Physica D 459 (2024) 134031

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

An extended Hasegawa–Mima equation for nonlinear drift wave turbulence in general magnetic configurations

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ARTICLE INFO

Communicated by T. Drivas

Keywords:
Nonlinear equation
Hamiltonian mechanics
Turbulence
Nonlinear stability
Hasegawa–Mima equation

ABSTRACT

We propose an extended Hasegawa–Mima equation describing the evolution of nonlinear drift wave turbulence in general magnetic configurations. Such HMGM equation can be derived within the kinetic framework of guiding center motion or from a two-fluid model of an ion-electron plasma by application of a drift wave turbulence ordering that does not involve conditions on spatial derivatives of magnetic field and plasma density. The HMGM equation is therefore appropriate to describe the evolution of drift wave turbulence in strongly inhomogeneous magnetized plasmas, such as magnetospheric and stellarator plasmas, involving complex magnetic field geometries and non-uniform plasma density distributions. We find conservation laws (mass, energy, and generalized enstrophy) of the HMGM equation, study its algebraic (Hamiltonian) structure, and prove a nonlinear stability criterion for steady solutions through the energy-Casimir method. We then apply these results to describe drift waves and infer the existence of stable toroidal zonal flows with radial shear in dipole magnetic fields.

1. Introduction

In this study, we propose the following generalization of the standard Hasegawa–Mima (HM) equation [1,2],

$$\frac{\partial}{\partial t}\left[\lambda A_e \chi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp \chi}{B^2}\right)\right] = \nabla\cdot\left[A_e\left(\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^{\chi}}{B^2} - 1\right)\boldsymbol{v}_E^{\chi}\right]. \quad (1)$$

Eq. (1) describes the nonlinear evolution of the field $\chi(\boldsymbol{x},t) = \varphi(\boldsymbol{x},t) + \frac{\sigma}{2}\boldsymbol{v}_E^2(\boldsymbol{x},t)$, physically representing the energy of a charged particle, caused by drift wave turbulence in an ion-electron plasma within a static magnetic field $\boldsymbol{B}(\boldsymbol{x}) \neq \boldsymbol{0}$ of arbitrary geometry. Here $\boldsymbol{x}$ are Cartesian coordinates in a region $\Omega \subseteq \mathbb{R}^3$, t is the time variable, $A_e(\boldsymbol{x})$ the leading order electron spatial density, $\sigma = m/Ze$ a physical constant with m and Ze ion mass and charge, $Z \in \mathbb{N}$, $\nabla_\perp = -B^{-2}\boldsymbol{B}\times(\boldsymbol{B}\times\nabla)$, and the velocity fields $\boldsymbol{v}_E^{\chi}(\boldsymbol{x},t)$ and $\boldsymbol{v}_E(\boldsymbol{x},t)$ are respectively defined as

$$\boldsymbol{v}_E^{\chi} = \frac{\boldsymbol{B}\times\nabla\chi}{B^2}, \quad \boldsymbol{v}_E = \frac{\boldsymbol{B}\times\nabla\varphi}{B^2}, \quad (2)$$

with $\boldsymbol{E}(\boldsymbol{x},t) = -\nabla\varphi(\boldsymbol{x},t)$ the electric field associated with the electrostatic potential $\varphi(\boldsymbol{x},t)$. The plasma is quasineutral and adiabatic, implying that $n_e = A_e e^{\lambda\varphi} = Zn_i$ with n_e and n_i the electron and ion densities, and $\lambda = e/k_BT_e$ a physical constant where k_BT_e denotes the temperature of the thermalized electron component.

In the following, we shall refer to Eq. (1) as Hasegawa–Mima equation in a general magnetic configuration, abbreviated HMGM equation. As we will discuss later on, the physical relevance of this equation for both astrophysical and fusion plasmas stems from its ability to capture nonlinear plasma regimes in which the effects of field inhomogeneities on the development of turbulence are not approximated to a given order, but appear in their entirety. At the same time, the HMGM equation retains the form of a single partial differential equation with well-defined algebraic structure and conservation laws. We therefore suggest that these features make the HMGM equation a simplified but effective and mathematically well-defined model of turbulence in the presence of enhanced field inhomogeneities.

When $\boldsymbol{B} = B_0\nabla z$, $\log A_e = \log A_{e0} + \beta x$, $B_0, A_{e0}, \beta \in \mathbb{R}$, $\beta L \sim \epsilon \ll 1$, the HMGM Eq. (1) reduces to the standard Hasegawa–Mima (HM) equation [1,2]

$$\frac{\partial}{\partial t}\left(\lambda\varphi - \frac{\sigma}{B_0^2}\Delta_{(x,y)}\varphi\right) = \frac{\sigma}{B_0^3}\left[\varphi, \Delta_{(x,y)}\varphi\right]_{(x,y)} + \frac{\beta}{B_0}\varphi_y. \quad (3)$$

In this notation, $[f,g]_{(x,y)} = f_x g_y - f_y g_x$, $\Delta_{(x,y)} = \partial_x^2 + \partial_y^2$, and lower indexes denote partial derivatives, for example $f_x = \partial f/\partial x$. The HM equation (3) is a nonlinear equation describing the turbulent behavior of electric potential and spatial density in a quasi-neutral plasma,

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https://doi.org/10.1016/j.physd.2023.134031
Received 12 October 2023; Received in revised form 11 December 2023; Accepted 18 December 2023
Available online 29 December 2023

made of hot thermalized electrons and cold ions, permeated by a strong, straight, and homogeneous magnetic field, and evolving over time scales long compared with the period of cyclotron motion. The nonlinearity of the HM equation is caused by the convection of the $\boldsymbol{E}\times\boldsymbol{B}$ velocity associated with the polarization drift. In the presence of an electron density gradient, solutions of the linearized HM equation are the characteristic drift waves, whose interaction gives rise to drift wave turbulence. The HM equation shares the same mathematical structure with the quasi-geostrophic equation characterizing atmospheric motion over the surface of rotating planets [3,4] due to the similarity between the Lorentz force and the Coriolis force, and contains 2-dimensional incompressible vorticity dynamics as a special case [5]. Despite its relative simplicity, the HM equation can describe essential features of 2-dimensional plasma and fluid turbulence [6], including onset of inverse turbulent cascades of energy [7–10] and self-organization of large scale structures and zonal flows [11–15]. In toroidally confined fusion plasmas, a zonal flow typically manifests as a poloidal flow, while in geophysical fluid dynamics, it primarily exhibits a latitudinal flow pattern.

One of the key assumptions behind the HM equation is that background magnetic field and electron spatial density change over spatial scales L_B and L_{A_e} that are large compared to the typical turbulence wavelength across the magnetic field $k_\perp^{-1}$, i.e. $k_\perp L_B \gg 1$ and $k_\perp L_{A_e} \gg 1$. This hypothesis effectively restricts the applicability of the HM equation to plasmas with a small density gradient and to magnetic fields with small curvature or field inhomogeneities (this does not mean that the HM equation is inconsistent at small wavenumbers, but simply that it cannot account for field inhomogeneities, which cannot be neglected when $k_\perp L_B \sim k_\perp L_{A_e} \sim 1$). However, experimental observations pertaining to plasmas confined by dipole magnetic fields [16–18] suggest the existence of drift wave turbulence and zonal flows in systems where both the electron spatial density and the magnetic field are characterized by strong gradients over spatial scales comparable to that of electric field and density fluctuations (these low frequency fluctuations are often referred to as entropy modes [19]). In principle, an accurate description of electromagnetic turbulence in such setting could be obtained with the aid of nonlinear gyrokinetic theory [20–22]. Nevertheless, it is natural to ask whether the HM equation can be generalized to allow strong magnetic field and density inhomogeneities while maintaining a single governing equation for the electric potential φ. In [23] this question has been answered positively, and a generalized Hasegawa–Mima (GHM) equation has been obtained from a two-fluid model [24] of an ion-electron plasma in the form below:

$$\frac{\partial}{\partial t}\left[\lambda A_e\varphi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\varphi}{B^2}\right)\right] = \nabla\cdot\left[A_e\left(\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E}{B^2}-1\right)\boldsymbol{v}_E\right]. \tag{4}$$

A two-fluid plasma model represents a physical system where both the ion and electron components are governed by fluid equations. The ordering used in [23] to derive the GHM equation (4) only involves one ordering condition on the spatial derivatives of the magnetic field and the electron spatial density, effectively extending the range of the HM equation to general magnetic field geometries. In Section 2 we will see that this remaining ordering condition on spatial derivatives of magnetic field and spatial density used to obtain the GHM Eq. (4) in [23] can be removed at the price of replacing the electrostatic potential φ with the reduced charged particle energy $\chi = \varphi + \frac{\sigma}{2}\boldsymbol{v}_E^2$ as dynamical variable. The resulting governing equation is the HMGM Eq. (1), which exhibits the same mathematical structure of the GHM equation (4).

Some clarification is needed with regard to the naming of the GHM equation (4). In the plasma physics community, the name 'generalized Hasegawa–Mima equation' has been used to identify generalizations of the standard Hasegawa–Mima (HM) equation [1,2] that allow a different electron adiabatic response between drift waves and zonal flows [25–31]. In the same context, sometimes the names 'extended Hasegawa–Mima equation' or 'modified Hasegawa–Mima equation' are used as well [32]. However, the word 'generalization' in the naming of the GHM equation (4) does not refer to the electron adiabatic response (which is the same as that of the standard HM equation), but to the degree at which the background magnetic field is allowed to depart from a straight magnetic field. The naming HMGM equation chosen for Eq. (1) hopefully removes the ambiguity arising from the use of the word 'generalized'.

Generalizations of the HM equation including field inhomogeneities have also been obtained within the framework of gyrokinetic theory [33–36]. Here, the generalized equations follow from the usual limit of cold ions and adiabatic electrons. In these models the magnetic field is typically restricted to specific geometries (slab, cylindrical, or axially symmetric configurations), or its gradient is small over the characteristic turbulence wavelength $k_\perp^{-1}$ across the magnetic field, which is taken to be of the order of the ion gyroradius (the radius of the circular motion of an ion around a magnetic field) ρ, i.e. $k_\perp\rho \sim 1$. As a result, such generalized equations contain corrections specific to a certain magnetic field or corrections up to a given order in the field inhomogeneities. Furthermore, a detailed analysis of algebraic structure, conservation laws, and stability properties (which represent the main task undertaken in this study) of these generalized HM equations appears not to be available in the literature. In contrast, the HMGM Eq. (1) can be derived from two-fluid theory (as the original HM equation) and for any magnetic and density configurations. Therefore, the range of applicability of the HMGM equation does not depend on the ion gyroradius ρ, and the effect of field inhomogeneities appears in its entirety in the equation.

Our aim in this paper is to derive the HMGM Eq. (1) and characterize its mathematical properties, including invariants, Hamiltonian structure [37], and nonlinear stability of steady solutions. Furthermore, we wish to determine whether zonal flows can form in complex magnetic geometries (e.g. a dipole magnetic field), and characterize drift waves in such configurations. Here, Hamiltonian structure refers to the representation of the governing equations in Hamiltonian form, which is characterized by the action of a Poisson bracket on a Hamiltonian function, while nonlinear stability characterizes dynamical systems that remain close to a steady state under the effect of small perturbations (see Section 5 for rigorous definitions).

The present paper is organized as follows. In Section 2 we derive the HMGM Eq. (1) by considering the evolution of the phase space distribution function of a magnetized plasma according to the guiding center equations of motion (the guiding center represents the average position of a charged particle during its circular motion around a magnetic field). This is done by expanding the Euler–Lagrange equations arising from the Northrop guiding center Lagrangian [38,39] under an appropriate guiding center drift wave turbulence ordering. In Section 3, we derive the same HMGM equation from a two-fluid drift wave turbulence ordering. In Section 4, we discuss the constants of motion of the HMGM equation. In Section 5, we examine the algebraic structure of the HMGM equation, and obtain sufficient conditions on magnetic field $\boldsymbol{B}$ and electron spatial density A_e under which the HMGM equation defines a noncanonical Hamiltonian system. These results are consistent with the Hamiltonian structure of the standard HM equation [40–43]. We also prove a theorem concerning the nonlinear stability of steady solutions of the HMGM equation by applying the energy-Casimir method [44–46]. This result generalizes Arnold's stability criterion for a 2-dimensional fluid flow [47]. In Section 6 we show that toroidal zonal flows can form in dipole magnetic fields, and characterize the angular frequency of drift waves in dipole geometry. Concluding remarks are given in Section 7.

Finally, throughout the text notations like $f(\ldots)$ will often be used where $(\ldots)$ is interpreted as a factor rather than an argument.

Table 1
Guiding center ordering required for the existence of the first adiabatic invariant μ (see [38]).

Order	Dimensionless	Fields	Distances	Rates	Velocities
ϵ^{-1}		$\boldsymbol{B}, \boldsymbol{E}_\perp$		ω_c	
1		$\boldsymbol{E}_\parallel$	L	$\boldsymbol{v}/L, \boldsymbol{v}_{\boldsymbol{E}}/L, \tau^{-1}$	$\boldsymbol{v}, \boldsymbol{v}_{\boldsymbol{E}}$
ϵ	$\rho/L,\ (\omega_c\tau)^{-1}$		ρ	$\boldsymbol{v}_\nabla/L, \boldsymbol{v}_\kappa/L, \boldsymbol{v}_{\rm pol}/L$	$\boldsymbol{v}_\nabla, \boldsymbol{v}_\kappa, \boldsymbol{v}_{\rm pol}$

2. Derivation of the HMGM equation within the kinetic framework of guiding center motion

In this section, we derive the HMGM equation for an ion-electron plasma obeying the guiding-center equations of motion under an appropriate drift wave turbulence ordering. This rather technical derivation clarifies how the HMGM is related to the guiding-center framework, and it represents an independent result from the two-fluid theory developed in Section 3. To make this section self-contained, all definitions will be given again.

2.1. Guiding center ordering

We consider a guiding-center plasma made of ions and electrons within a region $\Omega \subseteq \mathbb{R}^3$ permeated by a static magnetic field $\boldsymbol{B}(\boldsymbol{x}) \neq \boldsymbol{0}$ with modulus B, and where $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$ are Cartesian coordinates. Let $\boldsymbol{E} = -\nabla\varphi$ denote the electric field, with $\varphi(\boldsymbol{x}, t)$ the electric potential (t is the time variable), $\boldsymbol{E}_\perp$ the component of $\boldsymbol{E}$ perpendicular to $\boldsymbol{B}$, and $\boldsymbol{E}_\parallel$ the component of $\boldsymbol{E}$ parallel to $\boldsymbol{B}$. The small ordering parameter for the guiding-center expansion will be denoted by $\epsilon > 0$, the spatial scale of the system by L, the time scale of the system by τ, the ion gyroradius by ρ, and the ion cyclotron frequency by $\omega_c = ZeB/m$ where $Z \in \mathbb{N}$ and Ze and m are the ion electric charge and mass respectively. Let $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{\rho}$ denote the ion guiding center position, $\boldsymbol{b} = \boldsymbol{B}/B$ the unit vector along $\boldsymbol{B}$, $\mu = m\left(\boldsymbol{v}_\perp - \boldsymbol{v}_{\boldsymbol{E}}\right)^2/2B$ the lowest order magnetic moment, $\boldsymbol{v} = \dot{\boldsymbol{x}}$ the charged particle velocity, $\boldsymbol{v}_\perp = \boldsymbol{b}\times(\boldsymbol{v}\times\boldsymbol{b})$ the charged particle velocity perpendicular to $\boldsymbol{B}$, and

$$\boldsymbol{v}_{\boldsymbol{E}} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \tag{5}$$

the $\boldsymbol{E}\times\boldsymbol{B}$ velocity. Then, the ion guiding center equations of motion obtained from the Northrop phase space guiding center Lagrangian [39] given in Appendix A of [38] are

$$m\dot{u}\boldsymbol{b} = Ze\left(\boldsymbol{E}' + \dot{\boldsymbol{X}}\times\boldsymbol{B}'\right), \tag{6a}$$

$$\dot{\mu} = 0, \tag{6b}$$

$$\dot{\zeta} = \omega_c, \tag{6c}$$

which can be equivalently written as

$$\dot{\boldsymbol{X}} = u\frac{\boldsymbol{B}'}{B'_\parallel} + \boldsymbol{E}'\times\frac{\boldsymbol{b}}{B'_\parallel}, \tag{7a}$$

$$\dot{u} = \frac{Ze}{m}\frac{\boldsymbol{B}'\cdot\boldsymbol{E}'}{B'_\parallel}, \tag{7b}$$

$$\dot{\mu} = 0, \tag{7c}$$

$$\dot{\zeta} = \omega_c, \tag{7d}$$

where ζ is the gyrophase and

$$u = \dot{\boldsymbol{X}}\cdot\boldsymbol{b}, \tag{8a}$$

$$Ze\varphi' = Ze\varphi + \mu B + \frac{m}{2}\boldsymbol{v}_{\boldsymbol{E}}^2, \tag{8b}$$

$$\boldsymbol{A}' = \boldsymbol{A} + \frac{m}{Ze}\left(u\boldsymbol{b} + \boldsymbol{v}_{\boldsymbol{E}}\right) \tag{8c}$$

$$\boldsymbol{E}' = -\nabla\varphi' - \frac{\partial\boldsymbol{A}'}{\partial t}, \tag{8d}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \quad \boldsymbol{B}' = \nabla\times\boldsymbol{A}', \quad B'_\parallel = \boldsymbol{B}'\cdot\boldsymbol{b}. \tag{8e}$$

Here, we observe that u represents the component of the guiding center velocity parallel to $\boldsymbol{B}$. Furthermore, notice that the following guiding center drift velocities $\boldsymbol{v}_{\boldsymbol{E}}$ ($\boldsymbol{E}\times\boldsymbol{B}$ drift), $\boldsymbol{v}_\nabla$ (∇B drift), and $\boldsymbol{v}_\kappa$ (curvature drift) are contained in the right-hand side of (7a) according to

$$\boldsymbol{v}'_{\boldsymbol{E}} = \frac{\boldsymbol{b}\times\nabla\varphi}{B'_\parallel}, \quad \boldsymbol{v}'_\nabla = \frac{\mu}{Ze}\frac{\boldsymbol{b}\times\nabla B}{B'_\parallel}, \quad \boldsymbol{v}'_\kappa = \frac{mu^2}{ZeB'_\parallel}\nabla\times\boldsymbol{b}, \tag{9}$$

where the $'$ symbol is used to emphasize that the correction $B'_\parallel$ of the magnetic field B caused by the term $m\nabla\times\left(u\boldsymbol{b}+\boldsymbol{v}_{\boldsymbol{E}}\right)/Ze$ in $\boldsymbol{B}'$ is used in these formulas. Similarly, the polarization drift $\boldsymbol{v}_{\rm pol}$ is included in (7a) according to

$$\boldsymbol{v}'_{\rm pol} = \frac{m}{Ze}\frac{\boldsymbol{b}\times\frac{\partial\boldsymbol{v}_{\boldsymbol{E}}}{\partial t}}{B'_\parallel} + \boldsymbol{v}'_{\boldsymbol{E}} - \boldsymbol{v}_{\boldsymbol{E}} + \frac{m}{2Ze}\frac{\boldsymbol{b}\times\nabla\boldsymbol{v}_{\boldsymbol{E}}^2}{B'_\parallel}. \tag{10}$$

The physical meaning carried by this expression will become clear later. Eq. (7a) also includes a further drift term

$$\boldsymbol{v}'_* = \frac{mu}{ZeB'_\parallel}\nabla\times\boldsymbol{v}_{\boldsymbol{E}}, \tag{11}$$

which originates from the effective magnetic field $\frac{m}{Ze}\nabla\times\boldsymbol{v}_{\boldsymbol{E}}$ associated with $\boldsymbol{E}\times\boldsymbol{B}$ motion. The total guiding center velocity can thus be written as

$$\dot{\boldsymbol{X}} = u\frac{\boldsymbol{B}}{B'_\parallel} + \boldsymbol{v}_{\boldsymbol{E}} + \boldsymbol{v}'_\nabla + \boldsymbol{v}'_\kappa + \boldsymbol{v}'_{\rm pol} + \boldsymbol{v}'_*. \tag{12}$$

For completeness we recall that the gyroradius, which defines the coordinate transformation $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{\rho}$, is an oscillatory (gyrophse dependent) term given by

$$\boldsymbol{\rho} = \frac{m}{Ze}\frac{\boldsymbol{b}\times\left(\boldsymbol{v}-\boldsymbol{v}_{\boldsymbol{E}}\right)}{B}. \tag{13}$$

This term is removed from the Northrop guiding center phase space Lagrangian density $\mathcal{L}_{Ngc}$ by appropriate subtraction of total time derivatives (on this point, see section III.C and appendix A of [38]). Here, the Northrop guiding center phase space Lagrangian density $\mathcal{L}_{Ngc}$ is obtained by expansion of the charged particle phase space Lagrangian density $\mathcal{L} = \mathcal{L}_{Ngc} + o(\epsilon)$ according to the guiding center ordering. Since the Northrop guiding center phase space Lagrangian density $\mathcal{L}_{Ngc}$ is independent of $\boldsymbol{\rho}$, it is also independent of the gyrophase ζ, leading to conservation of the conjugate momentum μ by the Noether theorem. The guiding-center ordering required for the conservation of the magnetic moment μ is given in table I of [38], which we report in Table 1. This ordering represents the starting point that we will use to construct a more restrictive ordering leading to the HMGM equation. Note that the ion gyroradius $\rho = |\boldsymbol{\rho}|$ appearing in table 1 is different from the sound radius

$$\rho_s = \frac{c_s}{\omega_c} = \sqrt{\frac{k_B T_e}{m}}\frac{m}{ZeB}, \tag{14}$$

where k_B denotes the Boltzmann constant and T_e the electron temperature. We conclude by observing that the ordering parameter ϵ arises from the physical constant

$$\sigma = \frac{m}{Ze} \ll 1, \tag{15}$$

which is small for elementary particles such as ions and electrons. The constant σ^{-1} always multiplies the electromagnetic fields within the charged particle phase space Lagrangian density $\mathcal{L}(\boldsymbol{x}, \boldsymbol{v}, t) = \left(\sigma^{-1}\boldsymbol{A} + \boldsymbol{v}\right)\cdot\dot{\boldsymbol{x}} - \frac{1}{2}\boldsymbol{v}^2 - \sigma^{-1}\varphi$, which is the reason why $\boldsymbol{E}$ and $\boldsymbol{B}$ are treated as large fields in Table 1.

2.2. Derivation of the HMGM equation from a drift wave turbulence ordering within guiding center theory

We start by assuming that the guiding center ordering presented in Table 3 holds, and gradually impose additional conditions to obtain the relevant drift wave turbulence ordering. From now on we set $\tau = \tau_d$, with τ_d the drift turbulence time scale. Recall that $\tau_d\omega_c \sim \epsilon^{-1}$. Let $f(\boldsymbol{p},\boldsymbol{x},t)$ denote the ion distribution function in the canonical phase space $(\boldsymbol{p},\boldsymbol{x})$ of charged particle dynamics. The distribution function f satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \boldsymbol{p}}\cdot(\dot{\boldsymbol{p}}f) - \frac{\partial}{\partial \boldsymbol{x}}\cdot(\dot{\boldsymbol{x}}f) + \left(\frac{df}{dt}\right)_c, \tag{16}$$

where the last term on the right-hand side describes particle collisions. Introducing the ion spatial density $n(\boldsymbol{x},t) = \int_{\mathbb{R}^3} f\, d\boldsymbol{p}$, integrating Eq. (16) with respect to the momentum variables, and assuming $\lim_{|\boldsymbol{p}|\to\infty} f = 0$, we obtain the ion continuity equation

$$\frac{\partial n}{\partial t} = -\nabla\cdot(\langle\dot{\boldsymbol{x}}\rangle n), \tag{17}$$

where

$$\langle\dot{\boldsymbol{x}}\rangle = \frac{1}{n}\int_{\mathbb{R}^3} f\dot{\boldsymbol{x}}\, d\boldsymbol{p} = \frac{1}{n}\int_{\mathbb{R}^3} f\left(\dot{\boldsymbol{X}}+\dot{\boldsymbol{\rho}}\right) d\boldsymbol{p} = \left\langle\dot{\boldsymbol{X}}\right\rangle, \tag{18}$$

is the ensemble averaged ion velocity at a given position $\boldsymbol{x} = \boldsymbol{X}+\boldsymbol{\rho}$. Notice that in the last passage we used the fact that by hypothesis $\boldsymbol{\rho}$ is a cyclotron phase dependent (oscillatory) term whose time average identically vanishes (on this point, see e.g. [38]). Since it will also be assumed that the system is fluctuating around some equilibrium state (in particular, electrons follow a Boltzmann distribution, and the plasma is quasineutral), we may enforce an ergodic hypothesis by exchanging time averages with ensemble averages so that $\lim_{T\to\infty}\frac{1}{T}\int_0^T\dot{\boldsymbol{\rho}}\,dt = \frac{1}{n}\int_{\mathbb{R}^3} f\dot{\boldsymbol{\rho}}\,d\boldsymbol{p} = \boldsymbol{0}$, which gives (18). Note that since $\dot{\boldsymbol{X}}$ contains all guiding center drifts (including the polarization drift), and since the volume element in momentum space $d\boldsymbol{p}$ is spanned by the guiding center variables μ, u, and ζ, the hypothesis (18) can be physically interpreted in the sense that, upon averaging with respect to the cyclotron phase ζ, the oscillatory contribution due to $\dot{\boldsymbol{\rho}}$ vanishes. It is also important to stress that in Eq. (18) there is no ambiguity related to the discrepancy between particle and guiding center positions [48] because the value of the integral only relies on the assumption that $\langle\dot{\boldsymbol{\rho}}\rangle = 0$. In particular, notice that the term on the right-hand side is not the guiding center velocity but its ensemble average. We also remark that the collision term in (16) vanishes upon integration in momentum space because we assume that collisions result in deflections in velocity space that do not change the local particle number. Now suppose that the parallel velocity u is small (the time scale τ_b of dynamics along $\boldsymbol{B}$ is long):

$$u\frac{\tau_d}{L} \sim \frac{\tau_d}{\tau_b} \sim \epsilon^2. \tag{19}$$

Note that consistency with (7b) requires that the component $E'_\parallel$ of $\boldsymbol{E}'$ along $\boldsymbol{B}'$ is small, i.e. $\boldsymbol{B}'\cdot\boldsymbol{E}'/B'_\parallel E_\perp \sim \epsilon^3$. Physically, this means that the electric field experienced by a charged particle along the magnetic field is negligible. Although this is a rather stringent condition, it is analogous to the hypothesis used in the derivation of the standard HM equation where the parallel ion inertia is neglected [1]. We also observe that the same condition could be enforced by introducing a parallel length scale $L_\parallel$, and by demanding that $E'_\parallel/E_\perp \sim L/L_\parallel \sim \epsilon^3$. Let us now consider all contributions to the continuity Eq. (17) that are greater than ϵ^2. From the ordering condition (19) it readily follows that the only surviving terms in (7a) are those involving the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity and the ∇B drift. In particular, observing that $B'_\parallel = B(1+o(\epsilon))$, we have

$$\dot{\boldsymbol{X}} = \boldsymbol{v}_{\boldsymbol{E}} + \boldsymbol{v}'_\nabla + \boldsymbol{v}_{\rm pol} + o\left(\epsilon^2\right), \tag{20}$$

where the polarization drift $\boldsymbol{v}_{\rm pol}$ now has expression

$$\boldsymbol{v}_{\rm pol} = \sigma\frac{\boldsymbol{b}\times\frac{d\boldsymbol{v}_{\boldsymbol{E}}}{dt}}{B}, \quad \frac{d\boldsymbol{v}_{\boldsymbol{E}}}{dt} = \frac{\partial\boldsymbol{v}_{\boldsymbol{E}}}{\partial t} + \boldsymbol{v}_{\boldsymbol{E}}\cdot\nabla\boldsymbol{v}_{\boldsymbol{E}}, \tag{21}$$

and where we used the fact that

$$\begin{aligned}\boldsymbol{v}_{\boldsymbol{E}} + \boldsymbol{v}'_{\rm pol} &= \boldsymbol{v}_{\boldsymbol{E}} - \sigma\frac{\boldsymbol{b}\cdot\nabla\times\boldsymbol{v}_{\boldsymbol{E}}}{B}\boldsymbol{v}_{\boldsymbol{E}} + \sigma\frac{\boldsymbol{b}\times\left(\frac{\partial\boldsymbol{v}_{\boldsymbol{E}}}{\partial t}+\frac{1}{2}\nabla\boldsymbol{v}_{\boldsymbol{E}}^2\right)}{B} + o\left(\epsilon^2\right)\\ &= \boldsymbol{v}_{\boldsymbol{E}} + \boldsymbol{v}_{\rm pol} + o\left(\epsilon^2\right).\end{aligned} \tag{22}$$

We see that the polarization drift $\boldsymbol{v}_{\rm pol}$ is that average particle velocity resulting from a non-vanishing average acceleration $d\boldsymbol{v}_{\boldsymbol{E}}/dt$ across the magnetic field.

In the following, we shall also demand the energy μB of cyclotron dynamics to be small, so that $\boldsymbol{v}'_\nabla$ becomes a higher order correction. More precisely, denoting with $\langle\mu B\rangle = n^{-1}\int_{\mathbb{R}^3} f\mu B\,d\boldsymbol{p}$ the ensemble averaged kinetic energy of cyclotron dynamics, and defining an associated temperature T_c according to $\langle\mu B\rangle = k_BT_c$ with k_B the Boltzmann constant, we demand T_c to satisfy

$$\frac{k_BT_c}{\frac{m}{2}\boldsymbol{v}_{\boldsymbol{E}}^2} \sim \epsilon. \tag{23}$$

The ordering conditions (19) and (23) can be regarded as the usual drift wave turbulence ordering requirement of cold ions. The guiding-center velocity thus becomes

$$\dot{\boldsymbol{X}} = \boldsymbol{v}_{\boldsymbol{E}} + \boldsymbol{v}_{\rm pol} + o\left(\epsilon^2\right) = \frac{\boldsymbol{b}\times\nabla\varphi}{B} + \sigma\frac{\boldsymbol{b}\times\frac{d\boldsymbol{v}_{\boldsymbol{E}}}{dt}}{B} + o\left(\epsilon^2\right). \tag{24}$$

It should be noted that (19) and (23) describe an ion plasma with temperature anisotropy. Nevertheless, temperature isotropy can be obtained by enforcing the stronger ordering condition $2k_BT_c/m\boldsymbol{v}_{\boldsymbol{E}}^2 \sim \epsilon^4$.

To proceed further, it is convenient to introduce the orthogonal gradient operator

$$\nabla_\perp = -\boldsymbol{b}\times(\boldsymbol{b}\times\nabla). \tag{25}$$

Although the expression (24) is convenient to highlight the usual guiding center drift contributions separately, $o\left(\epsilon^2\right)$ order terms must be rearranged in $\boldsymbol{v}_{\boldsymbol{E}}+\boldsymbol{v}_{\rm pol}$ in order for the reduced (drift wave) Hamiltonian $\chi = \varphi + \frac{\sigma}{2}\boldsymbol{v}_{\boldsymbol{E}}^2$ arising from the expansion of the Northrop guiding center Hamiltonian $H_{Ngc} = \frac{\sigma}{2}u^2 + \varphi + \mu B + \frac{\sigma}{2}\boldsymbol{v}_{\boldsymbol{E}}^2 = \chi + o(\epsilon)$ to be an exact constant of motion in the case of time-independent electromagnetic fields. To this end, one can verify that Eq. (24) can be equivalently written as

$$\begin{aligned}&\dot{\boldsymbol{X}} = \dot{\boldsymbol{X}}_{dw} + o\left(\epsilon^2\right), \quad \dot{\boldsymbol{X}}_{dw} = \frac{\boldsymbol{b}\times\nabla\left(\varphi+\frac{\sigma}{2}\boldsymbol{v}_{\boldsymbol{E}}^2\right)}{B''_\parallel} - \sigma\frac{\nabla_\perp\varphi_t}{B^2},\\ &B''_\parallel = B\left(1+\sigma\frac{\boldsymbol{b}\cdot\nabla\times\boldsymbol{v}_{\boldsymbol{E}}}{B}\right),\end{aligned} \tag{26}$$

with χ an exact integral of the first order term $\dot{\boldsymbol{X}}_{dw}$ when $\varphi_t = \partial\varphi/\partial t = 0$.

Noting that $\dot{\boldsymbol{X}}_{dw}$ is a pure spatial function, the ensemble averaged ion velocity (18) at a given position $\boldsymbol{x} = \boldsymbol{X}+\boldsymbol{\rho}$ is

$$\langle\dot{\boldsymbol{x}}\rangle = \frac{1}{n}\int_{\mathbb{R}^3} f\left(\dot{\boldsymbol{X}}+\dot{\boldsymbol{\rho}}\right) d\boldsymbol{p} = \dot{\boldsymbol{X}}_{dw} + o\left(\epsilon^2\right). \tag{27}$$

Next, consider the density $n_e(\boldsymbol{x},t)$ of the electron component. We assume that n_e follows a Boltzmann distribution with temperature T_e, i.e.

$$n_e = A_e(\boldsymbol{x})\exp\{\lambda\varphi(\boldsymbol{x},t)\}, \quad \lambda = \frac{e}{k_BT_e} \tag{28}$$

where $A_e(\boldsymbol{x})$ is a spatial function. If we further demand the ion-electron plasma to be quasi-neutral, we have the following condition:

Table 2
Drift wave turbulence ordering used for the derivation of the HMGM equation (1) within the guiding-center framework.

Order	Dimensionless	Fields	Distances	Rates	Velocities
ϵ^{-1}		$\boldsymbol{B}, \boldsymbol{E}_\perp$		ω_c	
1		A_e	L	$\tau_d^{-1}, \boldsymbol{v}_E/L$	$\boldsymbol{v}_E$
ϵ	$\lambda\varphi, \rho/L, (\omega_c\tau_d)^{-1}, k_BT_c/\frac{m}{2}\boldsymbol{v}_E^2$		ρ	$\boldsymbol{v}_{\rm pol}/L$	$\boldsymbol{v}_{\rm pol}$
ϵ^2	$\frac{m}{2}\boldsymbol{v}_E^2/k_BT_e, \tau_d/\tau_b$	$E'_\parallel$		$\boldsymbol{v}_\nabla/L, u/L, \tau_b^{-1}$	$\boldsymbol{v}_\nabla, u$
ϵ^5				$\boldsymbol{v}_\kappa/L$	$\boldsymbol{v}_\kappa$

Table 3
Drift wave turbulence ordering for the derivation of the HMGM equation (1) from a two-fluid model.

Order	Dimensionless	Fields	Distances	Rates	Velocities
1		$\boldsymbol{B}, A_e$	L	ω_c	
ϵ	$\lambda\varphi, \omega_c^{-1}\partial_t$	$\boldsymbol{E}_\perp$		$\tau_d^{-1}, \boldsymbol{v}_E/L$	$\boldsymbol{v}_E$
ϵ^2	τ_d/τ_b			$\boldsymbol{v}_{\rm pol}/L$	$\boldsymbol{v}_{\rm pol}$
ϵ^3		$E_\parallel, P$		$\tau_b^{-1}, v_\parallel/L$	$v_\parallel$

$$Zn(\boldsymbol{x},t) = n_e(\boldsymbol{x},t). \tag{29}$$

Then, the continuity equation for the ion density reads as

$$Z\left[\frac{\partial n}{\partial t} + \nabla\cdot(\langle\dot{\boldsymbol{x}}\rangle n)\right] = \frac{\partial n_e}{\partial t} + \nabla\cdot\left(\dot{\boldsymbol{X}}_{dw} n_e\right) + o\left(\epsilon^2\right) = 0, \tag{30}$$

where we used Eq. (27). Substituting Eq. (28), Eq. (30) can be rearranged as

$$\begin{aligned}\lambda A_e\frac{\partial\varphi}{\partial t} =& -\lambda\nabla\varphi\cdot A_e\dot{\boldsymbol{X}}_{dw} - \nabla\cdot\left(A_e\dot{\boldsymbol{X}}_{dw}\right) + o\left(\epsilon^2\right)\\ =&\lambda\sigma A_e\nabla\varphi\cdot\left(\frac{\nabla_\perp\varphi_t}{B^2} + \frac{\boldsymbol{b}\times\nabla\boldsymbol{v}_E^2}{2B''_\parallel}\right) - \nabla\cdot\left[A_e\frac{\boldsymbol{b}\times\nabla\left(\varphi+\frac{\sigma}{2}\boldsymbol{v}_E^2\right)}{B''_\parallel}\right]\\ &+\sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\varphi_t}{B^2}\right) + o\left(\epsilon^2\right).\end{aligned} \tag{31}$$

Next, we demand the electron component to be hot compared to the ion component, i.e.

$$\frac{\frac{m}{2}\boldsymbol{v}_E^2}{k_BT_e} \sim \epsilon^2. \tag{32}$$

Since $\lambda = e/k_BT_e$, it follows that $\lambda\varphi \sim \epsilon$, while the first term on the right-hand side of Eq. (31) scales as ϵ^2. Eq. (31) thus reduces to

$$\frac{\partial}{\partial t}\left[\lambda A_e\varphi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\varphi}{B^2}\right)\right] = -\nabla\cdot\left[A_e\frac{\boldsymbol{b}\times\nabla\left(\varphi+\frac{\sigma}{2}\boldsymbol{v}_E^2\right)}{B''_\parallel}\right] + o\left(\epsilon^2\right). \tag{33}$$

We now express Eq. (33) in terms of reduced Northrop guiding center Hamiltonian $\chi = \varphi + \sigma\boldsymbol{v}_E^2/2$. We have

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\lambda A_e\left(\varphi+\frac{\sigma}{2}\boldsymbol{v}_E^2\right) - \sigma\nabla\cdot\left[A_e\frac{\nabla_\perp\left(\varphi+\frac{\sigma}{2}\boldsymbol{v}_E^2\right)}{B^2}\right]\right\}\\ &= -\nabla\cdot\left[A_e\frac{\boldsymbol{b}\times\nabla\left(\varphi+\frac{\sigma}{2}\boldsymbol{v}_E^2\right)}{B''_\parallel}\right] + o\left(\epsilon^2\right).\end{aligned} \tag{34}$$

Recalling that $B''_\parallel = B\left(1+\sigma\frac{\boldsymbol{b}\cdot\nabla\times\boldsymbol{v}_E}{B}\right)$ we thus arrive at the equation

$$\begin{aligned}\frac{\partial}{\partial t}\left[\lambda A_e\chi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right)\right] &= -\nabla\cdot\left(A_e\frac{\boldsymbol{b}\times\nabla\chi}{B''_\parallel}\right) + o\left(\epsilon^2\right)\\ &= \nabla\cdot\left\{A_e\left[\sigma\frac{\boldsymbol{b}\cdot\nabla\times\left(\frac{\boldsymbol{b}\times\nabla\chi}{B}\right)}{B} - 1\right]\frac{\boldsymbol{b}\times\nabla\chi}{B}\right\}\\ &\quad + o\left(\epsilon^2\right).\end{aligned} \tag{35}$$

Notice that $\chi = \varphi + o(1)$. Using the expression $\boldsymbol{v}_E^\chi = B^{-2}\boldsymbol{B}\times\nabla\chi$ at first order Eq. (35) reduces to the following closed equation for the variable χ,

$$\frac{\partial}{\partial t}\left[\lambda A_e\chi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right)\right] = \nabla\cdot\left[A_e\left(\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2} - 1\right)\boldsymbol{v}_E^\chi\right], \tag{36}$$

which is the HMGM Eq. (1). Here, we observe that in (36) the term $\nabla\cdot\left(A_e\boldsymbol{v}_E^\chi\right) = \nabla\chi\cdot\nabla\times\left(A_e\boldsymbol{B}/B^2\right)$ must scale as $\sim\epsilon$ to be consistent with the other terms in the equation. Hence, either $\nabla\times\left(A_e\boldsymbol{B}/B^2\right)$ scales as $\sim\epsilon^2$, or the effective electric field $-\nabla\chi\sim\epsilon^{-1}$ is mostly orthogonal to the vector field $\nabla\times\left(A_e\boldsymbol{B}/B^2\right)\sim\epsilon$. We stress however that the behavior of the term $\nabla\cdot\left(A_e\boldsymbol{v}_E^\chi\right)$ is a consequence of the ordering used to obtain Eq. (36), and not an ordering condition required to arrive at (36). It should also be emphasized that the smallness of $\nabla\cdot\left(A_e\boldsymbol{v}_E\right)$ does not imply that $\nabla A_e\cdot\boldsymbol{v}_E$ is a small quantity as well. As an example, setting $A_e\propto B^2$ and $\nabla\times\boldsymbol{B}=\boldsymbol{0}$ gives $\nabla\cdot\left(A_e\boldsymbol{v}_E\right)=0$ while $\nabla A_e\cdot\boldsymbol{v}_E$ can be large.

The ordering used to derive the HMGM Eq. (36) is summarized in Table 2. For completeness, we emphasize that A_e is treated as a ~ 1 term for simplicity, although the present theory is independent of the magnitude of A_e. Furthermore, observe that the sound radius (14) satisfies $\rho_s/L \sim 1$ in the ordering of Table 2. Finally, if we compare the two-fluid HMGM ordering of Table 3 with the guiding center ordering of Table 2, one first notices that $\boldsymbol{B}$ scales as $\epsilon^0\sim 1$ in the two-fluid case, while it is treated as a ϵ^{-1} term in the guiding center ordering. This difference does not change the order of the ratio $v_{\boldsymbol{E}}/v_{\rm pol}$, and it is therefore not essential (in fact, all dimensionless ratios have the same order in both orderings; compare the column 'dimensionless' in Tables 2 and 3 with the fluid pressure P playing the role of the cyclotron temperature T_c).

3. Two-fluid derivation of the HMGM equation

As anticipated in the introduction, the HMGM Eq. (1) can be obtained by modeling an ion-electron plasma as a two-fluid system. The derivation follows the same steps of section 2 in [23] up to equation (2.26) therein. These steps are therefore omitted here, and can be summarized as the expansion to second order of the ion momentum and

Table 4
Drift wave turbulence ordering used for the derivation of the GHM equation (4) from a two-fluid model in [23].

Order	Dimensionless	Fields	Distances	Rates	Velocities
1		$\boldsymbol{B}, A_e$	L	ω_c	
ϵ	$\lambda\varphi, \omega_c^{-1}\partial_t$	$\boldsymbol{E}_\perp$		$\tau_d^{-1}, \boldsymbol{v}_E/L$	$\boldsymbol{v}_E$
ϵ^2	τ_d/τ_b			$\boldsymbol{v}_{\rm pol}/L$	$\boldsymbol{v}_{\rm pol}$
ϵ^3	$\sigma\left\lvert\nabla\boldsymbol{v}_E^2\cdot\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)\right\rvert/A_e\omega_c$	$E_\parallel, P$		$\tau_b^{-1}, v_\parallel/L$	$v_\parallel$

Table 5
Drift wave turbulence ordering used for the derivation of the standard HM equation (3) in a straight homogeneous magnetic field from a two-fluid model.

Order	Dimensionless	Fields	Distances	Rates	Velocities
1		$\boldsymbol{B}, A_e$	L	ω_c	
ϵ	$\lambda\varphi, \omega_c^{-1}\partial_t, L\nabla\log B, L\nabla\log A_e$	$\boldsymbol{E}_\perp$		$\tau_d^{-1}, \boldsymbol{v}_E/L$	$\boldsymbol{v}_E$
ϵ^2	τ_d/τ_b			$\boldsymbol{v}_{\rm pol}/L$	$\boldsymbol{v}_{\rm pol}$
ϵ^3		$E_\parallel, P$		$\tau_b^{-1}, v_\parallel/L$	$v_\parallel$

continuity equations under the two-fluid drift wave turbulence ordering of Table 3 and the hypothesis of a quasineutral plasma with cold ions and adiabatic electrons, $n_e = A_e e^{\lambda\varphi} = Zn_i$. Here, $\epsilon > 0$ denotes a small ordering parameter, $\omega_c = ZeB/m$ the ion cyclotron frequency, τ, τ_d, τ_b a reference time scale, the drift wave turbulence time scale, and the time scale of bounce motion, L a characteristic scale length for the system, $\boldsymbol{E}_\perp$ the component of $\boldsymbol{E}$ perpendicular to $\boldsymbol{B}$, $\boldsymbol{E}_\parallel = \boldsymbol{E} - \boldsymbol{E}_\perp$, $v_\parallel$ the ion fluid velocity parallel to $\boldsymbol{B}$, $\boldsymbol{v}_{\rm pol} = \sigma B^{-2}\boldsymbol{B}\times\left(d\boldsymbol{v}_E/dt\right)$ the polarization drift, and P the ion fluid pressure. This expansion leads to the following equation for the electrostatic potential φ:

$$\frac{\partial}{\partial t}\left[\lambda A_e\varphi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\varphi}{B^2}\right)\right] = \nabla\cdot\left[A_e\left(\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E}{B^2}-1\right)\boldsymbol{v}_E - \sigma A_e\frac{\boldsymbol{B}\times\nabla\boldsymbol{v}_E^2}{2B^2}\right] + o\left(\epsilon^3\right). \tag{37}$$

As discussed in [23], although the second order part of Eq. (37) exactly preserves the total mass of the system, it does not exactly preserve energy. This is because, in general, the expansion of an equation to a given order only guarantees conservation laws to hold up to the order of the expansion (on this point see e.g. pp. 731–732 of [38]), and higher order terms are needed to enforce exact conservation laws. In order to retain φ as dynamical variable and enforce exact conservation of energy, one therefore needs to assume that the last second order term on the right-hand side of (37), which is responsible for violation of conservation of energy, is a third order term,

$$\frac{\sigma}{A_e\omega_c}\left\lvert\nabla\boldsymbol{v}_E^2\cdot\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)\right\rvert\sim\epsilon^3. \tag{38}$$

Enforcing (38), one thus arrives at the GHM equation (4) of [23], which possesses exact mass and energy invariants. The ordering required to obtain the GHM equation (4), which includes the ordering condition (38), is summarized in Table 4.

The ordering condition (38) effectively restricts the allowed combinations of magnetic field $\boldsymbol{B}$ and plasma density A_e. This restriction can be removed at the price of replacing the electrostatic potential φ with the reduced charged particle energy $\chi = \varphi + \sigma\boldsymbol{v}_E^2/2$ as dynamical variable. Indeed, expressing Eq. (37) in terms of χ, and using the two-fluid drift wave turbulence ordering of Table 3 without enforcing (38), one obtains

$$\frac{\partial}{\partial t}\left[\lambda A_e\chi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right)\right] = \nabla\cdot\left[A_e\left(\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2}-1\right)\boldsymbol{v}_E^\chi\right] + o\left(\epsilon^3\right), \tag{39}$$

which is precisely the HMGM Eq. (1) plus higher order corrections. Since the two-fluid drift wave turbulence ordering of Table 3 used to obtain the HMGM Eq. (1) does not include conditions on partial derivatives of magnetic field $\boldsymbol{B}$ and plasma density A_e, there is no restriction on the geometry of $\boldsymbol{B}$ and A_e in this model equation. Furthermore, since the HMGM Eq. (1) and the GHM equation (4) share the same mathematical structure, the HMGM equation is endowed with the same type of exact invariants of the GHM equation. The invariants of the HMGM equation will be discussed in the next section.

For completeness, the two-fluid drift wave turbulence ordering required for the derivation of the standard HM equation (3) is given in Table 5. Observe that the standard HM ordering of Table 5 is stricter than the two-fluid GHM ordering of Table 4 and the two-fluid HMGM ordering of Table 3. In particular, the HM conditions on the spatial changes in B and A_e, $L\nabla\log B \sim L\nabla\log A_e \sim \epsilon$ are relaxed through the condition (38) in the case of the GHM equation (4), and they are completely removed in the case of the HMGM Eq. (1).

4. Conservation laws

In this section, we show that the derived HMGM Eq. (1) preserves both mass and energy. Furthermore, we identify a third invariant associated with the vorticity of the flow in a more general form than the one obtained in [23], and discuss its relationship with the generalized enstrophy encountered in the standard HM equation. Quantities are ordered according to the two-fluid ordering of Section 3.

Since the HMGM Eq. (1) and the GHM equation (4) share the same mathematical structure, we already know that the invariants of the HMGM Eq. (1) can be obtained by replacing φ with χ in the expressions of the invariants of the GHM equation (4). It is however useful to recall the physical origin of these quantities. First observe that the total ion mass can be written as

$$\mathcal{M}_\Omega = \frac{m}{Z}\int_\Omega A_e e^{\lambda\varphi}\,d\boldsymbol{x}. \tag{40}$$

Since $\lambda\varphi\sim\epsilon$, we may expand the exponential in powers of $\lambda\varphi$ according to $e^{\lambda\varphi} = 1 + \lambda\chi + o\left(\epsilon^2\right)$ and consider the conservation of the first order term,

$$M_\Omega = \frac{m}{Z}\int_\Omega A_e\left(1+\lambda\chi\right)d\boldsymbol{x}. \tag{41}$$

Using (1) we have

$$\frac{dM_\Omega}{dt} = \frac{m}{Z}\int_{\partial\Omega}A_e\left[\sigma\frac{\nabla_\perp\chi_t}{B^2} - \left(1-\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2}\right)\boldsymbol{v}_E^\chi\right]\cdot\boldsymbol{n}\,dS = -\frac{m}{Z}\int_{\partial\Omega}A_e\boldsymbol{V}_{dw}\cdot\boldsymbol{n}\,dS, \tag{42}$$

where $\boldsymbol{n}$ denotes the unit outward normal to the bounding surface $\partial\Omega$, dS the surface element on $\partial\Omega$, and we defined

$$\boldsymbol{V}_{dw} = \left(1-\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2}\right)\boldsymbol{v}_E^\chi - \sigma\frac{\nabla_\perp\chi_t}{B^2}. \tag{43}$$

Table 6
Invariants of the HMGM equation (1). In the definition of enstrophy W_Ω, the quantity w is an arbitrary function of $\lambda\chi - \sigma\omega/A_e$ with $\omega = \nabla\cdot\left(A_e B^{-2}\nabla_\perp\chi\right)$. Note that the field condition $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) = \boldsymbol{0}$ is a sufficient (not necessary) condition for the existence of an enstrophy invariant (see discussion in sec. 5 for details).

Invariant	Expression	Field conditions	Boundary conditions
Mass M_Ω	$\int_\Omega A_e(1+\lambda\chi)\,d\boldsymbol{x}$	none	$A_e\boldsymbol{V}_{dw}\cdot\boldsymbol{n}=0$
Energy H_Ω	$\frac{1}{2}\int_\Omega A_e\left(\lambda\chi^2+\sigma\frac{\lvert\nabla_\perp\chi\rvert^2}{B^2}\right)d\boldsymbol{x}$	none	$A_e\chi\boldsymbol{V}_{dw}\cdot\boldsymbol{n}=0$
Enstrophy W_Ω	$\int_\Omega A_e w\,d\boldsymbol{x}$	$\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)=\boldsymbol{0}$	$wA_e\boldsymbol{v}_E^\chi\cdot\boldsymbol{n}=0$

The boundary integral (42) vanishes under suitable boundary conditions, such as $A_e = 0$ on $\partial\Omega$ or $\boldsymbol{V}_{dw}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$.

Next, observe that the leading order ion Hamiltonian is given by

$$H = \frac{m}{2}\boldsymbol{v}^2 + Ze\varphi = Ze\varphi + \frac{m}{2}\boldsymbol{v}_E^2 + o(\epsilon). \tag{44}$$

Therefore, at leading order the total ion energy satisfies

$$\mathcal{H}_\Omega = Ze\int_{\Omega\times\mathbb{R}^3} f\left(\varphi + \frac{\sigma}{2}\boldsymbol{v}_E^2\right)d\boldsymbol{p}d\boldsymbol{x} + o(\epsilon) = e\int_\Omega A_e e^{\lambda\varphi}\left(\varphi+\frac{\sigma}{2}\boldsymbol{v}_E^2\right)d\boldsymbol{x} + o(\epsilon). \tag{45}$$

Dividing this expression by $k_B T_e$ we obtain

$$\frac{\mathcal{H}_\Omega}{k_B T_e} = \int_\Omega A_e\left(1+\lambda\varphi+\frac{1}{2}\lambda^2\varphi^2\right)\left(\lambda\varphi+\frac{\sigma}{2}\lambda\boldsymbol{v}_E^2\right)d\boldsymbol{x} + o\left(\epsilon^4\right) = \int_\Omega A_e\left(\lambda\varphi+\lambda^2\varphi^2+\frac{\sigma}{2}\lambda\boldsymbol{v}_E^2\right)d\boldsymbol{x}+o\left(\epsilon^4\right). \tag{46}$$

It follows that

$$\frac{\mathcal{H}_\Omega}{k_B T_e} - \frac{Z}{m}\mathcal{M}_\Omega = \frac{1}{2}\int_\Omega A_e\left(\lambda^2\varphi^2+\lambda\sigma\boldsymbol{v}_E^2-2\right)d\boldsymbol{x}+o\left(\epsilon^3\right) = \frac{1}{2}\int_\Omega A_e\left(\lambda^2\chi^2+\lambda\sigma\boldsymbol{v}_E^{\chi 2}-2\right)d\boldsymbol{x}+o\left(\epsilon^3\right). \tag{47}$$

Since A_e is a spatial function, we thus expect the HMGM energy

$$H_\Omega = \frac{1}{2}\int_\Omega A_e\left(\lambda\chi^2+\sigma\frac{\lvert\nabla_\perp\chi\rvert^2}{B^2}\right)d\boldsymbol{x}, \tag{48}$$

to be a constant of motion. From Eq. (1), one can verify that

$$\frac{dH_\Omega}{dt} = \int_\Omega \chi\frac{\partial}{\partial t}\left[\lambda A_e\chi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right)\right]d\boldsymbol{x} + \sigma\int_{\partial\Omega}A_e\chi\frac{\nabla_\perp\chi_t}{B^2}\cdot\boldsymbol{n}\,dS = \int_{\partial\Omega}A_e\chi\left[\sigma\frac{\nabla_\perp\chi_t}{B^2} - \left(1-\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2}\right)\boldsymbol{v}_E^\chi\right]\cdot\boldsymbol{n}\,dS = -\int_{\partial\Omega}A_e\chi\boldsymbol{V}_{dw}\cdot\boldsymbol{n}\,dS. \tag{49}$$

Again, this boundary integral vanishes under suitable boundary conditions, such as $A_e = 0$ on $\partial\Omega$, $\chi = 0$ on $\partial\Omega$, or $\boldsymbol{V}_{dw}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$.

An additional invariant, associated with the vorticity $\nabla\times\boldsymbol{v}_E^\chi$, exists when the magnetic field and the electron spatial density satisfy the condition $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) = \boldsymbol{0}$. To see this, define the quantity

$$\omega = \nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right) = A_e\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2} + \nabla\chi\cdot\frac{\boldsymbol{B}}{B^2}\times\left[\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)\right]. \tag{50}$$

Next, observe that whenever

$$\frac{\boldsymbol{B}}{B^2}\times\left[\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)\right] = \boldsymbol{0}, \tag{51}$$

which implies that $A_e\boldsymbol{B}/B^2$ is a Beltrami field, the following identity holds

$$\omega = \nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right) = A_e\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2}. \tag{52}$$

The derived HMGM Eq. (1) can thus be written in the form

$$\frac{\partial}{\partial t}\left[\lambda A_e\chi - \sigma\omega\right] = -\nabla\cdot\left[\left(1-\sigma\frac{\omega}{A_e}\right)A_e\boldsymbol{v}_E^\chi\right]. \tag{53}$$

On the other hand, functionals of the form

$$W_\Omega = \int_\Omega A_e w\left(\lambda\chi - \sigma\frac{\omega}{A_e}\right)d\boldsymbol{x}, \tag{54}$$

where $w\left(\lambda\chi - \sigma\omega/A_e\right)$ is any function of $\lambda\chi-\sigma\omega/A_e$, satisfy

$$\frac{dW_\Omega}{dt} = \int_\Omega w'\frac{\partial}{\partial t}\left(\lambda A_e\chi - \sigma\omega\right)d\boldsymbol{x} = -\int_\Omega w'\left[\nabla\left(\lambda\chi-\sigma\frac{\omega}{A_e}\right)\cdot A_e\boldsymbol{v}_E^\chi + \left(1-\sigma\frac{\omega}{A_e}\right)\nabla\cdot\left(A_e\boldsymbol{v}_E^\chi\right)\right]d\boldsymbol{x}, \tag{55}$$

with $w' = dw/d\left(\lambda\chi-\sigma\omega/A_e\right)$. Now observe that

$$\nabla\cdot\left(A_e\boldsymbol{v}_E^\chi\right) = \nabla\chi\cdot\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right). \tag{56}$$

Hence, if we further demand that

$$\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right) = \boldsymbol{0}, \tag{57}$$

which is a special case of (51), we find that

$$\frac{dW_\Omega}{dt} = -\int_{\partial\Omega} wA_e\boldsymbol{v}_E^\chi\cdot\boldsymbol{n}\,dS. \tag{58}$$

This boundary integral vanishes whenever $wA_e\boldsymbol{v}_E^\chi\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. The quantity W_Ω can be identified with the generalized enstrophy encountered in the standard Hasegawa–Mima equation if the boundary condition above is satisfied through A_e, i.e. $A_e = 0$ on $\partial\Omega$. Indeed, choosing $w = \left(\lambda\chi-\sigma\omega/A_e\right)^2$ and integrating by parts gives

$$W_\Omega = 2\lambda H_\Omega + \sigma\int_\Omega\left\{\lambda A_e\frac{\lvert\nabla_\perp\chi\rvert^2}{B^2} + \sigma A_e^{-1}\left[\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right)\right]^2\right\}d\boldsymbol{x}. \tag{59}$$

In the following, we shall refer to W_Ω as generalized enstrophy, or simply enstrophy. It is worth observing that the condition (57) implies (Poincaré lemma) that the magnetic field locally defines the normal direction of a surface $C = \text{constant}$, i.e. $\boldsymbol{B}\parallel\nabla C$ for some appropriate function C and sufficiently small neighborhood $U\subseteq\Omega$. The invariants of the HMGM equation are summarized in Table 6. Finally, we emphasize that Eq. (57) is only a sufficient condition for the existence of an enstrophy invariant. This is because (57) is a sufficient condition for the HMGM equation to define a Hamiltonian system. See the discussion in Section 5 for more details.

5. Algebraic structure and nonlinear stability

In this section, we discuss the algebraic structure of the HMGM Eq. (1), and the resulting nonlinear stability properties of steady solutions. In particular, we are concerned with the conditions under which Eq. (1) can be written in the form

$$\frac{\partial\eta}{\partial t} = \left\{\eta, H_\Omega\right\}, \tag{60}$$

where $\eta = \lambda A_e\chi - \sigma\omega$ and $\{\cdot,\cdot\}$ denotes a Poisson bracket [37] acting on functionals of η.

Conservation of energy H_Ω suggests that the HMGM Eq. (1) has an antisymmetric bracket structure. An antisymmetric bracket (also called an almost Poisson bracket [49]) shares the same properties of a Poisson bracket, except that it does not satisfy the Jacobi identity that characterizes the Poisson algebra of Hamiltonian systems. This antisymmetric bracket structure will be sufficient to carry out the nonlinear stability analysis of Section 5.3. The purpose of Section 5.1 is to show that for general $\boldsymbol{B}$ and A_e the HMGM equation is endowed with such antisymmetric bracket structure, although the fulfillment of the Jacobi identity is not guaranteed. In practice, this means that in general the HMGM equation does not define a Hamiltonian system. Sufficient conditions on $\boldsymbol{B}$ and A_e for the HMGM to possess a Poisson bracket and thus to define a Hamiltonian system will be obtained in Section 5.2.

The reason why the validity of the Jacobi identity depends on the geometry of the magnetic field and the plasma density can be found in the underlying properties of $\boldsymbol{E}\times\boldsymbol{B}$ drift dynamics, i.e. the three-dimensional dynamical system defined by

$$\dot{\boldsymbol{X}} = \boldsymbol{v}_{\boldsymbol{E}} = \frac{\boldsymbol{B}\times\nabla\varphi}{B^2}. \tag{61}$$

It is well established [50–52] that the equations of motion (61) define a Hamiltonian system only when the magnetic field has a vanishing helicity density,

$$\boldsymbol{B}\cdot\nabla\times\boldsymbol{B} = 0, \tag{62}$$

although the potential energy $Ze\varphi$, which represents the energy of the system, is a constant of motion for any $\boldsymbol{B}$ because $\dot{\boldsymbol{X}}\cdot\nabla\varphi = 0$. Indeed, the condition (62) is nothing but the Jacobi identity for the Poisson operator $B^{-2}\boldsymbol{B}\times$. On the other hand, the average velocity of a charged particle in the present setting is represented by the sum of $\boldsymbol{E}\times\boldsymbol{B}$ velocity and polarization drift, $\boldsymbol{v} = \boldsymbol{v}_{\boldsymbol{E}}+\boldsymbol{v}_{\rm pol}$ (on this point, see section 2 or [23]). Since the HMGM Eq. (1) arises as the continuity equation for plasma density under the velocity $\boldsymbol{v}$ above, and since the polarization drift $\boldsymbol{v}_{\rm pol}$ is a higher-order contribution compared to the $\boldsymbol{E}\times\boldsymbol{B}$ velocity $\boldsymbol{v}_{\boldsymbol{E}}$, it follows that the HMGM equation can only be Hamiltonian if the $\boldsymbol{E}\times\boldsymbol{B}$ equation of motion (61) is itself a Hamiltonian system.

The Frobenius theorem [53] of differential geometry informs us that the vanishing of the magnetic helicity density expressed by (62) is equivalent to the integrability condition for the magnetic field $\boldsymbol{B}$. When (62) holds, local functions λ and C can be found such that $\boldsymbol{B} = \lambda\nabla C$, implying that $\boldsymbol{B}$ defines the normal of the surface $C = \text{constant}$. In section 4.2, we demonstrate that the HMGM Eq. (1) transitions to a Hamiltonian system whenever $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) = \boldsymbol{0}$. This condition is precisely the integrability condition (62), with $\lambda = B^2/A_e$ serving as the integration factor. This result establishes the mathematical soundness of the HMGM equation and clarifies that its Hamiltonian nature is contingent upon the integrability of the magnetic field.

We also stress that while (62) is a necessary condition for the existence of a Hamiltonian structure, $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) = \boldsymbol{0}$ is only a sufficient condition. Indeed, the integrability condition (62) locally amounts to $\nabla\times\left(\mu A_e\boldsymbol{B}/B^2\right) = \boldsymbol{0}$ for some $\mu(\boldsymbol{x})$. In general, the corresponding Poisson bracket will include μ, and an associated Casimir (enstrophy) invariant will arise as a result of the restriction of the dynamics to the planes $C = \text{constant}$. Nevertheless, this case is not pursued here to simplify the exposition. As an example, the standard HM equation with non-constant density $\log A_e = \log A_{e0} + \beta x$, $\beta L \ll 1$, and straight magnetic field $\boldsymbol{B} = B_0\nabla z$, gives $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) = -\beta A_e B_0^{-1}\nabla y$ as well as $\nabla\cdot\left(A_e\boldsymbol{v}_{\boldsymbol{E}}\right) = -\beta A_e B_0^{-1}\partial\varphi/\partial y$ and $\mu \propto A_e^{-1}$. The corresponding Poisson bracket and Casimir invariant can be found in [41].

In principle, Hamiltonian reductions of the two-fluid equations governing the ion-electron plasma could yield a 1-field HMGM equation with a Hamiltonian structure in three spatial dimensions for any magnetic field. However, this simplification would overlook crucial contributions stemming from the compressibility of the $\boldsymbol{E}\times\boldsymbol{B}$ flow $A_e\boldsymbol{v}_{\boldsymbol{E}}$, namely the term $\nabla\cdot\left(A_e\boldsymbol{v}_{\boldsymbol{E}}\right) = \nabla\varphi\cdot\nabla\times\left(A_e\boldsymbol{B}/B^2\right)$, which precludes, in general, the existence of an invariant (Liouville) measure for the system (61). Another approach to restoring the Hamiltonian structure in the case of a non-integrable magnetic field involves reintroducing the parallel velocity $v_{\parallel}$ (detailed mathematical explanations of this Hamiltonian extension can be found in [54]). However, this method would necessitate an increase in the system's phase space to 4 dimensions and invalidate the cold ions assumption.

It should also be emphasized that the HMGM Eq. (1) represents a case of non-Hamiltonian reduction [55], as often occurs with nonholonomically constrained mechanical systems [56,57], of which $\boldsymbol{E}\times\boldsymbol{B}$ dynamics (61), which is at the basis of important transport and turbulent phenomena, is a key plasma physics example. The Nosé–Hoover thermostat in molecular dynamics [58], the Chaplygin sleigh in rigid body dynamics [59,60], the Heisenberg system [60], and the Landau–Lifshitz equation describing magnetization evolution in ferromagnets [50,61] are just a few examples spanning various disciplines. These mechanical systems often exhibit intriguing dynamics associated with the violation of Lioville's theorem and the non-Hamiltonian structure of the phase space. We thus expect the HMGM Eq. (1) to display such non-Hamiltonian effects when the magnetic field is not integrable.

5.1. Antisymmetric bracket structure

First, define the second order linear partial differential operator $\mathcal{D}$ according to

$$\mathcal{D}\chi = \eta = \lambda A_e\chi - \sigma\omega = \lambda A_e\chi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right). \tag{63}$$

In the following, we shall assume the inverse operator $\mathcal{D}^{-1}$ mapping η to $\chi\in\mathfrak{X}$ to be well defined by appropriate choice of the space of solutions $\mathfrak{X}$. Next, consider the bracket

$$\{F,G\} = \int_\Omega A_e\left(1-\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_{\boldsymbol{E}}^{\chi}}{B^2}\right)\nabla\left(\frac{\delta F}{\delta\eta}\right)\cdot\frac{\boldsymbol{B}}{B^2}\times\nabla\left(\frac{\delta G}{\delta\eta}\right)d\boldsymbol{x}, \tag{64}$$

acting on functionals $F,G\in\mathfrak{X}^*$, where $\mathfrak{X}^*$ denotes the dual space of $\mathfrak{X}$. Assuming variations $\delta\chi$ and the electron spatial density A_e to vanish on the boundary, and noting that

$$\frac{\delta H_\Omega}{\delta\eta} = \int_\Omega\frac{\delta H_\Omega}{\delta\chi(\boldsymbol{x}',t)}\frac{\delta}{\delta\eta(\boldsymbol{x},t)}\mathcal{D}^{-1}\eta(\boldsymbol{x}',t)\,d\boldsymbol{x}' = \mathcal{D}^{-1}\frac{\delta H_\Omega}{\delta\chi} = \chi, \tag{65}$$

where H_Ω is the energy given in (48), one can verify that the HMGM Eq. (1) can be written in the form (60) through the bracket (64).

It is also clear that the bracket (64) possesses an antisymmetric bracket structure. Indeed, the bracket (64) is bilinear and alternating (and thus antisymmetric), and it also satisfies the Leibniz rule. In formulae,

$$\{aF+bG,H\} = a\{F,H\}+b\{G,H\},$$
$$\{H,aF+bG\} = a\{H,F\}+b\{H,G\}, \tag{66a}$$
$$\{F,F\} = 0, \tag{66b}$$
$$\{F,G\} = -\{G,F\}, \tag{66c}$$
$$\{FG,H\} = F\{G,H\}+\{F,H\}G, \tag{66d}$$

for all $a,b\in\mathbb{R}$ and $F,G,H\in\mathfrak{X}^*$. If one could futher show that the Jacobi identity holds,

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\} = 0. \tag{67}$$

the bracket (64) would also qualify as a Poisson bracket. Unfortunately, it turns out that this bracket does not satisfy the Jacobi identity for arbitrary $\boldsymbol{B}$ and A_e. To see this, it is useful to introduce the following notation for the Jacobi identity,

$$\{F,\{G,H\}\}+\circlearrowleft = 0, \tag{68}$$

where $\circlearrowleft$ represents summation of even permutations. Furthermore, we shall denote functional derivatives as $F_\eta = \delta F/\delta\eta$, and define the quantity

$$\boldsymbol{\beta} = A_e\left(1-\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_{\boldsymbol{E}}^{\chi}}{B^2}\right)\frac{\boldsymbol{B}}{B^2}. \tag{69}$$

Notice that $\boldsymbol{\beta} = \boldsymbol{\beta}[\eta]$ is a functional of η. Omitting the range of integration, the Jacobi identity for the bracket (64) now reads

$$\{F,\{G,H\}\}+\circlearrowright=\int\nabla F_\eta\cdot\boldsymbol{\beta}\times\nabla\frac{\delta}{\delta\eta}\left(\int\nabla G_\eta\cdot\boldsymbol{\beta}\times\nabla H_\eta\,d\boldsymbol{x}\right)d\boldsymbol{x}+\circlearrowright. \tag{70}$$

Terms involving second order functional derivatives of F, G, and H vanish (on this point see e.g. [62]). For example, the term

$$\int\nabla\left(G_{\eta\eta}\delta\eta\right)\cdot\boldsymbol{\beta}\times\nabla H_\eta\,d\boldsymbol{x}=-\int\delta\eta G_{\eta\eta}\nabla H_\eta\cdot\nabla\times\boldsymbol{\beta}\,d\boldsymbol{x}, \tag{71}$$

gives rise to the following contribution to the Jacobi identity,

$$-\int\nabla F_\eta\cdot\boldsymbol{\beta}\times\nabla\left(G_{\eta\eta}\nabla H_\eta\cdot\nabla\times\boldsymbol{\beta}\right)d\boldsymbol{x}=-\int G_{\eta\eta}\nabla H_\eta\cdot\nabla\times\boldsymbol{\beta}\nabla F_\eta\cdot\nabla\times\boldsymbol{\beta}\,d\boldsymbol{x}, \tag{72}$$

where we used the hypothesis that the electron density vanishes on the boundary, $A_e=0$ on $\partial\Omega$, so that $\boldsymbol{\beta}=\boldsymbol{0}$ on $\partial\Omega$ and boundary terms evaluate to zero. On the other hand, the following term occurring in the permutation $\{H,\{F,G\}\}$,

$$\int\nabla F_\eta\cdot\boldsymbol{\beta}\times\nabla\left(G_{\eta\eta}\delta\eta\right)d\boldsymbol{x}=\int\delta\eta G_{\eta\eta}\nabla F_\eta\cdot\nabla\times\boldsymbol{\beta}\,d\boldsymbol{x}, \tag{73}$$

contributes to the Jacobi identity with

$$\int\nabla H_\eta\cdot\boldsymbol{\beta}\times\nabla\left(G_{\eta\eta}\nabla F_\eta\cdot\nabla\times\boldsymbol{\beta}\right)d\boldsymbol{x}=\int G_{\eta\eta}\nabla H_\eta\cdot\nabla\times\boldsymbol{\beta}\nabla F_\eta\cdot\nabla\times\boldsymbol{\beta}\,d\boldsymbol{x}, \tag{74}$$

which cancels with (72). It follows that the only surviving terms in the Jacobi identity are those involving functional derivatives of $\boldsymbol{\beta}$. In particular, we must evaluate the integral

$$\int\nabla G_\eta\cdot\delta\boldsymbol{\beta}\times\nabla H_\eta\,d\boldsymbol{x}. \tag{75}$$

To this end, it is useful to define the quantities

$$\zeta=1-\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2},\quad\theta=-\sigma\nabla G_\eta\cdot\frac{\boldsymbol{B}}{B^2}\times\nabla H_\eta, \tag{76}$$

so that

$$\begin{aligned}\int\nabla G_\eta\cdot\delta\boldsymbol{\beta}\times\nabla H_\eta\,d\boldsymbol{x}&=\int A_e\theta\frac{\boldsymbol{B}}{B^2}\cdot\nabla\times\left(\frac{\boldsymbol{B}\times\nabla\delta\chi}{B^2}\right)d\boldsymbol{x}\\&=\int\frac{\boldsymbol{B}\times\nabla\delta\chi}{B^2}\cdot\nabla\times\left(A_e\theta\frac{\boldsymbol{B}}{B^2}\right)d\boldsymbol{x}\\&=\int\nabla\mathcal{D}^{-1}\delta\eta\cdot\nabla\times\left(A_e\theta\frac{\boldsymbol{B}}{B^2}\right)\times\frac{\boldsymbol{B}}{B^2}\,d\boldsymbol{x}\\&=\int\delta\eta\mathcal{D}^{-1}\nabla\cdot\left\{\frac{\boldsymbol{B}}{B^2}\times\left[\nabla\times\left(\theta A_e\frac{\boldsymbol{B}}{B^2}\right)\right]\right\}d\boldsymbol{x}\end{aligned} \tag{77}$$

and the Jacobi identity can be written as

$$\begin{aligned}\{F,\{G,H\}\}+\circlearrowright&=\int A_e\zeta\nabla F_\eta\cdot\frac{\boldsymbol{B}}{B^2}\times\nabla\mathcal{D}^{-1}\nabla\\&\quad\cdot\left\{\frac{\boldsymbol{B}}{B^2}\times\left[\nabla\times\left(\theta A_e\frac{\boldsymbol{B}}{B^2}\right)\right]\right\}d\boldsymbol{x}+\circlearrowright\\&=\int\mathcal{D}^{-1}\nabla\cdot\left\{\frac{\boldsymbol{B}}{B^2}\times\left[\nabla\times\left(\theta A_e\frac{\boldsymbol{B}}{B^2}\right)\right]\right\}\nabla F_\eta\\&\quad\cdot\nabla\times\left(\zeta A_e\frac{\boldsymbol{B}}{B^2}\right)d\boldsymbol{x}+\circlearrowright.\end{aligned} \tag{78}$$

Since the value of the parameters σ and λ is not specified, terms proportional to different powers of σ must cancel separately. The Jacobi identity above contains terms scaling as $\sigma\mathcal{D}^{-1}$, terms scaling as $\sigma^2\mathcal{D}^{-1}$, and terms scaling as $\sigma^3\mathcal{D}^{-1}$. From the first group of terms, we obtain the condition

$$\begin{aligned}&\int\mathcal{D}^{-1}\nabla\cdot\left\{\frac{\boldsymbol{B}}{B^2}\times\left[\nabla\times\left(\theta A_e\frac{\boldsymbol{B}}{B^2}\right)\right]\right\}\nabla F_\eta\cdot\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)d\boldsymbol{x}+\circlearrowright\\&=\int\left\{\mathcal{D}^{-1}\nabla\cdot\left\{\frac{\boldsymbol{B}}{B^2}\times\left[\nabla\times\left(\theta A_e\frac{\boldsymbol{B}}{B^2}\right)\right]\right\}\nabla F_\eta+\circlearrowright\right\}\cdot\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)d\boldsymbol{x}=0.\end{aligned} \tag{79}$$

We therefore see that a sufficient condition for this quantity to vanish is that the magnetic field $\boldsymbol{B}$ and the spatial density A_e satisfy

$$\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)=\boldsymbol{0}. \tag{80}$$

Now observe that when (80) holds, the surviving terms in the Jacobi identity (78) are

$$\begin{aligned}\{F,\{G,H\}\}+\circlearrowright&=\int\mathcal{D}^{-1}\nabla\cdot\left(A_e\frac{\nabla_\perp\theta}{B^2}\right)\nabla F_\eta\cdot\nabla\zeta\times A_e\frac{\boldsymbol{B}}{B^2}\,d\boldsymbol{x}+\circlearrowright\\&=\frac{1}{\sigma}\int\left[-\theta+\lambda\mathcal{D}^{-1}\left(A_e\theta\right)\right]\nabla F_\eta\cdot\nabla\zeta\times A_e\frac{\boldsymbol{B}}{B^2}\,d\boldsymbol{x}+\circlearrowright.\end{aligned} \tag{81}$$

On the other hand, the condition (80) implies that there exists some local function C such that $A_e\boldsymbol{B}/B^2=\nabla C$ (Poincaré lemma). The first term within the integrand involving $-\theta$ can therefore be locally written as

$$\begin{aligned}&A_e^{-1}\nabla G_\eta\cdot\nabla C\times\nabla H_\eta\nabla F_\eta\cdot\nabla\zeta\times\nabla C+\circlearrowright=A_e^{-1}\\&\left[\frac{\partial G_\eta}{\partial x}\left(\frac{\partial C}{\partial y}\frac{\partial H_\eta}{\partial z}-\frac{\partial C}{\partial z}\frac{\partial H_\eta}{\partial y}\right)+\frac{\partial G_\eta}{\partial y}\left(\frac{\partial C}{\partial z}\frac{\partial H_\eta}{\partial x}-\frac{\partial C}{\partial x}\frac{\partial H_\eta}{\partial z}\right)\right.\\&\left.+\frac{\partial G_\eta}{\partial z}\left(\frac{\partial C}{\partial x}\frac{\partial H_\eta}{\partial y}-\frac{\partial C}{\partial y}\frac{\partial H_\eta}{\partial x}\right)\right]\\&\left[\frac{\partial F_\eta}{\partial x}\left(\frac{\partial\zeta}{\partial y}\frac{\partial C}{\partial z}-\frac{\partial\zeta}{\partial z}\frac{\partial C}{\partial y}\right)+\frac{\partial F_\eta}{\partial y}\left(\frac{\partial\zeta}{\partial z}\frac{\partial C}{\partial x}-\frac{\partial\zeta}{\partial x}\frac{\partial C}{\partial z}\right)\right.\\&\left.+\frac{\partial F_\eta}{\partial z}\left(\frac{\partial\zeta}{\partial x}\frac{\partial C}{\partial y}-\frac{\partial\zeta}{\partial y}\frac{\partial C}{\partial x}\right)\right]\\&+\circlearrowright=0.\end{aligned} \tag{82}$$

Unfortunately, the term in (81) containing $\mathcal{D}^{-1}\left(A_e\theta\right)$ appears to represent an obstruction to the Jacobi identity that cannot be trivially removed. This fact suggests that in order to fulfill the Jacobi identity when condition (80) holds, the bracket (64) itself must be modified. This modification is discussed below.

5.2. Poisson bracket structure

The aim of this subsection is to show that when the magnetic field $\boldsymbol{B}$ and the plasma density A_e satisfy the integrability condition (80), the following alternative bracket is a Poisson bracket,

$$\{F,G\}'=\int_\Omega\eta\nabla\left(\frac{\delta F}{\delta\eta}\right)\cdot\frac{\boldsymbol{B}}{B^2}\times\nabla\left(\frac{\delta G}{\delta\eta}\right)d\boldsymbol{x}. \tag{83}$$

Observe that the bracket (83) satisfies the antisymmetric bracket axioms (66) by the same arguments used for the bracket (64). In addition, if (80) holds, the GHM equation (1) can be written in the equivalent form

$$\frac{\partial\eta}{\partial t}=\left\{\eta,H_\Omega\right\}'. \tag{84}$$

Furthermore, by repeating the same steps as above the Jacobi identity for the new bracket (83) can be evaluated to be

$$\{F,\{G,H\}'\}'+\circlearrowright=\int A_e\nabla\left(\frac{\eta}{A_e}\right)\cdot\frac{\boldsymbol{B}}{B^2}\times\nabla F_\eta\nabla G_\eta\cdot\frac{\boldsymbol{B}}{B^2}\times\nabla H_\eta\,d\boldsymbol{x}+\circlearrowright=0, \tag{85}$$

which vanishes by the same calculation used in Eq. (82). We have thus shown that the antisymmetric bracket (83) is a Poisson bracket whenever Eq. (80) holds. It should not be surprising that (80) is exactly the same condition for the conservation of generalized enstrophy W_Ω (see Table 4). Indeed, the functional W_Ω is a Casimir invariant of the Poisson bracket (83),

$$\begin{aligned}\frac{dW_\Omega}{dt}&=\left\{W_\Omega,H_\Omega\right\}'\\&=\int\frac{\eta}{A_e}\nabla w'\cdot A_e\frac{\boldsymbol{B}}{B^2}\times\nabla\frac{\delta H_\Omega}{\delta\eta}\,d\boldsymbol{x}\\&=\int_{\partial\Omega}A_e\left[\int\frac{\eta}{A_e}w''d\left(\frac{\eta}{A_e}\right)\right]\frac{\boldsymbol{B}}{B^2}\times\nabla\frac{\delta H_\Omega}{\delta\eta}\cdot\boldsymbol{n}\,dS=0\quad\forall H_\Omega.\end{aligned} \tag{86}$$

In the last passage, we used the boundary condition $A_e = 0$ on $\partial\Omega$. We stress again that, however, the bracket (83) cannot be used to generate the HMGM system (1) when the condition (80) does not hold. We also remark that the mass M_Ω encountered in Eq. (41) is a Casimir invariant of both brackets, i.e.

$$\frac{dM_\Omega}{dt} = \left\{M_\Omega, H_\Omega\right\} = \left\{M_\Omega, H_\Omega\right\}' = 0 \quad \forall H_\Omega. \tag{87}$$

In this calculations we used the fact that the boundary condition $A_e = 0$ on $\partial\Omega$ implies that

$$\delta M_\Omega = \int_\Omega \left[\lambda A_e \delta\chi - \nabla\cdot\left(A_e \frac{\nabla_\perp \delta\chi}{B^2}\right)\right] d\boldsymbol{x} = \int_\Omega \delta\eta\, d\boldsymbol{x}. \tag{88}$$

It is worth observing that the condition (80) implies that the magnetic field satisfies the Frobenius integrability condition $\boldsymbol{B}\cdot\nabla\times\boldsymbol{B} = 0$ because $\boldsymbol{B} = A_e^{-1}B^2\nabla C$ locally. Furthermore, it also implies that the $\boldsymbol{E}\times\boldsymbol{B}$ velocity $\boldsymbol{v}_E^\chi$ multiplied by the spatial density A_e is divergence free, $\nabla\cdot\left(A_e\boldsymbol{v}_E^\chi\right) = \nabla\chi\cdot\nabla\times\left(A_e B^{-2}\boldsymbol{B}\right) = 0$. Notice also that (80) can always be satisfied for a vacuum field $\boldsymbol{B} = \nabla C$ by setting $A_e \propto B^2$. Finally, when A_e is a constant the HMGM Eq. (1) defines a noncanonical Hamiltonian system provided that the magnetic field $\boldsymbol{B}$ satisfies

$$\nabla\cdot\boldsymbol{B} = 0, \quad \nabla\times\left(\frac{\boldsymbol{B}}{B^2}\right) = \boldsymbol{0}. \tag{89}$$

Nontrivial examples of such configurations in different geometries can be found in [23].

5.3. Nonlinear stability criterion for steady solutions

The aim of this subsection is to elucidate the nonlinear stability properties of steady solutions of the HMGM Eq. (1) with the aid of the energy-Casimir method [44–47].

First, notice that steady solutions $\chi_0(\boldsymbol{x})$ of the HMGM Eq. (1) can be characterized in terms of critical points of the energy-Casimir functional

$$\mathfrak{H}_\Omega = H_\Omega + \gamma M_\Omega + \nu W_\Omega, \tag{90}$$

where γ, ν are spatial constants, and ν is taken to be zero for configurations violating the Poisson bracket condition (80). Here, a critical point η_0 of a functional $\mathfrak{H}_\Omega[\eta]$ is a point of the domain of $\mathfrak{H}_\Omega$ where its first variation with respect to the variable η identically vanishes, $\delta\mathfrak{H}_\Omega\left[\eta_0\right] = 0$. Indeed, when $\delta\mathfrak{H}_\Omega = 0$, from (60) one sees that $\eta_t = \partial\eta/\partial t = 0$. Let $\chi(\boldsymbol{x},t)$ denote a solution of the HMGM Eq. (1). A critical point χ_0 is nonlinearly stable provided that for every $\epsilon > 0$ there exists a norm $\|\cdot\|_1$ on the space of solutions $\mathfrak{X}$ and a $\delta > 0$ such that $\left\|\chi(\boldsymbol{x},0) - \chi_0(\boldsymbol{x})\right\|_1 < \delta$ implies

$$\left\|\chi(\boldsymbol{x},t) - \chi_0(\boldsymbol{x})\right\|_2 < \epsilon \quad \forall t \geq 0, \tag{91}$$

where $\|\cdot\|_2$ is a further norm on the state space $\mathfrak{X}$. Notice that the nonlinear stability described by (91) only ensures that the solution χ remains close to the critical point in the norm $\|\cdot\|_2$.

Theorem 1 (*Nonlinear Stability of Steady Solutions of the HMGM Equation*). *Let $\chi_0(\boldsymbol{x}) \in C^2(\Omega)$ denote a critical point of the energy-Casimir functional $\mathfrak{H}_\Omega$. If the condition $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) = \boldsymbol{0}$ of Eq. (80) holds, assume that the function $w\left(\eta/A_e\right)$ appearing within the integrand of the Casimir invariant W_Ω is twice differentiable in its argument, and that it satisfies*

$$0 < c_m \leq \nu w'' = \nu\frac{d^2 w}{d\left(\eta/A_e\right)^2} \leq c_M < \infty, \tag{92}$$

with c_m and c_M real constants. If $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) \neq \boldsymbol{0}$ set $\nu = 0$. Further assume that $\boldsymbol{B}, A_e \in C^2\left(\bar\Omega\right)$, that their minima satisfy $B_m, A_{em} > 0$, and that the HMGM Eq. (1) admits a solution $\chi(\boldsymbol{x},t) \in C^2(\Omega\times[0,t))$ for all $t \geq 0$ such that $\delta\chi = \chi - \chi_0 = 0$ and $A_e = 0$ on the boundary $\partial\Omega$. Then, the critical point χ_0 is nonlinearly stable: there exists a positive real constant $\mathfrak{C}$ such that

$$\left\|\chi(t) - \chi_0\right\|_\perp^2 \leq \mathfrak{C}\left\|\chi(0) - \chi_0\right\|_\perp^2 \quad \forall t \geq 0, \tag{93}$$

with

$$\|\chi\|_\perp^2 = \begin{cases} \|\chi\|_{L^2(\Omega)}^2 + \|\nabla_\perp\chi\|_{L^2(\Omega)}^2 + \|\mathcal{D}\chi\|_{L^2(\Omega)}^2, & \text{if } \nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right) = \boldsymbol{0}, \\ \|\chi\|_{L^2(\Omega)}^2 + \|\nabla_\perp\chi\|_{L^2(\Omega)}^2 & \text{if } \nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right) \neq \boldsymbol{0}, \end{cases} \tag{94}$$

where $L^2(\Omega)$ denotes the standard L^2 norm in Ω and we used the abbreviated notation $\chi(t) = \chi(\boldsymbol{x},t)$.

Proof. We start by observing that key to the proof is the conservation of $\mathfrak{H}_\Omega$. Indeed, the energy-Casimir method consists in finding norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathfrak{X}$ so that the following chain of inequalities holds:

$$\begin{aligned} &C\left\|\chi(0) - \chi_0\right\|_1^2 \geq \left|\mathfrak{H}_\Omega\left[\chi(0)\right] - \mathfrak{H}_\Omega\left[\chi_0\right]\right| \\ &= \left|\mathfrak{H}_\Omega\left[\chi(t)\right] - \mathfrak{H}_\Omega\left[\chi_0\right]\right| \geq C'\left\|\chi(t) - \chi_0\right\|_2^2, \end{aligned} \tag{95}$$

where C, C' are positive real constants. To derive these inequalities for the case $\nu \neq 0$ (corresponding to $\nabla\times\left(A_e\boldsymbol{B}/B^2\right) = \boldsymbol{0}$) we rely on a standard result: setting $\eta = \mathcal{D}\chi$, $\eta_0 = \mathcal{D}\chi_0$, and $\delta\eta = \eta - \eta_0$, Taylor's theorem asserts that

$$w\left(\frac{\eta}{A_e}\right) = w\left(\frac{\eta_0}{A_e}\right) + w'\left(\frac{\eta_0}{A_e}\right)\frac{\delta\eta}{A_e} + w''\left(\frac{\tilde\eta}{A_e}\right)\frac{\delta\eta^2}{2A_e^2}, \tag{96}$$

with $\tilde\eta$ between η_0 and η and $w' = dw/d\left(\eta/A_e\right)$. Since $A_e \in C^2\left(\bar\Omega\right)$ and $A_e \geq A_{em} > 0$, A_e attains a positive maximum $A_{eM} < \infty$ in $\bar\Omega$. Using $0 < c_m \leq \nu w'' \leq c_M < \infty$ and $0 < A_{em} \leq A_e \leq A_{eM} < \infty$ we thus obtain

$$\frac{c_m}{2A_{eM}}\|\delta\eta\|_{L^2(\Omega)}^2 \leq \int_\Omega \frac{\nu\delta\eta^2}{2A_e} w''\left(\frac{\tilde\eta}{A_e}\right) d\boldsymbol{x} \leq \frac{c_M}{2A_{em}}\|\delta\eta\|_{L^2(\Omega)}^2. \tag{97}$$

Now observe that

$$\begin{aligned} &\mathfrak{H}_\Omega\left[\chi(t)\right] - \mathfrak{H}_\Omega\left[\chi_0\right] = \\ &\int_\Omega A_e\left\{\frac{\lambda}{2}\left(2\chi_0\delta\chi + \delta\chi^2\right) + \sigma\frac{2\nabla_\perp\delta\chi\cdot\nabla_\perp\chi_0 + \left|\nabla_\perp\delta\chi\right|^2}{2B^2} + \gamma\lambda\delta\chi\right\} d\boldsymbol{x} \\ &+ \nu\int_\Omega A_e\left[w'\left(\frac{\eta_0}{A_e}\right)\frac{\delta\eta}{A_e} + w''\left(\frac{\tilde\eta}{A_e}\right)\frac{\delta\eta^2}{2A_e^2}\right] d\boldsymbol{x} \\ &= \int_\Omega A_e\left\{\frac{\lambda}{2}\left(2\chi_0\delta\chi + \delta\chi^2\right) - \frac{\sigma}{A_e}\delta\chi\nabla\cdot\left(A_e\frac{\nabla_\perp\chi_0}{B^2}\right) + \sigma\frac{\left|\nabla_\perp\delta\chi\right|^2}{2B^2} + \gamma\lambda\delta\chi\right\} d\boldsymbol{x} \\ &+ \nu\int_\Omega A_e\left[w'\left(\frac{\eta_0}{A_e}\right)\frac{\delta\eta}{A_e} + w''\left(\frac{\tilde\eta}{A_e}\right)\frac{\delta\eta^2}{2A_e^2}\right] d\boldsymbol{x}. \end{aligned} \tag{98}$$

However, by hypothesis χ_0 solves the critical equation for $\mathfrak{H}_\Omega$

$$\lambda A_e\left(\chi_0 + \gamma\right) - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi_0}{B^2}\right) + \nu\mathcal{D}w' = \mathcal{D}\left(\chi_0 + \gamma + \nu w'\right) = 0. \tag{99}$$

Hence, the difference (98) reduces to

$$\mathfrak{H}_\Omega\left[\chi(t)\right] - \mathfrak{H}_\Omega\left[\chi_0\right] = \int_\Omega A_e\left\{\frac{\lambda}{2}\delta\chi^2 + \sigma\frac{\left|\nabla_\perp\delta\chi\right|^2}{2B^2} + \frac{\nu\delta\eta^2}{2A_e^2}w''\left(\frac{\tilde\eta}{A_e}\right)\right\} d\boldsymbol{x}. \tag{100}$$

Using (97), it readily follows that

$$\frac{1}{2}\left(\lambda A_{em}\left\|\delta\chi\right\|^2_{L^2(\Omega)}+\frac{\sigma A_{em}}{B_M^2}\left\|\nabla_\perp\delta\chi\right\|^2_{L^2(\Omega)}+\frac{c_m}{A_{eM}}\left\|\mathcal{D}\delta\chi\right\|^2_{L^2(\Omega)}\right)$$
$$\leq \mathfrak{H}_\Omega\left[\chi(t)\right]-\mathfrak{H}_\Omega\left[\chi_0\right]$$
$$=\mathfrak{H}_\Omega\left[\chi(0)\right]-\mathfrak{H}_\Omega\left[\chi_0\right]\leq\frac{1}{2}\left(\lambda A_{eM}\left\|\delta\chi_0\right\|^2_{L^2(\Omega)}+\frac{\sigma A_{eM}}{B_m^2}\left\|\nabla_\perp\delta\chi_0\right\|^2_{L^2(\Omega)}\right.$$
$$\left.+\frac{c_M}{A_{em}}\left\|\mathcal{D}\delta\chi_0\right\|^2_{L^2(\Omega)}\right), \tag{101}$$

where $\delta\chi_0=\chi(0)-\chi_0$ and $B_M<\infty$ is the maximum of B. We have thus shown that

$$\left\|\chi(t)-\chi_0\right\|^2_\perp\leq\mathfrak{C}\left\|\chi(0)-\chi_0\right\|^2_\perp\qquad\forall t\geq 0, \tag{102}$$

for some positive real constant $\mathfrak{C}$. The case $\nu=0$ (corresponding to $\nabla\times\left(A_e\boldsymbol{B}/B^2\right)\neq\boldsymbol{0}$) follows in a similar fashion and the theorem is proven. □

The following remarks are useful.

Remark 1. Theorem 1 generalizes Arnold's result concerning the stability of a two dimensional ideal fluid flow [47]. Indeed, Arnold's case can be recovered by setting $\boldsymbol{B}=\nabla z$, $A_e=\sigma=1$, and $\lambda=0$. In this setting we have

$$\eta=-\Delta_{(x,y)}\chi, \tag{103}$$

so that the critical point Eq. (99) reduces to

$$\chi_0+\nu w'\left(-\Delta_{(x,y)}\chi_0\right)=0. \tag{104}$$

Hence, using Arnold's notation,

$$\nu w''=\frac{\nabla\chi_0}{\nabla\Delta_{(x,y)}\chi_0}=-\frac{\nabla\chi_0}{\nabla\eta_0}. \tag{105}$$

Remark 2. According to Theorem 1 steady states of the HMGM Eq. (1) corresponding to $\nu=0$ are nonlinearly stable, provided that the hypothesis of Theorem 1 pertaining to regularity and boundary conditions hold true. Notice also that $\nu=0$ when the magnetic field $\boldsymbol{B}$ and the electron spatial density A_e do not satisfy the condition (80) and the generalized enstrophy W_Ω is not a constant of motion.

6. Zonal flows and drift waves in dipole magnetic fields

As outlined in the introduction, one of the motivations behind the development of HMGM Eq. (1) is the understanding of drift wave turbulence in complex magnetic geometries, such as that of a magnetospheric plasma. The purpose of this section is to show that the theory developed in this paper points to the existence of stable toroidal zonal flows with radial velocity shear within dipole magnetic fields, and to characterize drift waves in dipole geometry. To see this, we first observe that a dipole magnetic field is a vacuum field outside the central region containing the electric current generating it. Furthermore, it is axially symmetric. We may therefore write

$$\boldsymbol{B}=\nabla\zeta(r,z)=\nabla\Psi(r,z)\times\nabla\phi, \tag{106}$$

where (r,ϕ,z) denote cylindrical coordinates, $\zeta(r,z)$ the magnetic potential, and $\Psi(r,z)$ the flux function. It is convenient to work with magnetic coordinates (ζ,Ψ,ϕ). The Jacobian determinant of this coordinate change is

$$\nabla\zeta\cdot\nabla\Psi\times\nabla\phi=B^2. \tag{107}$$

Then, it follows that

$$\boldsymbol{v}_E^\chi=\frac{\chi_\Psi}{B^2}\nabla\zeta\times\nabla\Psi-\frac{\chi_\phi}{B^2}\nabla\phi\times\nabla\zeta, \tag{108a}$$

$$A_e\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2}=\frac{A_e}{B^2}\nabla\cdot\nabla_\perp\chi, \tag{108b}$$

where, as usual, lower indexes denote partial derivatives, e.g. $\chi_\Psi=\partial\chi/\partial\Psi$. Hence, the HMGM Eq. (1) can be written as

$$\frac{\partial}{\partial t}\left[\lambda A_e\chi-\sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right)\right]=B^2\left[\chi,\frac{A_e}{B^2}\left(\sigma\frac{\Delta_\perp\chi}{B^2}-1\right)\right]_{(\Psi,\phi)}, \tag{109}$$

where we introduced the linear differential operators $\Delta_\perp=\nabla\cdot\nabla_\perp$ and $[f,g]_{(\Psi,\phi)}=f_\Psi g_\phi-f_\phi g_\Psi$. Notice that this equation is two-dimensional, i.e. it can be considered as a closed system within a surface given by a level set of ζ, with the function ζ effectively behaving as an external parameter. It follows that steady solutions χ_0 of Eq. (109) satisfy

$$\frac{A_e}{B^2}\left(\sigma\frac{\Delta_\perp\chi_0}{B^2}-1\right)=f\left(\chi_0,\zeta\right), \tag{110}$$

with $f\left(\chi_0,\zeta\right)$ some function of χ_0 and ζ. We have shown that steady solutions with given values of mass and generalized enstrophy can be equivalently characterized in terms of critical points of the energy-Casimir function, which, according to Eq. (99), are given by

$$\mathcal{D}\left(\chi_0+\gamma+\nu w'\right)=0, \tag{111}$$

where γ,ν are spatial constants and $w'=dw/d\left(\eta/A_e\right)$. In this context, a steady zonal flow solution is described by the condition $\chi_{0\phi}=0$, implying a toroidal flow $\boldsymbol{v}_E^\chi=\chi_{0\Psi}\partial_\phi$ where $\partial_\phi=B^{-2}\nabla\zeta\times\nabla\Psi$ denotes the tangent vector in the ϕ direction. Evidently, Eqs. (110) and (111) admit such configurations provided that A_e is axially symmetric (since the dipole magnetic field $\boldsymbol{B}$ is axially symmetric). Observe that in this case on the equatorial plane $z=0$ the toroidal $\boldsymbol{E}\times\boldsymbol{B}$ velocity has radial shear since $\boldsymbol{v}_E^\chi(r,z=0)=\chi_{0\Psi}(r,z=0)\,\partial_\phi$. The stability properties of these zonal flow solutions can be deduced from Theorem 1 of Section 6. In particular, they will depend on the specific value of the generalized vorticity W_Ω in the case in which the density A_e satisfies (80), i.e. $A_e\propto B^2$ (configurations of this type are predicted by equilibrium statistical mechanics because the invariant (Liouville) measure associated with $\boldsymbol{E}\times\boldsymbol{B}$ dynamics in a vacuum field is $B^2\,d\boldsymbol{x}$ [52]). Otherwise $\nu=0$, and zonal flows are expected to be nonlinearly stable. It should be emphasized that the characteristic spatial scale of χ_0 is related to that of magnetic field $\boldsymbol{B}$ and electron spatial density A_e, while the zonal nature of the solution stems from the axial symmetry of these fields. Nevertheless, exception made for the case in which A_e satisfies (80), the generalized enstrophy W_Ω is not necessarily a constant, and inverse energy cascade toward small wave numbers is not available in the usual form. The turbulent mechanism by which zonal flow solutions can be formed in general magnetic fields therefore requires a separate discussion. A crucial role should be played by boundary conditions for χ_0, especially when $\nu=0$ and there is no constraint arising from generalized enstrophy, since trivial boundary conditions, such as Dirichlet boundary conditions, result in trivial steady states $\chi_0+\gamma=0$.

We conclude this section by describing the drift wave in a dipole magnetic field. Assume that the electron spatial density $A_e=A_e(\zeta,\Psi)$ is axially symmetric. Let

$$\chi_d=\xi(\zeta,\Psi)\exp\left\{-\mathrm{i}\left(\ell\phi+\omega t\right)\right\} \tag{112}$$

represent the drift wave with $\ell\in\mathbb{Z}$, $\xi(\zeta,\Psi)$ a real function of ζ and Ψ, and $\omega\in\mathbb{R}$. Linearizing Eq. (109) with respect to χ_d we thus obtain the following equation for ξ,

$$\frac{1}{A_e}\nabla\cdot\left(A_e\frac{\nabla_\perp\xi}{B^2}\right)+\xi\left[\frac{\ell}{\sigma\omega}\frac{\partial}{\partial\Psi}\log\left(\frac{A_e}{B^2}\right)-\frac{\ell^2}{r^2B^2}-\frac{\lambda}{\sigma}\right]=0. \tag{113}$$

Conversely, the angular frequency ω can be expressed as

$$\omega=\frac{\ell\frac{\partial}{\partial\Psi}\log\left(\frac{A_e}{B^2}\right)}{\frac{\sigma\ell^2}{r^2B^2}+\lambda-\frac{\sigma}{A_e\xi}\nabla\cdot\left(A_e\frac{\nabla_\perp\xi}{B^2}\right)}. \tag{114}$$

In order to estimate the magnitude of ω, consider the simplified case in which ℓ is small and $\log\left(A_e/B^2\right)$ is a weak function of Ψ, and consider its Taylor expansion around Ψ_0. Then, we may set $\xi = \xi(\zeta)$ to find

$$\omega \approx \frac{\ell \beta_{\Psi_0}}{\lambda} = \ell \beta_{\Psi_0} T_e[eV], \quad \beta_{\Psi_0} = \left[\frac{\partial}{\partial \Psi}\log\left(\frac{A_e}{B^2}\right)\right]_{\Psi=\Psi_0}. \tag{115}$$

Here, $T_e[eV] = \lambda^{-1}$ is the electron temperature expressed in electron-volt. Notice that the term $\log A_e$ is the one responsible for the usual drift wave in the HM equation. Remarkably, even in the presence of a constant electron spatial density A_e, an inhomogeneous magnetic field can sustain a geometric drift wave through the spatial dependence of B. For a dipole magnetic field $B \sim 1\,T$ in a trap with size $L \sim 1\,m$, a roughly constant electron spatial density A_e, $\ell = 1$, and an electron temperature of 1 keV, one obtains $\omega \approx 10^3$ Hz.

These results regarding zonal flows and drift waves in dipole geometry align well with experimental observations. Indeed, toroidal plasma flows have been observed in magnetic dipole traps [18], and low-frequency turbulent fluctuations have been measured experimentally in both the RT-1 and LDX experiments [17,19,63,64]. For instance, [17] reports the observation of electrostatic, density, and magnetic fluctuations with characteristic frequencies of ~ 1 kHz, which are consistent with the values obtained from the dispersion relation (114). These low-frequency fluctuations are believed to be responsible for the inward (toward the center of the dipole field) diffusion of charged particles, which is associated with violation of the third adiabatic invariant.

Measuring low-frequency magnetic fluctuations in a dipole magnetic field experiment like the RT-1 device [65,66] could provide further insights into the applicability of the present theory. In this setting, magnetic fluctuations δB are several orders of magnitude smaller than the background magnetic field ($\delta B \sim 10^{-5}$ T versus $B \sim 10^{-2}$ T to 1 T). Therefore, electrostatic and magnetic fluctuations can be modeled separately using the HMGM Eq. (1) and an induction equation. By measuring the correlation between electrostatic and magnetic fluctuations and comparing the theoretical predictions for the fluctuations frequency in the two models, one could gain a clearer understanding of drift wave turbulence in strongly inhomogeneous configurations.

Due to the HMGM equation's versatility in handling inhomogeneous magnetic fields and densities and its relative simplicity (it is a 1-field equation in three spatial dimensions), we envision its application to studying drift wave turbulence and self-organizing phenomena in physical systems involving complex magnetic geometries and heterogeneous plasma distributions. Examples include magnetospheric plasmas and radiation belts, which are often characterized by hot electrons and cold ions [67] and low-frequency fluctuations associated with the violation of the third adiabatic invariant [68], or stellarators [69], which exhibit inherently three-dimensional magnetic configurations that are expected to significantly affect turbulence and transport properties.

It is now useful to make some considerations on the effects brought by the spatial dependence of the last term in the denominator of the dispersion relation (114), the quantity $Q = \sigma\nabla\cdot\left(A_e B^{-2}\nabla_\perp \xi\right)/A_e\xi$. First, we observe that it comprises the real functions ξ, A_e, and B, as well as the real constant σ. Hence, this term cannot be imaginary, implying that there are no instability or dissipation mechanisms associated with it. Considering the parallelism between the nonlinear stability criterion proved in theorem 1 and the equivalent result for the original HM equation [44], these facts suggest that the stability physics of the HMGM Eq. (1) is essentially analogous to that of the HM equation once field inhomogeneities are appropriately taken into account. However, the spatial dependence of Q introduces distortions in wave propagation that result in attenuation or enhancement of the drift wave depending on the spatial position. To see this, consider a regime in which the term ℓ^2/r^2B^2 appearing in Eq. (113) is small compared with the other terms within the square bracket. This occurs when, for example, ℓ is small, rB is sufficiently large, or σ is small. Further assume that the ratio A_e/B^2 is a weak function of the magnetic flux Ψ,

$$\frac{A_e}{B^2} \sim 1 + \beta_{\Psi_0}\left(\Psi - \Psi_0\right), \tag{116}$$

with $\beta_{\Psi_0} \ll 1$ and Ψ_0 real constants. As previously mentioned, configurations with $A_e \approx B^2$ are expected from the equilibrium statistical mechanics of ensembles of charged particles whose dynamics is dominated by $\boldsymbol{E}\times\boldsymbol{B}$ motion [52]. Choosing ω to satisfy

$$\omega = \frac{\ell\beta_{\Psi_0}}{\lambda}, \tag{117}$$

Eq. (113) reduces to

$$\frac{\partial^2\xi}{\partial\Psi^2} + \left(\beta_{\Psi_0} + \frac{\Delta\Psi}{|\nabla\Psi|^2}\right)\frac{\partial\xi}{\partial\Psi} = 0, \tag{118}$$

where we used the orthogonality of the coordinate system (ζ,Ψ,ϕ) and the identity $B^2 = |\nabla\zeta|^2$. Now consider the case $\beta_{\Psi_0} < 0$ (the cases $\beta_{\Psi_0} = 0$ and $\beta_{\Psi_0} > 0$ can be solved in a similar manner). Let us examine the behavior of the solution at $z = 0$ and in proximity of $r = 0$. Since the flux function has expression $\Psi = Mr^2/\left(r^2+z^2\right)^{3/2}$ in cylindrical coordinates, with M the magnetic moment of the dipole field, one can verify that the ratio $\Delta\Psi/|\nabla\Psi|^2 \sim r$ is small in proximity of $r = 0$, and therefore negligible in (118). It readily follows that $\xi = \xi_0\exp\left(-\beta_{\Psi_0}\Psi\right)$ for some $\xi_0 \in \mathbb{R}$ and the complete drift wave takes the form

$$\chi_d = \xi_0\exp\left(\frac{\left|\beta_{\Psi_0}\right|M}{r}\right)\exp\left\{-i\ell\left(\phi + \frac{\beta_{\Psi_0}}{\lambda}t\right)\right\}, \tag{119}$$

in this region. The amplitude of the drift wave (119) becomes smaller at large radii, while it grows exponentially for smaller values of r (this divergence is caused by the divergence of the point dipole field at the origin, and would disappear for a non-divergent flux function Ψ). This example clearly shows that the inhomogeneity of the magnetic field may result in attenuation or enhancement of drift waves depending on the spatial position. In principle, such behavior could be applied to concentrate/trap the bulk of wave and turbulence activity by tailoring the field inhomogeneity (in the present setting, changing the magnetic moment M affects the distribution of drift waves in proximity of the center $(r,z) = (0,0)$ of the dipole field).

Finally, we observe that the standard dispersion relation for the drift wave in a straight homogeneous magnetic field can be recovered by setting $\boldsymbol{B} = B_0\nabla z$, $\xi = \exp\left\{ik_x x\right\}$, $\ell = -k_y L$, $\log A_e = \log A_{e0} + \beta x$, $\zeta = B_0 z$, $\Psi = B_0 L x$, and $\phi = y/L$ with $B_0, \omega, k_x, k_y, A_{e0}, \beta \in \mathbb{R}$ and $L\beta \sim \epsilon \ll 1$ in Eq. (109). In this case, we have

$$\omega = -\frac{k_y\beta}{\lambda B_0 + \sigma\frac{k_x^2+k_y^2}{B_0}}. \tag{120}$$

7. Concluding remarks

The Hasegawa–Mima equation in general magnetic configuration (HMGM Eq. (1)) is a single nonlinear equation describing the evolution of electrostatic turbulence in inhomogeneous plasmas immersed in a static magnetic field with arbitrary geometry. The HMGM equation serves as a generalization of the standard Hasegawa–Mima (HM) equation for drift wave turbulence in a straight homogeneous magnetic field, and it can be applied as a simple toy model of turbulence in ion-electron plasmas characterized by strong inhomogeneities of both the magnetic field $\boldsymbol{B}$ and the electron spatial density A_e. In particular, the equation can account for turbulence occurring over spatial scales comparable to the characteristic spatial scales of the background magnetic field, and it can be used to model electrostatic turbulence in systems with irregular geometries, such as the dipole magnetic field of a planetary magnetosphere or the confining magnetic field of a stellarator.

In this study, we first derived the HMGM equation from a drift wave turbulence ordering within the kinetic framework of guiding center motion and from a two-fluid plasma model with cold ions and adiabatic electrons, and examined the invariants of the HMGM equation. We then studied the algebraic properties of the equation, and found conditions under which the HMGM equation possesses a noncanonical Hamiltonian structure: the antisymmetric bracket (83) becomes a Poisson bracket whenever the magnetic field $\boldsymbol{B}$ and the electron spatial density A_e fulfill the integrability condition (80). This same condition is required for the conservation of generalized enstrophy W_Ω, which is a Casimir invariant of the Poisson bracket (83). Using the algebraic structure of the HMGM equation, we applied the energy-Casimir method to obtain a nonlinear stability criterion for steady solutions of the HMGM equation (Theorem 1). This result implies that sufficiently regular solutions of the HMGM equation, whose initial conditions are sufficiently close to critical points of the energy-Casimir function (90) characterized either by $\nu = 0$ or (92), remain close to these critical points at all later times. In addition, we showed that radially sheared stable toroidal zonal flows may be created in dipole magnetic fields, and characterized the angular frequency of magnetospheric drift waves, which explicitly depends on the magnetic field geometry.

Amidst the novel effects induced by field inhomogeneities in the HMGM equation, we anticipate enhanced transport linked to magnetic helicity to play a pivotal role. While $\boldsymbol{E}\times\boldsymbol{B}$ and polarization drifts maintain orthogonality to the magnetic field, non-vanishing helicity density ($\boldsymbol{B}\cdot\nabla\times\boldsymbol{B} \neq 0$) implies (Frobenius theorem [53]) that $\boldsymbol{B}$ fails to define the normal of a two-dimensional surface. Consequently, charged particle orbits effectively span three spatial dimensions, implying amplified ergodicity and phase space mixing. We therefore foresee drift wave turbulence in magnetic fields exhibiting non-vanishing helicity density, such as Beltrami fields [70,71], commonly observed in astrophysical and laboratory plasmas, to exhibit anomalous characteristics arising from the breakdown of enstrophy conservation associated with the loss of Hamiltonian structure and the subsequent absence of inverse energy cascades that underpin zonal flow formation.

Finally, we remark that for practical purposes (e.g. numerical implementation), the solution χ of the HMGM equation can be used to approximate $\varphi \approx \chi$ since both χ and $\varphi = \chi + o\left(\epsilon^2\right)$ scale as $o(\epsilon)$ in the two-fluid ordering of Table 3.

CRediT authorship contribution statement

Naoki Sato: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Writing – original draft, Writing – review & editing, Supervision. **Michio Yamada:** Conceptualization, Methodology, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Naoki Sato reports financial support was provided by Japan Society for the Promotion of Science.

Data availability

No data was used for the research described in the article.

Funding

The research of NS was partially supported by JSPS, Japan KAKENHI Grant No. 21K13851 and 22H04936.

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