution $\boldsymbol{w}\in C^\infty(\Omega)$ of system (5). To see this, first recall that ρ is a smooth solution of equation (43), and thus the vector field (63) fulfills equation (28) in the hollow torus Ω. Since $\nabla\cdot\boldsymbol{w} = 0$, the vector field (63) therefore solves system (5) in Ω. Furthermore, the vector field (63) is non-vanishing since

$$\langle\Theta_\mu\rangle = \int_D \Theta_\mu\, d\mu d\nu = \int_D \left(1+\rho_\mu\right)\, d\mu d\nu = 4\pi^2. \tag{64}$$

Recalling that the partial derivative Θ_μ is smooth and that $\boldsymbol{w} = J\left(\Theta_\mu\partial_\nu - \Theta_\nu\partial_\mu\right)$ it follows that $\boldsymbol{w}\neq\boldsymbol{0}$ in some open set within Ω. It may happen however that the solution $\boldsymbol{w}$ is a curl-free (vacuum) solution $\nabla\times\boldsymbol{w} = \boldsymbol{0}$, or a Beltrami field $\nabla\times\boldsymbol{w} = \hat{h}\boldsymbol{w}$ for some proportionality coefficient $\hat{h}(\boldsymbol{x})\neq 0$. Nevertheless, denoting with $f(\Psi)\neq 0$ any smooth function of the variable Ψ such that $\partial f/\partial\Psi\neq 0$, it readily follows that in such scenario the vector field $\boldsymbol{w}' = f(\Psi)\boldsymbol{w}$ is a nontrivial solution of (5). Indeed, recalling that by construction $\boldsymbol{w}\cdot\nabla\Psi = 0$, one has

$$\left(\nabla\times\boldsymbol{w}'\right)\times\boldsymbol{w}' = -\frac{1}{2}\frac{\partial f^2}{\partial\Psi}\boldsymbol{w}^2\nabla\Psi\neq\boldsymbol{0}, \quad \nabla\cdot\boldsymbol{w}' = \frac{\partial f}{\partial\Psi}\nabla\Psi\cdot\boldsymbol{w} = 0. \tag{65}$$

$\square$

***Remark* 1.** *In the original formulation of the problem* (1)*, the domain Ω is a torus. However, the result of theorem 1 applies to a hollow torus. For the solution $\boldsymbol{w}$ of theorem 1 to hold in the hollow region as well, the vector field $\boldsymbol{w}$ must be well defined when approaching the toroidal axis. This is often the case, as it will be shown in the example constructed in section 6.*

***Remark* 2.** *In the study of the vorticity equation for fluid flows over two-dimensional surfaces parametrized by Ψ and embedded in three-dimensional Euclidean space, the relationship between the component of the vorticity $\omega^\Psi = \boldsymbol{\omega}\cdot\nabla\Psi$ and the stream function Θ is precisely $\nabla\cdot[\nabla\Psi\times(\nabla\Theta\times\nabla\Psi)] = -\omega^\Psi$ (see [31]). The result of lemma 1 thus implies that one can solve for the stream function Θ knowing the vorticity ω^Ψ. Notice in particular that the topology of the level sets of Ψ does not need to be toroidal.*

5 Example of numerical solution

The aim of this section is to provide a numerical example of solution of equation (43). This example will also clarify the role played by periodic boundary conditions in ensuring the regularity of the solution $\boldsymbol{w}$ of (1) and its derivatives. To this end, we consider a family of toroidal surfaces corresponding to level sets of the function

$$\Psi = \frac{1}{2}\left(r - r_0\right)^2 + \frac{1}{2}\mathcal{E}\left(z - h\right)^2, \tag{66}$$

with $h = h\left(\varphi, z\right)$, $r_0 > 0$ and $\mathcal{E} > 0$ real constants, and (r, φ, z) cylindrical coordinates. The constant r_0 represents the major radius of the torus. When $\mathcal{E} = 1$ and $h = 0$, level sets of (66) correspond to axially symmetric toroidal surfaces enclosing a toroidal volume Ω with circular cross-section $\Sigma_\varphi = \{\boldsymbol{x} \in \Omega : \varphi = \varphi_0 \in [0, 2\pi)\}$. If $\mathcal{E} \neq 1$ the cross-sections Σ_φ depart from circles, while a non-zero h can be regarded as a displacement of the toroidal axis in the vertical direction. Figure 2 shows examples of toroidal surfaces obtained as level sets of (66). Notice that an appropriate choice of the function h breaks the rotational (axial) symmetry of the

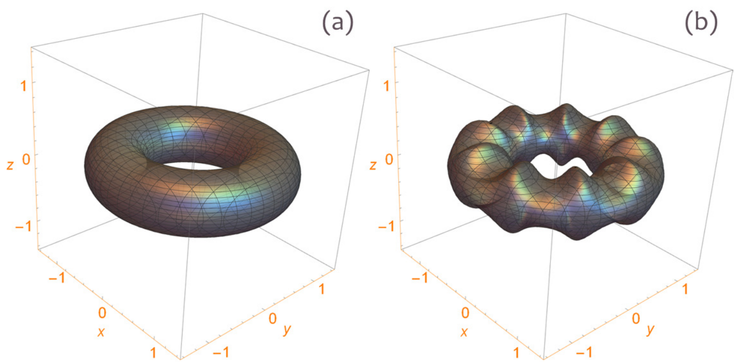

Figure 2: (a) Axially symmetric torus corresponding to the level set $\Psi = 0.08$ with $r_0 = 1$, $\mathcal{E} = 1$, and $h = 0$ in equation (66). (b) Torus corresponding to the level set $\Psi = 0.08$ with $r_0 = 1$, $\mathcal{E} = 1.6$, $h = 0.3z\sin\left(9\varphi\right)$ in equation (66).

surface. More generally, it is possible to construct toroidal surfaces that are not invariant under continuous Euclidean isometries (combinations of translations and rotations). For example, setting $h = \epsilon z \sin\left(m\varphi\right)$ with $\epsilon > 0$ a real constant and $m \neq 0$ an integer, the corresponding toroidal surface is not invariant under continuous Euclidean isometries. Indeed, the Lie-derivative

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi = \left(\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}\right) \cdot \nabla\Psi, \tag{67}$$

where $\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{\xi}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, is the generator of continuous Euclidean isometries in $\mathbb{R}^3$, vanishes only if $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$. This can be verified by evaluating (67) on the planes $z = 0$, $\varphi = 0$, and $\varphi = \pi/2$, which respectively give the conditions $a_x = a_y = 0$, $b_x = b_z = 0$, and $b_y = a_z = 0$. Therefore, the example shown in figure 2(b) is not invariant under continuous Euclidean isometries.

In order to construct a solution $\boldsymbol{w}$ of system (43), we must now define angle coordinates $\mu, \nu \in [0, 2\pi)$ spanning the toroidal surfaces Ψ. In particular, we consider the curvilinear coordinates $(\mu, \nu, \Psi) = (\varphi, \vartheta, \Psi)$ with

$$\vartheta = \arctan\left(\frac{z}{r - r_0}\right), \tag{68}$$

the poloidal angle. The contravariant components of the metric tensor can be evaluated to be

$$\begin{aligned} &g^{\varphi\varphi} = \frac{1}{r^2}, \quad g^{\varphi\vartheta} = 0, \quad g^{\vartheta\vartheta} = \frac{1}{z^2 + \left(r - r_0\right)^2}, \quad g^{\varphi\Psi} = -\mathcal{E}\frac{z - h}{r^2}h_\varphi, \\ &g^{\vartheta\Psi} = \frac{r - r_0}{z^2 + \left(r - r_0\right)^2}\left[\mathcal{E}\left(z - h\right)\left(1 - h_z\right) - z\right], \quad g^{\Psi\Psi} = \left(r - r_0\right)^2 + \mathcal{E}^2\left(z - h\right)^2\left[\left(1 - h_z\right)^2 + \frac{h_\varphi^2}{r^2}\right]. \end{aligned} \tag{69}$$

The covariant components are

$$g_{\varphi\varphi}=\frac{g^{\vartheta\vartheta}g^{\Psi\Psi}-\left(g^{\vartheta\Psi}\right)^2}{J^2},\quad g_{\varphi\vartheta}=\frac{g^{\vartheta\Psi}g^{\varphi\Psi}}{J^2},\quad g_{\vartheta\vartheta}=\frac{g^{\varphi\varphi}g^{\Psi\Psi}-\left(g^{\varphi\Psi}\right)^2}{J^2},$$
$$g_{\varphi\Psi}=-\frac{g^{\varphi\Psi}g^{\vartheta\vartheta}}{J^2},\quad g_{\vartheta\Psi}=-\frac{g^{\varphi\varphi}g^{\vartheta\Psi}}{J^2},\quad g_{\Psi\Psi}=\frac{g^{\varphi\varphi}g^{\vartheta\vartheta}}{J^2}. \tag{70}$$

We also have

$$J=\nabla\varphi\cdot\nabla\vartheta\times\nabla\Psi=\frac{\left(r-r_0\right)^2+\mathcal{E}z\left(1-h_z\right)\left(z-h\right)}{r\left[z^2+\left(r-r_0\right)^2\right]}. \tag{71}$$

Next, let us consider a vertical axial displacement $h=\epsilon z\sin\left(m\varphi\right)$. In this case, the inverse coordinate transformation reads

$$z^2=\frac{2\Psi\sin^2\vartheta}{\cos^2\vartheta+\mathcal{E}\left[1-\epsilon\sin\left(m\varphi\right)\right]^2\sin^2\vartheta},\quad \left(r-r_0\right)^2=\frac{2\Psi\cos^2\vartheta}{\cos^2\vartheta+\mathcal{E}\left[1-\epsilon\sin\left(m\varphi\right)\right]^2\sin^2\vartheta}. \tag{72}$$

Using (72), the metric coefficients (70) and the Jacobian (71) can be expressed explicitly as functions of (μ,ν,Ψ). Hence, we may attempt to solve equation (43) in the doubly periodic domain D. If a periodic solution ρ with periodic derivatives could be found, the corresponding vector field $\boldsymbol{w}=\nabla\Psi\times\nabla\Theta$ with $\Theta=\mu+\rho$ would provide the desired solution in Ω. However, this task is not trivial, since the space of solutions is effectively restricted to functions $\rho=\sum_{m,n}c_{mn}\left(\Psi\right)e^{\mathrm{i}(m\mu+n\nu)}$ that are represented by a convergent Fourier series in the variables μ and ν. Nevertheless, numerical solutions can be obtained in a rather straightforward fashion by sacrificing the continuity of the partial derivatives ρ_μ and ρ_ν (and thus the continuity of $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$, recall (45)) on the points $\boldsymbol{x}\in\Omega$ corresponding to $(\mu,\nu)\in\partial D$. Indeed, coupling equation (43) with Dirichlet boundary conditions

$$\rho\left(0,\nu,\Psi\right)=\rho\left(2\pi,\nu,\Psi\right)=\rho\left(\mu,0,\Psi\right)=\rho\left(\mu,2\pi,\Psi\right)=0, \tag{73}$$

results in a usual elliptic problem that can be approached with standard numerical tools. The corresponding solution ρ will be periodic in the variables μ and ν, although only the partial derivative of ρ tangential to the boundary ∂D will be periodic, while the normal component will not. For completeness, we also note that the regularity of the function ρ at the boundary ∂D is obstructed by the corners of the square domain D, which give $\rho\in C^{1,\alpha}(\bar{D})$ with Hölder coefficient $0<\alpha<1$ (see [32, 33]).

Figure 3 shows two examples of numerical solution of equation (43) with Dirichlet boundary conditions (73). Notice that while ρ is periodic in μ and ν, ρ_μ is periodic only between $\nu=0$ and $\nu=2\pi$, while it takes different values at $\mu=0$ and $\mu=2\pi$. Analogous considerations apply to ρ_ν, $\Theta_\mu=1+\rho_\mu$, and $\Theta_\nu=\rho_\nu$. As already explained at the end of section 3, this implies that the corresponding vector fields $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$, which are given by (45), will exhibit discontinuities at the points $\boldsymbol{x}\in\Omega$ corresponding to $(\mu,\nu)\in\partial D$. Finally, we remark that, in principle, the smooth vector field $\boldsymbol{w}$ constructed in theorem 1 could be numerically computed by expanding in Fourier series each term in equation (43) and by solving for the Fourier coefficients of the solution ρ.

6 Example of smooth solution and relation with anisotropic magnetohydrodynamics

In this section we construct an example of smooth solution $\boldsymbol{w}\in C^\infty\left(\Omega\right)$ of equation (1) such that $\left(\nabla\times\boldsymbol{w}\right)\times\boldsymbol{w}\neq\boldsymbol{0}$ in a toroidal domain Ω with the aid of proposition 1.

Note that the proof of proposition 1 follows by direct evaluation of $\left(\nabla\times\boldsymbol{w}\right)\times\boldsymbol{w}$ and $\nabla\cdot\boldsymbol{w}$. Recall also that the dimension of the linear space $L^2_H\left(\Omega\right)$ is given by the genus of $\partial\Omega$. For a toroidal surface with genus 1 the space of harmonic vector fields $\boldsymbol{\xi}\in L^2_H\left(\Omega\right)$ is therefore 1-dimensional. We refer the reader to [34, 35] for additional details on harmonic vector fields, which arise in the context of Hodge decomposition of differential forms.

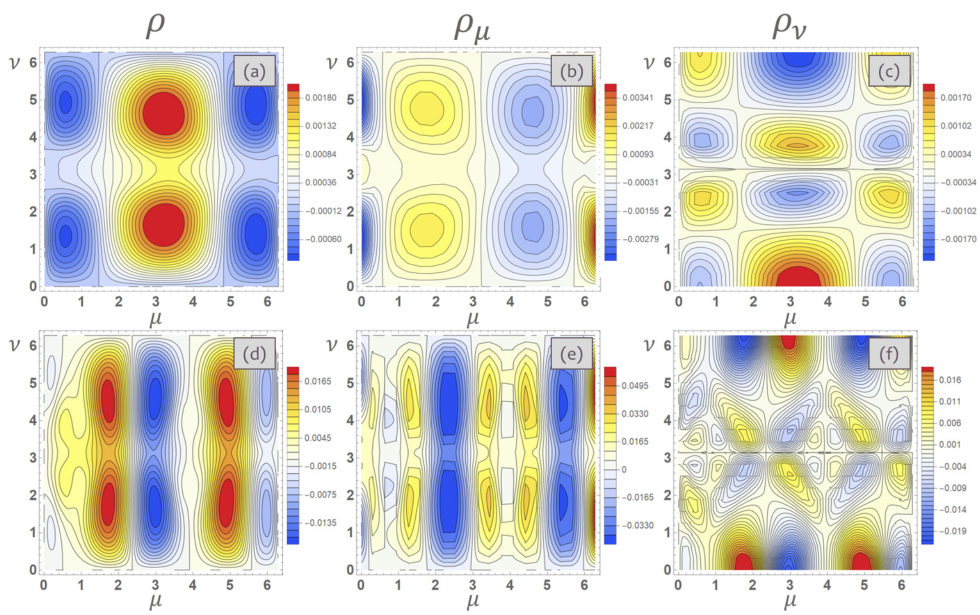

Figure 3: (a), (b), and (c): numerical solution ρ of equation (43) with Dirichlet boundary conditions (73) and its partial derivatives ρ_μ and ρ_ν for $r_0 = 1$, $m = 1$, $\mathcal{E} = 1.6$, $h = \epsilon z \sin(m\varphi)$, and $\epsilon = 0.03$ on the toroidal surface $\Psi = 0.16$, with Ψ given by equation (66). (d), (e), and (f): numerical solution ρ of equation (43) with Dirichlet boundary conditions (73) and its partial derivatives ρ_μ and ρ_ν for $r_0 = 1$, $m = 2$, $\mathcal{E} = 1.6$, $h = \epsilon z \sin(m\varphi)$, and $\epsilon = 0.3$ on the toroidal surface $\Psi = 0.08$, with Ψ given by equation (66).

Proposition 1 suggests that solutions of equations (1) can be obtained by identifying harmonic vector fields foliated by toroidal surfaces. The prototypical example of such vector field is the gradient $\boldsymbol{\xi}_0 = \nabla\varphi$ of the toroidal angle φ, which is tangential to axially symmetri tori parametrized by $\Psi_0 = \frac{1}{2}\left[(r-r_0)^2 + z^2\right]$. To break axial symmetry, we proceed as follows. First, we perturb the toroidal angle according to

$$\eta = \varphi + \epsilon\sigma, \tag{74}$$

where σ is chosen to be a harmonic function so that the vector field

$$\boldsymbol{\xi}_\epsilon = \nabla\eta = \nabla\varphi + \epsilon\nabla\sigma \in L^2_H(\Omega), \tag{75}$$

is a harmonic vector field in a toroidal domain Ω whose precise shape has yet to be determined. In particular, for sufficiently small $\epsilon > 0$, we expect to find a function Ψ_ϵ such that $\boldsymbol{\xi}_\epsilon \cdot \nabla\Psi_\epsilon = 0$ and the level sets of Ψ_ϵ define toroidal surfaces. Indeed, the limit $\epsilon \to 0$ corresponds to the axially symmetric case of the vector field $\boldsymbol{\xi}_0 = \nabla\varphi$ tangential to contours of Ψ_0. A simple choice for the perturbation is the harmonic function $\sigma = r^m \cos(m\varphi)$, $m \in \mathbb{Z}$, $m \neq 0$. For example, take $m = 1$ so that $\sigma = r\cos\varphi = x$. Then, the following orthogonality condition must be solved for Ψ_ϵ,

$$\boldsymbol{\xi}_\epsilon \cdot \nabla\Psi_\epsilon = \frac{1}{r^2}\left(1 - \epsilon r \sin\varphi\right)\frac{\partial\Psi_\epsilon}{\partial\varphi} + \epsilon\cos\varphi\frac{\partial\Psi_\epsilon}{\partial r} = 0. \tag{76}$$

One can verify that a solution is given by the function

$$\Psi_\epsilon = \frac{1}{2}\left[\left(re^{-\epsilon y} - r_0\right)^2 + z^2\right], \tag{77}$$

where $r_0 > 0$ is a real constant. Contours of Ψ_ϵ define toroidal surfaces as shown in figure 4. Notice also that both Ψ_ϵ and the vector field

$$\boldsymbol{w} = f(\Psi_\epsilon)\,\boldsymbol{\xi}_\epsilon, \tag{78}$$

are smooth within the toroidal volume Ω enclosed by Ψ_ϵ for a suitable choice of $f(\Psi_\epsilon)$. In addition,

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = -\frac{1}{2}\frac{\partial f^2}{\partial \Psi_\epsilon}\frac{(1-\epsilon y)^2 + \epsilon^2 x^2}{r^2}\nabla \Psi_\epsilon, \qquad \nabla \cdot \boldsymbol{w} = 0. \tag{79}$$

A plot of the vector field (78) for $f = \exp\{\Psi_\epsilon/2\}$ is given in figure 4. It should be noted that both $\boldsymbol{w}$ and $\nabla \times \boldsymbol{w}$ diverge when $r \to 0$, a fact that makes the constructed solution unphysical in $\mathbb{R}^3$ (an infinite current $\nabla \times \boldsymbol{w}$ would be needed on the vertical axis to sustain such a magnetic field $\boldsymbol{w}$). Nevertheless, this divergence is not worrisome as it is analogous to the divergence of the magnetic field $\boldsymbol{B} \propto \nabla\varphi$ generated by a straight current flowing along the vertical axis. Notice also that Ψ_ϵ and $|\nabla \Psi_\epsilon|$ diverge at large distances from the origin, but the corresponding divergences in $\boldsymbol{w}$ and $\nabla \times \boldsymbol{w}$ can be suppressed by appropriate choice of f.

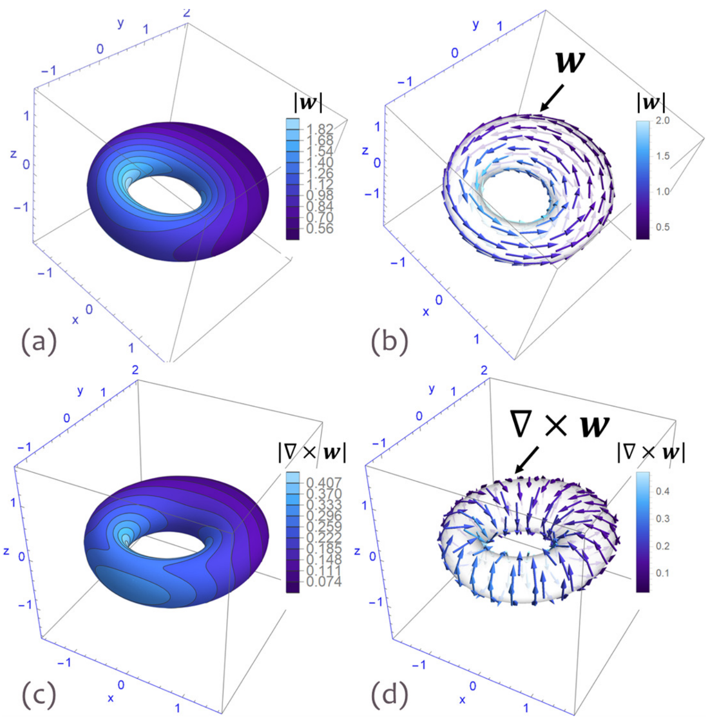

Figure 4: (a) Contour plot of $|\boldsymbol{w}|$ over the level set $\Psi_\epsilon = 0.08$. (b) Vector plot of $\boldsymbol{w}$ over the level set $\Psi_\epsilon = 0.08$. (c) Contour plot of $|\nabla \times \boldsymbol{w}|$ over the level set $\Psi_\epsilon = 0.08$. (d) Vector plot of $\nabla \times \boldsymbol{w}$ over the level set $\Psi_\epsilon = 0.08$. In (a), (b), (c) and (d) Ψ_ϵ is defined by equation (77) with $r_0 = 1$ and $\epsilon = 0.18$, and $\boldsymbol{w}$ is defined by equation (78) with $f = \exp\{\Psi_\epsilon/2\}$ and $\boldsymbol{\xi}_\epsilon = \nabla(\varphi + \epsilon r \cos\varphi)$. Observe that the vector field $\boldsymbol{w}$ shown here is a solution of (1) such that both the bounding surface Ψ_ϵ and the vector field $\boldsymbol{w}$ are not invariant under continuous Euclidean isometries.

Let us now verify that both Ψ_ϵ and the vector field $\boldsymbol{w}$ defined in equation (78) are not endowed with continuous Euclidean isometries. Following the same procedure of section 2, we must determine constant vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ such that the Lie derivative below vanishes,

$$\begin{aligned}\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon =& e^{-\epsilon y}\frac{re^{-\epsilon y} - r_0}{r}\left[(a_x + b_y z - b_z y)\,x + (a_y + b_z x - b_x z)\left(y - \epsilon r^2\right)\right] \\ &+ (a_z + b_x y - b_y x)\,z = 0\end{aligned} \tag{80}$$

Considering the section $\varphi = \pi/2$, one obtains the condition

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = (a_y - b_x z)\,e^{-\epsilon r}\left(re^{-\epsilon r} - r_0\right)(1-\epsilon r) + z\,(a_z + b_x r) = 0. \tag{81}$$

When $z = 0$, the expression above holds only if $a_y = 0$. Then, for $z \neq 0$ it follows that $a_z = b_x = 0$ as well. Similarly, at $\varphi = 0$, one has

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = (r - r_0)\left(a_x + b_y z - \epsilon b_z r^2\right) - b_y r z = 0. \tag{82}$$

When $z=0$, this quantity vanishes for arbitrary r provided that $a_x=b_z=0$. We therefore conclude that $b_y=0$ as well, and Ψ_ϵ is not invariant under continuous Euclidean isometries. To ascertain that the vector field $\boldsymbol{w}=f\left(\Psi_\epsilon\right)\boldsymbol{\xi}_\epsilon$ is not invariant under the same class of transformations, it is sufficient to study the symmetry of its modulus. Indeed, by standard vector identities

$$\boldsymbol{w}\cdot\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}=\frac{1}{2}\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}^2. \tag{83}$$

Hence, if the Lie-derivative of the modulus $\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}^2$ does not vanish, the Lie-derivative of the vector field $\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}$ does not vanish as well. Next, observe that

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}^2=\boldsymbol{w}^2\left[\left(\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}\right)\cdot\nabla\log f^2+\left(\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}\right)\cdot\nabla\log\left|\boldsymbol{\xi}_\epsilon\right|^2\right]=0. \tag{84}$$

Consider, for example, the case $f^2=\exp\left\{\Psi_\epsilon\right\}$. We have

$$\begin{aligned}\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}^2=&\boldsymbol{w}^2\left\{\left(a_x+b_yz-b_zy\right)\left[xe^{-\epsilon y}\frac{re^{-\epsilon y}-r_0}{r}+2x\left(\frac{\epsilon^2}{\left(1-\epsilon y\right)^2+\epsilon^2x^2}-\frac{1}{r^2}\right)\right]\right.\\&+\left(a_y+b_zx-b_xz\right)\left[e^{-\epsilon y}\frac{re^{-\epsilon y}-r_0}{r}\left(y-\epsilon r^2\right)-2\left(\frac{y}{r^2}+\epsilon\frac{1-\epsilon y}{\left(1-\epsilon y\right)^2+\epsilon^2x^2}\right)\right]\\&\left.\left(a_z+b_xy-b_yx\right)z\right\}=0.\end{aligned} \tag{85}$$

On the section $\varphi=\pi/2$, we obtain the condition

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}^2=\boldsymbol{w}^2\left\{\left(a_y-b_xz\right)\left[e^{-\epsilon r}\left(re^{-\epsilon r}-r_0\right)\left(1-\epsilon r\right)-2\left(\frac{1}{r}+\frac{\epsilon}{1-\epsilon r}\right)\right]+\left(a_z+b_xr\right)z\right\}. \tag{86}$$

Setting $z=0$ leads to $a_y=0$. Since r and z are not constants, it thus follows that the equation above can be satisfied only if $a_z=b_x=0$ as well. Next, on the section $\varphi=0$ we have

$$\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\boldsymbol{w}^2=\boldsymbol{w}^2\left\{\left(a_x+b_yz\right)\left[r-r_0+2r\left(\frac{\epsilon^2}{1+\epsilon^2r^2}-\frac{1}{r^2}\right)\right]-\epsilon b_zr\left[r\left(r-r_0\right)+\frac{2}{1+\epsilon^2r^2}\right]-b_yrz\right\}. \tag{87}$$

Considering the case $z=0$, it follows that $a_x=b_z=0$. But then $b_y=0$ as well. Hence, for $f^2=\exp\left\{\Psi_\epsilon\right\}$, the modulus $\boldsymbol{w}^2$ (and thus $\boldsymbol{w}$) is not invariant under continuous Euclidean isometries.

The solution constructed above can be generalized to a wider class of solutions with the aid of two-dimensional harmonic conjugate functions $P\left(x,y\right)$ and $Q\left(x,y\right)$ such that $\partial P/\partial x=\partial Q/\partial y$ and $\partial P/\partial y=-\partial Q/\partial x$. Explicitly, we can define the family of solutions $\boldsymbol{w}=f\left(\Psi_\epsilon\right)\boldsymbol{\xi}_\epsilon$ to (1) with

$$\boldsymbol{\xi}_\epsilon=\nabla\left[\varphi+\epsilon P\left(x,y\right)\right], \tag{88a}$$

$$\Psi_\epsilon=\frac{1}{2}\left\{\left[re^{-\epsilon Q(x,y)}-r_0\right]^2+z^2e^{-\epsilon S(z)}\right\}, \tag{88b}$$

where $S\left(z\right)$ is a function of z. For example, setting $P=e^{mx}\cos\left(my\right)$, $Q=e^{mx}\sin\left(my\right)$, $S=2\sin z$, with $m\in\mathbb{R}$, $m\neq 0$, generates solutions $\boldsymbol{w}=f\left(\Psi_\epsilon\right)\boldsymbol{\xi}_\epsilon$ of (1) without continuous Euclidean isometries. We also remark that such solutions do not possess discrete Euclidean isometries (reflections) as well. To see this, first note that Ψ_ϵ is no longer invariant under the transformation $z\to -z$. Invariance under other reflections can be excluded as follows. Let $\boldsymbol{n}=\left(n_x,n_y,n_z\right)\in\mathbb{R}^3$, $\boldsymbol{n}\neq\boldsymbol{0}$, denote the unit normal of a plane corresponding to a level set of the function $\zeta=\boldsymbol{n}\cdot\boldsymbol{x}$, i.e. $\boldsymbol{n}=\nabla\zeta$ with $\boldsymbol{n}^2=1$. Next, choose $\boldsymbol{t},\boldsymbol{v}\in\mathbb{R}^3$, with $\boldsymbol{t}\cdot\boldsymbol{v}=\boldsymbol{t}\cdot\boldsymbol{n}=\boldsymbol{v}\cdot\boldsymbol{n}=0$, $\boldsymbol{t}^2=\boldsymbol{v}^2=1$, define $\eta=\boldsymbol{t}\cdot\boldsymbol{x}$, $\theta=\boldsymbol{v}\cdot\boldsymbol{x}$, and perform the change of coordinates $\left(x,y,z\right)\to\left(\zeta,\eta,\theta\right)$. Notice that the set $\left(\zeta,\eta,\theta\right)$ is orthonormal by construction. A function $\Psi_\epsilon\left(\zeta,\eta,\zeta\right)$ is endowed with a reflection symmetry R_ζ by the plane ζ provided that

$$\Psi_\epsilon\left(\zeta,\eta,\theta\right)=R_\zeta\Psi_\epsilon\left(\zeta,\eta,\theta\right)=\Psi_\epsilon\left(-\zeta,\eta,\theta\right). \tag{89}$$

Using the fact that the Cartesian coordinates (x, y, z) are linear functions of the new coordinates (ζ, η, θ), one can verify that there exist no nontrivial choice of the vector $\boldsymbol{n}$ such that (89) is satisfied. Furthermore, the function Ψ_ϵ cannot be invariant under combinations of continuous and discrete Euclidean isometries, because after a reflection R_ζ one can always define a new set of Cartesian coordinates $\boldsymbol{x}' = (R_\zeta x, R_\zeta y, R_\zeta z)$ that preserve the functional form of Ψ_ϵ, i.e. $R_\zeta \Psi_\epsilon = \Psi_\epsilon\,(x \to R_\zeta x, y \to R_\zeta y, z \to R_\zeta z)$, implying that $\mathfrak{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}'}\Psi_\epsilon = \boldsymbol{0}$ if and only if $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$.

It should be noted that the vector fields (78) constructed above can be regarded as steady solutions of anisotropic magnetohydrodynamics,

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla \cdot \Pi, \qquad \nabla \cdot \boldsymbol{w} = 0 \quad \text{in} \;\; \Omega, \tag{90}$$

where the Cartesian components of the pressure tensor Π are given by [36, 37]

$$\Pi^{ij} = \left(P - \frac{1}{2}\gamma \boldsymbol{w}^2\right)\delta^{ij} + \gamma w^i w^j, \qquad i, j = 1, 2, 3, \tag{91}$$

with P a reference pressure field and γ the pressure anisotropy. Notice that equation (2) is recovered for $\gamma = 0$. The first equation in (90) expressing anisotropic force balance can be written as

$$(1-\gamma)\,(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla P - \frac{1}{2}\boldsymbol{w}^2 \nabla\gamma + (\boldsymbol{w} \cdot \nabla\gamma)\,\boldsymbol{w}. \tag{92}$$

Evidently, the vector field (78) satisfies (92) with

$$P = 0, \qquad \gamma = 1 - \frac{1}{f^2}. \tag{93}$$

Finally, we remark that (78) is well defined along the toroidal axis $re^{-\epsilon Q} \to r_0$, $z \to 0$ provided that f exists in this limit (this is the case of $f^2 = \exp\{\Psi_\epsilon\}$ considered above).

7 Considerations on magnetohydrodynamic equilibria, steady Euler flows, and quasisymmetry

In this last section we discuss some aspects pertaining to the application of the theory developed in the previous sections to the analysis of equation (2).

First recall that solutions $\boldsymbol{w}$ of equation (2) are solutions of equation (1). Therefore, the space of solutions of equation (2) is a subset of the space of solutions of equation (1). Next, observe that the difference between equation (1) and equation (2) is that while in the former the vector field $(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}$ is only required to lie along $\nabla\Psi$, in the latter these two vector fields must coincide. Hence, in addition to the orthogonality between $\nabla \times \boldsymbol{w}$ and $\nabla\Psi$ as described by equation (28), a further condition exists on the magnitude of the component of $(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}$ along $\nabla\Psi$. In particular, enforcing the Clebsch representation $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta$, equation (2) can be written as

$$[\nabla \times (\nabla\Psi \times \nabla\Theta)] \times (\nabla\Psi \times \nabla\Theta) = [\nabla\Theta \cdot \nabla \times (\nabla\Psi \times \nabla\Theta)]\,\nabla\Psi - [\nabla\Psi \cdot \nabla \times (\nabla\Psi \times \nabla\Theta)]\,\nabla\Theta = \nabla\Psi. \tag{94}$$

Hence, one obtains the system of equations

$$\nabla \cdot [\nabla\Theta \times (\nabla\Psi \times \nabla\Theta)] = -1, \qquad \nabla \cdot [\nabla\Psi \times (\nabla\Theta \times \nabla\Psi)] = 0 \quad \text{in} \;\; \Omega, \tag{95a}$$
$$\Psi = \text{constant} \quad \text{on} \;\; \partial\Omega. \tag{95b}$$

While in the study of equation (43) the function Ψ was given, it is convenient to regard system (95) as coupled partial differential equations for the unknowns Ψ and Θ. Indeed, one expects that fixing Ψ will prevent, in general, the existence of regular solutions Θ fulfilling both equations in (95a). Notice that boundary conditions (95b) on Ψ have been imposed to ensure that $\boldsymbol{w}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. Evidently, a solution (Ψ,Θ) of system (95) provides a solution $\boldsymbol{w}=\nabla\Psi\times\nabla\Theta$ of equation (2).

It is worth observing that if the condition $\nabla\cdot\boldsymbol{w}=0$ is dropped in equation (2), it is possible to find explicit solutions of $(\nabla\times\boldsymbol{w})\times\boldsymbol{w}=\nabla\Psi$ that break axial symmetry. Considering axially symmetric toroidal surfaces corresponding to level sets of $\Psi=\frac{1}{2}\left[(r-r_0)^2+z^2\right]$, examples include vector fields of the type

$$\boldsymbol{w}=\sqrt{C-2\Psi^2}\nabla\vartheta+g\left(\varphi\right)\nabla\varphi, \tag{96}$$

with $\vartheta=\arctan\left(z/\left(r-r_0\right)\right)$ the poloidal angle, $\varphi=\arctan\left(y/x\right)$ the toroidal angle, $C>0$ a sufficiently large real constant, and g any periodic function of φ.

As in the case of equation (43), equations (95a) admits a variational formulation. The target energy functional is

$$E'_\Omega=\int_\Omega\left(\frac{1}{2}\left|\nabla\Psi\times\nabla\Theta\right|^2-\Psi\right)dV. \tag{97}$$

where the variable $\Psi=P$ plays the role of the mechanical pressure P in the context of magnetohydrodynamics, and corresponds to the sum $\Psi=-P-\frac{1}{2}\boldsymbol{v}^2$ in the hydrodynamic interpretation with $\boldsymbol{v}$ the fluid velocity. Assuming $\delta\Psi=\delta\Theta=0$ on $\partial\Omega$, we have

$$\delta E'_\Omega=-\int_\Omega\delta\Psi\left\{1+\nabla\cdot\left[\nabla\Theta\times\left(\nabla\Psi\times\nabla\Theta\right)\right]\right\}dV-\int_\Omega\delta\Theta\nabla\cdot\left[\nabla\Psi\times\left(\nabla\Theta\times\nabla\Psi\right)\right]dV. \tag{98}$$

Hence, stationary points of the functional E'_Ω assign solutions of (95a). Now suppose that solutions (Ψ,Θ) of (95) are sought in the Sobolev space $H^1\left(\Omega\right)$ with norm $\left|\left|\cdot\right|\right|_{H^1(\Omega)}$. From (97) it is clear that the functional E'_Ω is not coercive, i.e. it does not satisfy a condition of the form $E'_\Omega\geq c_1\left|\left|\Psi\right|\right|^2_{H^1(\Omega)}+c_2\left|\left|\Theta\right|\right|^2_{H^1(\Omega)}+C$ for some constants with $c_1,c_2,C\in\mathbb{R}$, $c_1>0$, and $c_2>0$. Indeed, the value of (97) can be kept finite, $|E'_\Omega|<\infty$, even if $\left|\left|\Psi\right|\right|_{H^1(\Omega)},\left|\left|\Theta\right|\right|_{H^1(\Omega)}\to\infty$ by setting $\Theta=\Psi$ while taking $\left|\left|\nabla\Psi\right|\right|_{L^2(\Omega)}=\left|\left|\nabla\Theta\right|\right|_{L^2(\Omega)}\to\infty$ where $\left|\left|\cdot\right|\right|_{L^2(\Omega)}$ denotes the standard $L^2\left(\Omega\right)$ norm. The lack of coercivity prevents the application of variational methods [38] to establish the existence of a relative minimizer of E'_Ω, and thus a solution of (95) in the relevant function space.

It is worth however explaining why the situation is different if the variable Ψ is fixed, i.e. if one considers equation (28) arising from the functional E_Ω of (32) in the context of equation (1). This will also provide explicit proof of the (weak) solvability of equation (43) for the unkwon ρ on each toroidal surface $\Psi=$ constant. Consider the setting of theorem 1 where Ψ is smooth, perform the change of variables $\Theta=\mu+\rho$, and use curvilinear coordinates $\left(x^1,x^2,x^3\right)=\left(\mu,\nu,\Psi\right)$ to express E_Ω as follows

$$E_\Omega=\frac{1}{2}\int_U d\Psi\int_D\left(\sum_{i,j=1}^{2}a_{ij}\rho_i\rho_j+2\sum_{i=1}^{2}a_{\mu i}\rho_i+a_{\mu\mu}\right)Jd\mu d\nu, \tag{99}$$

where a_{ij} are the components of the symmetric positive definite matrix A encountered in equation (37). Assuming that ρ is periodic in D, integration by parts gives

$$E_\Omega=\frac{1}{2}\int_U d\Psi\int_D\left[\sum_{i,j=1}^{2}Ja_{ij}\rho_i\rho_j-2\rho\sum_{i=1}^{2}\frac{\partial\left(Ja_{\mu i}\right)}{\partial x^i}+Ja_{\mu\mu}\right]d\mu d\nu. \tag{100}$$

For each $\Psi\in U$ we may therefore identify an energy functional

$$E_D=\frac{1}{2}\int_D\left[\sum_{i,j=1}^{2}Ja_{ij}\rho_i\rho_j-2\rho\sum_{i=1}^{2}\frac{\partial\left(Ja_{\mu i}\right)}{\partial x^i}+Ja_{\mu\mu}\right]d\mu d\nu\geq\lambda\left|\left|\nabla_{(\mu,\nu)}\rho\right|\right|^2_{L^2(D)}-2c\left|\left|\rho\right|\right|_{L^2(D)}-C. \tag{101}$$

Here, λ, c and C are positive real constants, and we used the strict ellipticity of Ja_{ij}. Recalling that $\langle\rho\rangle = 0$ and applying the Poincaré inequality, we further obtain

$$E_D \geq \frac{\lambda}{4}\,||\rho||^2_{H^1(D)} + \left(\frac{\lambda}{4}\,||\rho||_{H^1(D)} - 2c\right)||\rho||_{H^1(D)} - C \geq \frac{\lambda}{4}\,||\rho||^2_{H^1(D)} - 4\frac{c^2}{\lambda} - C. \tag{102}$$

This shows that E_D is a coercive functional with respect to the $H^1(D)$ norm since $E_D \to \infty$ when $||\rho||_{H^1(D)} \to \infty$. Since E_D is also sequentially lower-semicontinuous, for each Ψ there exist a relative minimizer $\rho \in H^1_{\mathrm{per}}(D)$ of the functional E_D, which corresponds to a solution of equation (43).

We conclude this section with an observation concerning the existence of quasisymmetric solutions of equation (2), i.e. solutions of equation (2) that satisfy the additional property

$$\boldsymbol{u} \times \boldsymbol{w} = \nabla g(\Psi), \quad \boldsymbol{u} \cdot \nabla \boldsymbol{w}^2 = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{103}$$

for some function $g(\Psi)$ such that $\nabla g \neq \boldsymbol{0}$ and some vector field $\boldsymbol{u}$ called the quasisymmetry of $\boldsymbol{w}$. The property (103) is a desirable feature for the confining magnetic field in nuclear fusion reactors known as stellarators, because it ensures steady confinement of the burning plasma within a finite volume of space [19]. In this regard, we have:

Proposition 2. *Suppose that $\boldsymbol{\xi} \in L^2_H(\Omega)$ is a harmonic vector field in a toroidal domain Ω foliated by nested toroidal surfaces corresponding to contours of a function $\Psi \in C^1(\bar{\Omega})$. Further assume that*

$$\boldsymbol{\xi} \cdot \nabla\Psi = 0 \quad \text{in } \Omega, \tag{104}$$

and that $|\boldsymbol{\xi}|^2 = |\boldsymbol{\xi}|^2(\Psi)$. Then, the vector field $\boldsymbol{w} = f(\Psi)\,\boldsymbol{\xi} \in H^1_{\sigma\sigma}(\Omega)$, with $f \in C^1(\bar{\Omega})$, solves (2) and is quasisymmetric with quasisymmetry

$$\boldsymbol{u} = \boldsymbol{\xi} \times \nabla\Psi \in L^2_\sigma(\Omega). \tag{105}$$

The proof of the above statement can be obtained by evaluating equations (2) and (103). We remark that, however, the requirement $|\boldsymbol{\xi}|^2 = |\boldsymbol{\xi}|^2(\Psi)$ that the modulus of the harmonic vector field $\boldsymbol{\xi}$ is a function of Ψ is a stringent condition related to the notion of isodynamic magnetic field [39]. Therefore, the existence of such configurations is nontrivial.

8 Concluding remarks

In this paper, we studied equation (1), which determines a solenoidal vector field with the property that both the vector field ans its curl are foliated by a family of nested toroidal surfaces. Equation (1) represents a generalization of an equation encountered in magnetohydrodynamics and fluid mechanics (equation (2)) describing equilibrium magnetic fields and steady Euler flows. At present, a general theory concerning the existence of solutions of equation (2) is not available due to the mathematical difficulty originating from its nontrivial characteristic surfaces. Analysis of the simpler problem posed by equation (1) may therefore provide useful insight into the nature of the space of solutions of equation (2).

In theorem 1 we showed that nontrivial solutions in the class $C^\infty(\Omega)$ of equation (1), where Ω is a hollow toroidal volume, always exist for a given family of smooth nested toroidal surfaces. The proof relies on the reduction of equation (1) to a two-dimensional linear elliptic second order partial differential equation (43) for each toroidal surface with the aid of Clebsch parameters. Regular periodic solutions for these equations exist by elliptic theory, and can be used to determine the desired smooth solution of problem (1). In section 5, an example of numerical solution was also computed, while in section 6 examples of smooth solutions in toroidal volumes were constructed analytically such that both the bounding surface and the solution are not invariant under continuous Euclidean isometries. Such solutions can be regarded as solutions of anisotropic magnetohydrodynamics (90).

The results obtained above concerning equation (1) entail a number of consequences for the problem described by equation (2). First, the formulation of equation (2) in terms of Clebsch potentials (equation

(95)) discussed in section 7 suggests that simultaneous optimization of the Clebsch potentials Ψ and Θ is needed to find solutions. That is, the shape of the toroidal surfaces Ψ (and possibly the profile of the domain Ω itself) should be adjusted together with the variable Θ to accommodate the solution within Ω. Secondly, if solutions are sought in the form $\boldsymbol{w} = f(\Psi)\,\boldsymbol{\xi}$ of (78), where $\boldsymbol{\xi}$ is a harmonic vector field in Ω, solving (2) amounts to finding a harmonic vector field that is foliated by toroidal surfaces Ψ and such that the modulus $|\boldsymbol{\xi}|^2$ is itself a function of Ψ. Finally, as observed in proposition 2 of section 7, if such kind of solution could be found, it would also guarantee quasisymmetry, and thus magnetic confinement of a plasma within a finite volume of space as desirable in nuclear fusion applications.

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Statements and declarations

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Competing interests

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