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Radiation by Plasma Waves in a Homogeneous Plasma.*

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§1. Introduction and summary

In the presence of inhomogeneities in density the longitudinal plasma waves and the electromagnetic waves are coupled and there results in the possible excitation of electromagnetic radiation from plasma oscillations. Since the work by Field[1)] many authors treated these problems. Also in a homogeneous plasma the density fluctuations of statistical nature exist and the transformation of plasma waves into the electromagnetic waves due to gradient coupling will be possible when the plasma waves are excited by such a mechanism as beam-injection and propagate in a homogeneous plasma. This possibility was first pointed out by Ginzburg and Zhelezniakov.[2)]

In this paper we shall be concerned with the latter processes. We shall consider a model that a progressive plasma wave is switched on in the homogeneous main plasma where collisions are to be neglected and no external magnetic field is impressed. It is further assumed that the electrons and the ions are nearly at thermal equilibrium and the mean energy of Coulomb interaction between the nearby electrons is small compared with the thermal energy. The energy of the incident plasma wave will be converted partly into the electromagnetic field by scatterings on the density fluctuations of the medium as it goes through the medium. By density fluctuations we mean the variations of the electron and ion densities which take place stochastically in space and time, (thermal fluctuations) as will as harmonically. The statistical ensemble means of their frequency Fourier transforms may be divided into two parts; rather a broad spectrum around $\omega \approx 0$ and a sharp peak at the plasma frequency ω_p. So the radiation processes are divided into two processes. The radiation due to the density fluctuation having broad spectrum around $\omega \approx 0$ has the frequency ν nearly equal to that of the incident plasma wave Ω,

$\nu \simeq \Omega \simeq \omega_p$. While the frequency of the radiation in the second process is $\nu = \overline{\omega} + \Omega \approx 2\,\omega_p$ where $\overline{\omega}$ is the mean frequency of the plasma oscillations in the main plasma.

The set of the Boltzmann-Vlasov equations and the Maxwell equations is used. For the incident plasma wave it is taken a simple plane wave of the form.

$$\vec{\mathcal{E}}_0 = \frac{\vec{K}}{K} U \exp\,[i(\vec{K}\vec{r} - \Omega t)] \,.$$

The basic equations for this and for the main plasma are given in §2. Taking the Landau damping out of $\vec{\mathcal{E}}_0$, the scattering of $\vec{\mathcal{E}}_0$ on the density fluctuations within a localized region of the main plasma is treated as a stationary problem in §3. This procedure was first used by Tidman and Weiss[3] in their work on the similar problems in non-uniform plasma. Under the Born approximation the equations for the scattered fields having both the longitudinal mode and the transverse mode are written down. In §4 a far distance solution of the scattered fields is obtained using a Fourier transform over space and time and some further approximations. In §5 and §6 the radiation intensities are expressed in terms of the statistical ensemble means of mass or charge fluctuations for which we refer to the calculations by Salpeter.[4]

In details the contributions of mass fluctuation and of thermal charge fluctuation for the radiation with the frequency $\nu \simeq \Omega$ are examined separately. We define the mass fluctuation as $\Delta N^M = (M/(M+m))\,\Delta N^i + (m/(M+m))\,\Delta N^e$ and the charge fluctuation as $\Delta N^c = Z\,\Delta N^i - \Delta N^e$, where ΔN^e and ΔN^i are the electron and the ion fluctuations in density, respectively, m and $-e$ are the mass and the charge of an electron, and M and Ze are those of an ion. The radiation intensity per unit time per unit volume due to the mass fluctuation is found to be for $Z = 1$,

$$Q^M(\nu \simeq \Omega) = \frac{1}{24\pi}\left(\frac{\omega_p^4}{n_o c^3}\right)\left(\frac{v_T}{v_{ph}}\right)(\varepsilon U^2) ,$$

where n_o is the mean number density of the electrons, v_T is their root mean square velocity, $v_{ph} = \Omega/K$ the phase velocity of the incident plasma wave, and $(\varepsilon U^2/2)$ is its energy density in the MKS unit system (ε = the permittivity in the vacuum). While the contribution of the thermal charge fluctuation is shown to be negligible compared with above. The angular distribution is of the type of a dipole radiation and the width of the frequency spectrum is due to the Doppler effect of the ion thermal motions.

The radiation intensity of the second process is calculated for $Z = 1$ as

$$Q^C(\nu \simeq \bar{\omega} + \Omega) = \frac{2}{5\sqrt{3}\pi}\left(\frac{\omega_p^4}{n_o c^3}\right)\left(\frac{v_T}{c}\right)^2 (\varepsilon U^2) .$$

For Q^C the harmonic fluctuation of charge, i.e., the medium plasma oscillations is responsible. The radiation has the angular distribution such as of a quadrupole radiation and the width of the frequency spectrum will be determined by the Landau damping factor of the medium plasma oscillations.

Our results are to be compared with the corresponding results given previously by Ginzburg and Zheleznïakov in the heuristic way.[2)]

The fact that their value of $Q^M(\nu \simeq \Omega)$ is exactly in agreement with ours, which confirms their plausible use of the elementary theory of scattering by dielectrics to this problem. On the other hand they estimated the corresponding $Q^C(\nu \simeq \bar{\omega} + \Omega)$ as

$$Q^C(G-Z) = \frac{1}{4\sqrt{3}\pi}\left(\frac{\omega_p^4}{n_o c^3}\right)\left(\frac{v_T}{v_{ph}}\right)^2 (\varepsilon U^2) .$$

As compared with ours their value is to be reduced about by the factor of $(v_{ph}/c)^2 = (\Omega/cK)^2$. The discrepancy indicates that the elementary theory of scattering by dielectrics used by Ginzburg and Zhelezniakov is of restricted applicability.

Finally in §7, our results are compared with other works on non-uniform model. As for $Q^M(\nu \simeq \Omega)$ Cohen derived the similar result by the continuum theory of radiation and waves in a continuous loss-free electron fluid.[5] Also the suitable version of the Tidman and Weiss theory leads substantially the same conclusion, in which they assumed the non-uniform static background of ions and dealed only with quantities about electrons. It seems rather of question that the electron fluid model by these authors give the almost correct values of Q^M, for which the mass fluctuation or the ion fluctuation is mainly responsible. The reason is that the explicit or impllicit assumption of static non-uniform background underlying these approaches ascertains the coincidence among the respective values.

It is apparent that the models of the static non-uniform ion distributions cannot excite such radiation of higher harmonics as deduced from our model.

Recently Dawson and Oberman[6] treated the same process from a different point of view. They calculated the ac conductivity for frequencies embracing the plasma frequency and found the same $Q^M(\nu \simeq \Omega)$ by using the Kirchhoff's law.

§2. Basic equations

We shall assume the set of the Boltzmann-Vlasov equations and the Maxwell equations valid for the present problems. Let the electron and the ion distribution functions be $f^e(\vec{r}, \vec{v}, t)$ and $f^i(\vec{r}, \vec{v}, t)$, and the electric and magnetic fields $\vec{E}$ and $\vec{B}$, then

$$\left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}\right)f^e - \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B})\vec{\nabla}_v f^e = 0 \qquad (1\text{-a})$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}\right)f^i + \frac{Ze}{M}(\vec{E} + \vec{v} \times \vec{B})\vec{\nabla}_v f^i = 0 \qquad (1\text{-a}')$$

$$\vec{\nabla}\vec{E} = \frac{e}{\varepsilon}\int (Zf^{i} - f^{e})d\vec{v} \qquad (1\text{-b})$$

$$\frac{1}{C^2}\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} - \mu e \int \vec{v}(Zf^{i} - f^{e})d\vec{v} \qquad (1\text{-c})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \vec{\nabla}\vec{B} = 0 \qquad (1\text{-d})$$

The MKS units system is used and ε and μ are the permittivity and the permeability in the vacuum, respectively.

For a given volume V containing N electrons and (N/Z) ions, the Debye length D and the plasma frequency ω_p are defined as

$$D = (\Theta/m)^{1/2}\,\omega_p^{-1}\,, \qquad (2)$$

$$\omega_p = (n_0 e^2/m\,\varepsilon)^{1/2}\,, \qquad (3)$$

where $n_0 = N/V$ is the mean number density of electrons and Θ is the temperature in units of the Boltzmann constant.

A. The Main Plasma

By $\delta F_1^{e,i}(\vec{r}, \vec{v}, t)$ we denote the electron and the ion distribution functions to describe the thermal fluctuations; the stochastic variations of charges $\delta\rho^{e,i}(\vec{r}, t)$ and of currents $\delta\vec{J}^{e,i}(\vec{r}, t)$.

Then

$$\delta\rho^{e} = -e\int \delta F_1^{e} d\vec{v} \quad \text{and} \quad \delta\vec{J}^{e} = -e\int \vec{v}\delta F_1^{e} d\vec{v} \qquad (4\text{-a})$$

$$\delta\rho^{i} = Ze\int \delta F_1^{i} d\vec{v} \quad \text{and} \quad \delta\vec{J}^{i} = Ze\int \vec{v}\delta F_1^{i} d\vec{v}\,. \qquad (4\text{-a}')$$

It is nataural to postulate the chage conservation in thermal fluctuations

$$\frac{\partial}{\partial t}\delta\cdot\rho^{e,i} + \mathrm{div}\,\delta\vec{J}^{e,i} = 0\,. \qquad (5)$$

Further, we assume

$$\left(\frac{\partial}{\partial t} + \vec{v}\vec{\nabla}\right)\delta F_1^{e,i} = 0\,. \qquad (6)$$

For the total distributions of the electrons and the ions we write

$$f^{e,i} = f_o^{e,i} + F_1^{e,i} + \delta F_1^{e,i} \quad , \tag{7}$$

where f_o^e and f_o^i are the Maxwellian distributions with the normalization $\int f_o^e(\vec{v})\, d\vec{v} = n_o$ and $\int f_o^i(\vec{v})\, d\vec{v} = n_o/Z$ and $F_1^{e,i}$ the perturbed distribution related to the electron plasma oscillations or to the ion oscillations. The distribution functions and the electric field $\vec{\varepsilon}_1$ satisfy the following linearized equations

$$\left(\frac{\partial}{\partial t} + \vec{v}\,\vec{\nabla}\right) F_1^e - \frac{e}{m}\, \vec{\mathcal{E}}_1 \vec{\nabla}_V f_o^e = 0 \quad , \tag{8-a}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\,\vec{\nabla}\right) F_1^i - \frac{Ze}{M}\, \vec{\mathcal{E}}_1 \vec{\nabla}_V f_o^i = 0 \quad , \tag{8-a'}$$

$$\mathrm{div}\, \vec{\mathcal{E}}_1 = \frac{e}{\varepsilon} \int (Z F_1^i - F_1^e)\, d\vec{v} + \frac{e}{\varepsilon} \int (Z \delta F_1^i - \delta F_1^e)\, d\vec{v} \; , \tag{8-b}$$

and

$$\frac{1}{c^2} \frac{\partial \mathcal{E}_1}{\partial t} = -\mu e \int \vec{v}\, (Z F_1^i - F_1^e)\, d\vec{v} - \mu e \int \vec{v}\, (Z \delta F_1^i - \delta F_1^e)\, d\vec{v} \; . \tag{8-c}$$

B. Incident Plasma Wave

It is only sufficient to consider the perturbed electron distribution $F_o^e(\vec{r}, \vec{u}, t)$ and the associated electric field $\vec{\varepsilon}_o(\vec{r}, t)$ and to write,

$$\left(\frac{\partial}{\partial t} + \vec{v}\,\vec{\nabla}\right) F_o^e - \frac{e}{m}\, \vec{\mathcal{E}}_o \vec{\nabla}_V f_o^e = 0 \; , \tag{9-a}$$

$$\mathrm{div}\, \vec{\mathcal{E}}_o = -\frac{e}{\varepsilon} \int F_o^e\, d\vec{v} \quad , \tag{9-b}$$

$$\frac{1}{c^2} \frac{\partial \vec{\mathcal{E}}_o}{\partial t} = \mu e \int \vec{v}\; F_o^e\, d\vec{v} \tag{9-c}$$

We shall choose for $\vec{\mathcal{E}}_o$ a simple plane wave of the form

$$\vec{\mathcal{E}}_o = \frac{\vec{K}}{K}\, U \exp\{i(\vec{K}\vec{r} - \Omega t)\} \; . \tag{10}$$

When the Landau damping is out of consideration, the dispersion formula is given by

$$\Omega^2 = \omega_p^2 + K^2 V_T^2 \quad \text{and} \quad V_T^2 = 3\, \Theta/m \tag{11}$$

§3. Interaction as a stationary Problem

Without loss of generality we may take the Landau damping out of the incident plasma wave. Thus we are free from initial value problem and can treat the interaction of the incident wave with a localized region of the main plasma as a stationary problem (see Fig.1). We shall make the Born approximation and write the distributions and the fields in the interaction as

$$f^{e,i}_{total} = G^{e,i} + (F_1^{e,i} + \delta F_1^{e,i}) + F_0^{e,i} + f_0^{e,i}$$

$$\vec{E}_{total} = \vec{E} + \vec{\mathcal{E}}_1 + \vec{\mathcal{E}}_0$$

The scattered fields $\vec{E}$ include both the transverse part and the longitudinal part. G^e and G^i are thier source functions. And we may assum $G \ll F_i$, F_1, F_0 and $E \ll \mathcal{E}_1$, $\mathcal{E}_0$.

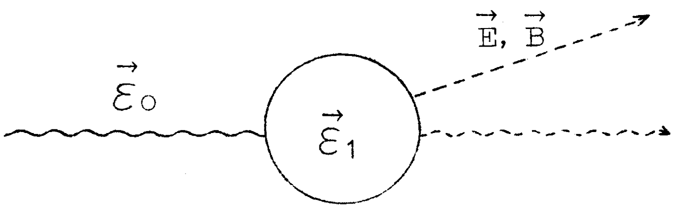

Fig. 1

Substituting $f^{e,i}_{total}$ and $\vec{E}_{total}$ into (1-a~d) and with use of (6), (8-a~c) and (9-a~c), under the above approximations, we find

$$\left(\frac{\partial}{\partial t}+\vec{v}\vec{\nabla}\right)G^e - \frac{e}{m}\vec{E}\cdot\vec{\nabla}_v f_0^e - \frac{e}{m}\vec{\mathcal{E}}_0\cdot\vec{\nabla}_v (F_1^e+\delta F_1^e) - \frac{e}{m}\vec{\mathcal{E}}_1\vec{\nabla}_v F_0^e = 0 \quad (12\text{-a})$$

$$\left(\frac{\partial}{\partial t}+\vec{v}\vec{\nabla}\right)G^i + \frac{Ze}{M}\vec{E}\cdot\vec{\nabla}_v f_0^i + \frac{Ze}{M}\vec{\mathcal{E}}_0\cdot\vec{\nabla}_v(F_1^i+\delta F_1^i) = 0 \quad (12\text{-a}')$$

$$\operatorname{div}\vec{E} = \frac{e}{\varepsilon}\int (ZG^i - G^e)\,d\vec{v}\,, \quad (12\text{-b})$$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu e\frac{\partial}{\partial t}\int \vec{v}(ZG^i-G^e)\,d\vec{v} + \frac{e}{\varepsilon}\vec{\nabla}\int (ZG^i-G^e)\,d\vec{v} \quad (12\text{-c})$$

The terms such as $(\vec{v}\times\vec{B})(\partial f_0^{e,i}/\partial\vec{v})$ are dropped due to the isotropy of $f_0^{e,i}\,\vec{v})$ in the velocity space. Also such terms expressing the non-linear self-interactions as $\vec{\mathcal{E}}_0\vec{\nabla}_V F_0^e$ and $\vec{\mathcal{E}}_1\vec{\nabla}_V(F_1^{e,i}+\delta F_1^{e,i})$ are out of our treatment and omitted.

We define

$$F_1^c = Z F_1^i - F_1^e$$

$$F_1^M = [M/(M+m)]\,F_1^i + [m/(M+m)]\,F_1^e\,. \tag{13}$$

F_1^c corresponds to the harmonically oscillatory part of charge fluctuation in the main plasma, and F_1^M to that of the mass fluctuation. The similar expressions are given for $\delta F_1^{e,i}$ and G. Taking account of $m/M \ll 1$, we rewrite (12-a) to (12-c) as

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}\right)G^c+\frac{e}{m}\vec{E}\cdot\vec{\nabla}_V\left(f_0^e+\frac{mZ^2}{M}f_0^i\right)$$

$$-\frac{e}{m}\vec{\mathcal{E}}_1\cdot\vec{\nabla}_V F_0^c-\frac{e}{m}\vec{\mathcal{E}}_c\cdot\vec{\nabla}_V\{F_1^c+\delta F_1^c-Z(F_1^M+\delta F_1^M)\}=0 \tag{14-a}$$

$$\mathrm{div}\,\vec{E}=\frac{e}{\varepsilon}\int G^c\,d\vec{v}\,, \tag{14-b}$$

$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}=\mu e\,\frac{\partial}{\partial t}\int\vec{v}G^c\,d\vec{v}+\frac{e}{\varepsilon}\,\vec{\nabla}\int G^c\,d\vec{v} \tag{14-c}$$

Note that $F_0^c=-F_0^e$ because of $F_0^i=0$.

§4. Transformed equations and far-distance solutions of the scattered fields.

We shall define a Fourier transform over space and time of a function $f(\vec{r},\,t)$

$$f(\vec{r},\,t)=\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}d\vec{k}\;f_\omega(\vec{k},\,\omega)e^{i(\vec{k}\vec{r}-\omega t)}$$

$$f_\omega(\vec{k},\,\omega)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}d\vec{r}\;f(\vec{r},\,t)e^{-i(\vec{k}\vec{r}-\omega t)} \tag{15}$$

So far as no confusion possibly arises, we use only the subscript ω for the transformed function.

Appling the Fourier transform to (14-a~c) and by some manipulations we obtain

$$\left[1-\frac{e^2}{m\varepsilon}\int d\vec{v}\,\frac{f_0^{\,e}+(mZ^2/M)f_0^{\,i}}{(\omega-\vec{k}\vec{v})^2}\right]\vec{k}\,\vec{E}_\omega$$

$$=-\frac{e^2}{m\varepsilon}\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}d\vec{r}\int_{-\infty}^{+\infty}dt\;e^{-i(\vec{k}\vec{r}-\omega t)}\left[\int d\vec{u}\,\frac{F_1^{\,c}+\delta F_1^{\,c}-Z(F_1^{\,M}+\delta F_1^{\,M})}{(\omega-\vec{k}\vec{v})^2}(\vec{k}\mathcal{E}_0)\right.$$

$$\left.+\int d\vec{v}\,\frac{F_0^{\,2}}{(\omega-\vec{k}\vec{u})^2}(\vec{k}\cdot\vec{\mathcal{E}}_1)\right]\quad, \tag{16}$$

$$\left[-k^2+\frac{\omega^2}{c^2}-\frac{\omega}{c^2}\frac{e^2}{m\varepsilon}\int d\vec{v}\,\frac{f_0^{\,e}+(mZ^2/M)f_0^{\,i}}{(\omega-kv)^2}\right]\vec{k}\times\vec{E}_\omega$$

$$=-\frac{\mu e^2}{m}\frac{\omega}{(2\pi)^4}\int_{-\infty}^{+\infty}d\vec{r}\int_{-\infty}^{+\infty}dt\;e^{-i(\vec{k}\vec{r}-\omega t)}$$

$$\cdot\left[\int d\vec{v}\left\{\frac{F_1^{\,c}+\delta F_1^{\,c}-Z(F_1^{\,M}+\delta F_1^{\,M})}{\omega-\vec{k}\vec{v}}\right\}\left\{\vec{k}\times\vec{\varepsilon}_0+\frac{(\vec{k}\times\vec{v})\vec{k}\vec{\mathcal{E}}_0}{\omega-\vec{k}\vec{v}}\right\}\right.$$

$$\left.+\int d\vec{v}\,\frac{F_0^{\,c}}{\omega-\vec{k}\vec{v}}\left\{\vec{k}\times\vec{\varepsilon}_1+\frac{(\vec{k}\times\vec{v})\vec{k}\vec{\mathcal{E}}_1}{\omega-\vec{k}\vec{v}}\right\}\right]. \tag{16'}$$

The general inversions of (16) and (16') are difficult and we are forced to make some further approximations; to consider only the contribution of $\vec{k}$ for $\vec{E}$ $(\vec{r},t)$ such that $(\vec{k}\cdot\vec{v})^2\ll\omega^2$ and to assume $F_1^{\,M}+\delta F_1^{\,M}$ isotropic in the velocity space. We expand (16) and (16') in power of $(\vec{k}\vec{v})/\omega$ and retain terms to the order of $(\vec{k}\cdot\vec{v})^2/\omega^2$, and use (8-b, c), (9-b, c) and such relations as

$$\int d\vec{r}\int dt\;e^{-i(\vec{k}\vec{r}-\omega t)}(\vec{k}\vec{\mathcal{E}})g(\vec{r}t)$$

$$=\int d\vec{r}\int dt\;e^{-i(\vec{k}\vec{r}-\omega t)}(-i\vec{\nabla})(g\vec{\mathcal{E}})=\frac{1}{\omega}\int d\vec{r}\int dt\;e^{-i(\vec{k}\vec{r}-\omega t)}\frac{\partial}{\partial t}\vec{\nabla}(g\vec{\mathcal{E}})$$

Finally we find the differential equations for $\vec{E}$

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\{\frac{\partial^2}{\partial t^2}+\omega_p^2-v_T^2\vec{\nabla}^2\}\vec{E}(r,t)=\frac{\partial}{\partial t}\vec{\nabla}\vec{S}(\vec{r},t) \tag{17}$$

$$\frac{\partial}{\partial t}\vec{\nabla}\times\{\frac{\partial^2}{\partial t^2}+\omega_p^2-c^2\nabla^2\}\vec{E}(\vec{r},t)=\frac{\partial}{\partial t}\vec{\nabla}\times\vec{S}(\vec{r},t) \tag{18}$$

$$\vec{S}=\frac{e}{m}[2\{\vec{\mathcal{E}}_1(\vec{\nabla}\cdot\vec{\mathcal{E}}_0)+\vec{\mathcal{E}}_0(\vec{\nabla}\cdot\vec{\mathcal{E}}_1)\}+(\vec{\mathcal{E}}_1\cdot\vec{\nabla})\vec{\mathcal{E}}_0+(\vec{\mathcal{E}}_0\vec{\nabla})\vec{\mathcal{E}}_1$$

$$-\frac{Ze}{\varepsilon}\vec{\mathcal{E}}_0\int d\vec{v}(F_1^M+\delta F_1^M)] \tag{19}$$

where v_T^2 is the mean-square velocity given in (11), and $\vec{E}$ and the source function $\vec{S}$ are composed of the longitudinal and the transverse parts,

$$\vec{E}=\vec{E}_L+\vec{E}_T \quad \text{and} \quad \vec{S}=\vec{S}_L+\vec{S}_T$$

Let us define

$$\begin{aligned}\vec{S}_T(\vec{r},t)&=\int_{-\infty}^{+\infty}\vec{S}_{T,\nu}(\vec{r})\,e^{-i\nu t}\,d\nu\,,\\ \vec{E}_T(\vec{r},t)&=\int_{-\infty}^{+\infty}\vec{E}_{T,\nu}(\vec{r})\,e^{-i\nu t}\,d\nu\,,\end{aligned} \tag{20}$$

with

$$\begin{aligned}\vec{E}^*_{T,\nu}&=\vec{E}_{T,-\nu}\\ \vec{S}^*_{T,\nu}&=\vec{S}_{T,-\nu}\end{aligned} \tag{21}$$

From (18) we have

$$(\vec{\nabla}^2+\frac{\nu^2}{c'^2})\vec{E}_{T,\nu}=-\frac{1}{c^2}\vec{S}_{T,\nu} \tag{22}$$

with

$$c'^2=c^2/(1-\frac{\omega_p^2}{\nu^2})\,.$$

As well known the far-distance solution of the radiation field is

$$\vec{E}_{T,\nu}(\vec{r}) = \frac{1}{4\pi c^2}\int_V \frac{e^{i\frac{\nu}{c'}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\vec{S}_{T,\nu}(\vec{r}')\,d\vec{r}'$$

$$\simeq \frac{e^{i\frac{\nu}{c'}r}}{4\pi c^2 r}\int_V e^{-i\frac{\nu}{c'}\vec{n}\vec{r}'}\vec{S}_{T,\nu}(\vec{r}')\,d\vec{r}' , \tag{23}$$

$$\vec{B}_{T,\nu} = \vec{n}\times\dot{\vec{E}}_{T,\nu}/c'$$

$$\vec{n} = \vec{r}/r .$$

The Poynting flux $\vec{\Sigma}$ for the radiation is, keeping in mind that the time dependency of field quantities is like $e^{-i\nu t}$,

$$\vec{\Sigma}(\vec{r}, t) = \frac{1}{2\mu}\vec{E}(\vec{r}, t)\times\vec{B}(\vec{r}, t) \tag{24}$$

And we obtain

$$\int_{-\infty}^{+\infty}(\vec{n}.\vec{\Sigma})dt = \frac{\pi}{\mu}\int_{-\infty}^{+\infty}d\nu\,\frac{1}{c'}\,|\vec{n}\times\vec{E}_{T,\nu}|^2 , \tag{25}$$

where T is chosen sufficiently long compared with the interaction time which is the linear dimension of local cells of density fluctuation divided by the phase velocity of the incident plasma wave, being of the order of or less than ω_p^{-1}.

The radiation intensity in the solid angle $d\vec{S}$ per unit time the very quantity we seek is given by

$$I(\vec{S})\,d\vec{S} = \frac{\pi r^2 d\vec{S}}{T\mu}\int_{-\infty}^{+\infty}d\nu\,\frac{1}{c'}\,|\vec{n}\times\vec{E}_{T,\nu}|^2$$

$$= \frac{d\vec{S}}{16\pi\mu c^4 T}\int_{-\infty}^{+\infty}d\nu\,\frac{1}{c'}\,\left|\vec{n}\times\int_V e^{-i\frac{\nu}{c'}\vec{n}\cdot\vec{r}'}\vec{S}_{T,\nu}(\vec{r}')\,d\vec{r}'\right|^2. \tag{26}$$

§5. Effects of the mass fluctuation and the thermal charge fluctuation.

We shall take F_1^c in (13) corresponding to the harmonically oscillatory part of charge fluctuation out of consideration.
We subtract the contribution of F_1^c from $\vec{S}$ in (19) and $\vec{\mathcal{E}}_1$ to be substituted in it is given by setting $F_1^c = 0$ in (8-b, c) as

$$\operatorname{div} \vec{\mathcal{E}}_1 = \frac{e}{\varepsilon} \int \delta F_1^c \, d\vec{v} \ . \qquad (8\text{-b}')$$

Let us define

$$\begin{aligned} \Delta N^M(\vec{r}, t) &= \Delta N_p^M(\vec{r}, t) + \Delta N_T^M(\vec{r}, t) , \\ \Delta N_p^M(\vec{r}, t) &= \int d\vec{v}\, F_1^M(\vec{r}, \vec{v}, t) , \\ \Delta N_T^{M,C}(\vec{r}, t) &= \int d\vec{v}\, \delta F_1^{M,C}(\vec{r}, \vec{v}, t) . \end{aligned} \qquad (27)$$

Since these have the time scales much different from the interaction time $\lesssim \omega_p^{-1}$, their frequency Fourier transforms have no resonant peak at ω_p. The ion organized motions corresponding to ΔN_p^M change so slowly during the period ω_p^{-1} that

$$\begin{aligned} \Delta N_p^M(\vec{k}, \omega) &= \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d\vec{r} \int_{-\infty}^{+\infty} dt \; e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \Delta N_p^M(\vec{r}, t) \\ &\cong \frac{1}{(2\pi)^3} \int d\vec{r} \; e^{-i\vec{k}\cdot\vec{r}} \langle \Delta N_p^M(\vec{r}) \rangle \int dt\, e^{i\omega t} \\ &= \langle \Delta N_p^M(\vec{k}) \rangle\, \delta(\omega) , \end{aligned} \qquad (28)$$

where $\delta(\omega)$ is the Dirac δ-function.

On the other hand ΔN_T^M and ΔN_T^C vary stochastically in space and time, much faster than ω_p^{-1}. We can smooth out them and get the approximate relations like above

$$\Delta N_T^{M,C}(\vec{k}, \omega) \cong \langle \Delta N_T^{M,C}(\vec{k}) \rangle\, \delta(\omega) \qquad (29)$$

It should be noted that the same expressions are deduced from the very different reasons. We may replace the time average above by the statistical ensemble mean.

Therefore one obtain from (8-b') and (29)

$$\vec{\mathcal{E}}_1(\vec{k},\ \omega) = \frac{e}{i\varepsilon}\ \frac{\vec{k}}{k^2}\ \delta(\omega) \langle \Delta N_T{}^{C}(\vec{k}) \rangle\ . \tag{30}$$

Substituting this and (10) with (27)~(29) into (19), we find

$$\vec{S} = \frac{e^2}{m\varepsilon}\int_{-\infty}^{+\infty} d\vec{k} \int_{-\infty}^{+\infty} d\omega\ e^{i(\vec{k}+\vec{K})\cdot\vec{r}-i(\omega+\Omega)t}\ U\delta(\omega)\{\vec{F}\langle \Delta N_T{}^{C}(\vec{k})\rangle - \frac{Z\vec{K}}{K}\langle \Delta N^{M}(\vec{k})\rangle\} \tag{31}$$

where

$$\langle \Delta N^{M}(\vec{k})\rangle = \langle \Delta N_p{}^{M}(\vec{k})\rangle + \langle \Delta N_T{}^{M}(\vec{k})\rangle$$

and

$$\vec{F}(\vec{k},\ \vec{K}) = \frac{1}{k^2K}(2K^2+\vec{k}\cdot\vec{K})\vec{k} + \frac{1}{k^2K}(2k^2+\vec{k}\cdot\vec{K})\vec{K}\ .$$

From the inversion of (20) and (31) one obtain

$$\vec{J} = \int_V e^{-i\vec{n}'\cdot\vec{r}'}\ \vec{S}_{\nu}(\vec{r}')d\vec{r}'$$

$$= \frac{e^2}{m\varepsilon}(2\pi)^3\int d\vec{k}\ \delta(\vec{k}+\vec{K}-\vec{n}')\delta(\nu-\Omega)\{\vec{F}\langle \Delta N_T{}^{C}(\vec{k})\rangle - \frac{Z\vec{K}}{K}\langle \Delta N^{M}(\vec{k})\rangle\}$$

with

$$\vec{n}' \equiv \frac{\nu}{c'}\vec{n} \qquad \text{and} \quad \delta(\vec{\ell}) \equiv \delta(\ell_x)\ \delta(\ell_y)\ \delta(\ell_z)\ .$$

Thus $\vec{E}_{T,\nu}$ in (23) will be

$$\vec{E}_{T,\nu} = -\frac{e^{i\frac{\nu}{c'}r}}{4\pi r}\ \frac{\omega_p^2}{c^2 n_o}\ U\ \frac{\vec{n}\times(\vec{n}\times\vec{K})}{K}\ (2\pi)^3\int d\vec{k}\ \delta(\vec{k}+\vec{K}-\vec{n}')\ \delta(\nu-\Omega)$$

$$\times\{\frac{2(k^2-K^2)}{k^2}\ \langle \Delta N_T{}^{C}(\vec{k})\rangle - Z\langle \Delta N^{M}(\vec{k})\rangle\}\ . \tag{32}$$

The radiation intensity $I(\vec{S})\,d\vec{s}$ in (26) is calculated as

$$I(\vec{S})d\vec{s} = \frac{\sin^2\theta\, d\vec{s}}{2(4\pi)^2}\left(\frac{\omega_p^2}{n_o c}\right)^2(\varepsilon U^2)\int d\nu\, \delta(\nu-\Omega)\frac{1}{c'}\left[\frac{4(k^2-K^2)^2}{k^4}\,|\Delta N_T^{C}(\vec{k})|^2 + Z^2\,|\Delta N^{M}(\vec{k})|^2\right] \quad (33)$$

$$\vec{k} = \vec{n}' - \vec{K}$$

where θ is the angle between the wave vector of the incident wave $\vec{K}$ and $\vec{n}$ and use is made of the identity

$$\int_{-\infty}^{\infty} d\nu\, \delta^2(\nu-\Omega) = \frac{T}{2\pi}\int_{-\infty}^{\infty} d\nu\, \delta(\nu-\Omega)\,.$$

As $\nu = \Omega \simeq \omega_p$ it is approximated that $|\vec{n}'| \ll K$ and

$$\frac{4(k^2-K^2)^2}{k^4} = \frac{4\{(\vec{K}-\vec{n}')^2-K^2\}^2}{(\vec{K}-\vec{n}')^4} \simeq \frac{4^2(\vec{n}'\cdot\vec{K})^2}{K^4}\,.$$

Now we can separate the contributions of ΔN_T^{C} and ΔN_T^{M} for $I(\vec{S})$ as

$$I(\Omega) = I^C(\vec{S}) + I^M(\vec{S})\ ,$$

$$I^C(\vec{S})d\vec{s} = \frac{1}{2\pi^2}\frac{\omega_p^4}{n_o c^3}\left(\frac{V_T}{C}\right)^2\left(\frac{V_T}{V_{ph}}\right)(\varepsilon U^2)(\sin^2\theta\,\cos^2\theta\, d\vec{s})\int d\nu\,\delta(\nu-\Omega)(2\pi)^6 \langle \Delta N_T^{C}(\vec{k}\simeq\vec{K})\rangle^2 \quad (34\text{-a})$$

$$I^M(\vec{S})d\vec{s} = \frac{1}{2(4\pi)^2}\frac{\omega_p^4}{n_o c^3}\left(\frac{V_T}{V_{ph}}\right)(\varepsilon U^2)(\sin^2\theta\, d\vec{s})\int d\nu\,\delta(\nu-\Omega)(2\pi)^6 Z^2\langle \Delta N^{M}(\vec{k}\simeq\vec{K})\rangle^2 \quad (34\text{-b})$$

where $v_{ph} = \Omega/K$ is the plase velocity of the incident wave and that of the radiation is taken as

$$c'(\nu) = c\left(1-\frac{\omega_p^2}{\nu^2}\right)^{-1/2} \cong C\ (V_{ph}/V_T)\,. \quad (35)$$

Finally the radiation intensity per unit time and unit volume is given by

$$Q(\nu) = \frac{1}{V}\int I(\vec{S})d\vec{s}, \quad (36)$$

$$Q^C(\nu) = \frac{4}{15\pi}\frac{\omega_p^4}{n_o c^3}\left(\frac{V_T}{c}\right)^2\left(\frac{V_T}{V_{ph}}\right)(\varepsilon U^2)\int d\nu\,\delta(\nu-\Omega)\left[\frac{(2\pi)^3}{\sqrt{N}}\langle\Delta N_T^{C}(\vec{k}\simeq\vec{K})\rangle\right]^2 \quad (37\text{-a})$$

$$Q^M(\nu) = \frac{1}{12\pi}\frac{\omega_p^4}{n_o c^3}(\frac{V_T}{V_{ph}})(\varepsilon U^2)\int d\nu\, \delta(\nu-\Omega)[\frac{(2\pi)^3}{\sqrt{N}}<\Delta N^M(\vec{k}\simeq\vec{K})>]^2. \qquad (37\text{-b})$$

The quantities in the bracket are nothing but the statistical ensemble average of density fluctuations and their expressions for the cases of interest have been derived by Salpeter.[4] To use the values by Salpeter it needs a caution that for $<\Delta N_T{}^C>^2$ the subtraction of the part due to the plasma oscillations from the corresponding coefficient to the whole charge fluctuation is necessary. The calculations are worked out in the Appendix.

With use of (A-7) and (A-8) we can evaluate Q^C and Q^M as,

$$Q^C \lesssim (\frac{\omega_p^4}{n_o c^3})(\frac{V_T}{C})^2(\frac{V_T}{V_{ph}})^5(\varepsilon U^2) , \qquad (38\text{-a})$$

$$Q^M = \frac{1}{12\pi}\frac{\omega_p^4}{n_o c^3}(\frac{V_T}{V_{ph}})(\varepsilon U^2)\int d\nu\, \delta(\nu-\Omega)\frac{1}{Z(Z+1)}. \qquad (38\text{-b})$$

Note that the ion fluctuations are mainly responsible for Q^M, while the electron's contribution Q^C is negligible compared with Q^M.

According to Ginzburg and Zhelezniakov,[2] it is convenient to introduce the effective cross-sections σ_{pe} and σ_t for a plasma bubble of the volume V in which plasma waves are scattered

$$\sigma_{pe}\, d\vec{s} = I(\vec{S})d\vec{s}/\Sigma_p ,$$
$$\sigma_t = \int \sigma_{pe}\, d\vec{s} = Q V/\Sigma_p, \qquad (39)$$

where Σ_p is the energy flux in the incident wave

$$\Sigma_p = (V_T^2/V_{ph})(\varepsilon U^2/2). \qquad (40)$$

From(38-a), (34-b) there result for $Z=1$

$$\sigma_{pc}^M(\nu\simeq\omega_p)\,d\vec{s} = \frac{1}{32\pi^2}\frac{\omega_p^4 V}{n_o c^3 V_T}\sin^2\theta\, d\vec{s} ,$$
$$\sigma_t^M(\nu\simeq\omega_p) = \frac{1}{12\pi}\frac{\omega_p^4 V}{n_o c^3 V_T} . \qquad (41)$$

We should like to comment on the frequency spectrum of the radiation. We have derived the radiated electromagnetic waves with the line spectrum at the frequency of the incident wave. This is due to our approximations in (28) and (29). Strictly speaking, the ensemble mean value of $\Delta N^M(\vec{k}, \omega)$ has a width of the order of $K V_T^i = K(m/M)^{1/2} V_T$ around $\omega = 0$, that is, the Doppler broadening due to the ion thermal motions. Thus it may be expected the radiation spectrum is presented by the substitution,

$$\int d\nu\, \delta(\nu-\Omega) \rightarrow \frac{\gamma_i}{\pi} \int d\nu \frac{1}{(\nu-\Omega)^2+\gamma_i^2} \tag{42}$$

with

$$\gamma_i \cong (m/M)^{1/2} K V_T = (m/M)^{1/2} (V_T/V_{ph})\, \Omega \ll \Omega .$$

§6. Effect of the charge fluctuation associated with the plasma oscillations.

We shall deal only with the effect of $F_1^c = ZF_1^i - F_1^e$ in (8-b) and (19) and set δF_1^c and $F_1^M + \delta F_1^M$ zero. We expand the corresponding electric field $\vec{\mathcal{E}}_1$ in the spectral resolution of the form

$$\vec{\mathcal{E}}_1(\vec{r}, t) = \int_{-\infty}^{+\infty} d\vec{k} \int_{-\infty}^{+\infty} d\omega\, \frac{\vec{k}}{k}\, \mathcal{E}_1(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{43}$$

with

$$\mathcal{E}_1^*(\vec{k}, \omega) = -\mathcal{E}_1(-\vec{k}, -\omega).$$

Substituting this and (10) into (19) and with use of the inversion of (20) one finds

$$\vec{J} = \int e^{-i\vec{n}'\cdot\vec{r}}\, \vec{S}_\nu(\vec{r}')\, d\vec{r}'$$

$$= \frac{ie}{m} (2\pi)^3 \int d\vec{k} \int d\omega\, \delta(\vec{k}+\vec{K}-\vec{n}')\, \delta(\nu-\Omega-\omega)\, U\, \mathcal{E}_1(\vec{k}, \omega) \vec{H}(\vec{k}, \vec{K}), \tag{44}$$

where

$$\vec{n}' \equiv \frac{\nu}{c'}\vec{n},$$

$$\vec{H}(\vec{k}\cdot\vec{K}) = \frac{1}{kK}(2K^2+\vec{k}\cdot\vec{K})\vec{k}+\frac{1}{kK}(2k^2+\vec{k}\cdot\vec{K})\vec{K}.$$

Then $\vec{E}_{T\nu}$ in (23) will be

$$\vec{E}_{T\nu} = -\frac{e^{in'r}}{4\pi r}\frac{ie}{mc^2}U\frac{\vec{n}\times(\vec{n}\times\vec{K})}{K}(2\pi)^3\int d\vec{k}\int d\omega\,\delta(\vec{k}+\vec{K}-\vec{n}')\,\delta(\nu-\omega-\Omega)$$
$$\times\left[\frac{2(k^2-K^2)}{k}\mathcal{E}_1(\vec{k},\omega)\right] \tag{45}$$

This indicates that the frequency spectrom of the radiation is very sharp at $\nu\simeq\omega+\Omega\simeq 2\omega_p$ since $\varepsilon_1(\vec{k},\omega)$ has a resonant peak at $\omega^2\approx\omega_p{}^2+k^2U^2{}_T$.

The radiation intencity $I(\vec{S})d\vec{s}$ in (26) is

$$I^c(\vec{S})d\vec{s} = \frac{\sin^2\theta\,d\vec{s}}{(2\pi)^2\mu}\left(\frac{e^2}{mc^2}\right)^2U\frac{\pi}{T}\int d\nu\frac{1}{c}\left(1-\frac{\omega_p{}^2}{\nu^2}\right)^{1/2}(2\pi)^6\left[\frac{2(k^2-K^2)}{k}|\mathcal{E}_1(\vec{k},\omega)|\right]^2 \tag{46}$$
$$\vec{k}=\vec{n}'-\vec{K}$$
$$\omega=\nu-\Omega$$

The upper suffix "c" emphasized the effect of charge fluctuation. Now $\nu=\omega+\Omega\simeq 2\omega_p$ and $|\vec{n}'|\ll K$, then

$$\left(\frac{k^2-K^2}{k}\right)^2 = \frac{\{(\vec{n}'-\vec{K})^2-K^2\}^2}{(\vec{n}'-\vec{K})^2}\cong\frac{4(\vec{n}'\cdot\vec{K})^2}{K^2}\quad.$$

$I^c(\vec{S})d\vec{s}$ is turned out to be

$$I^c(\vec{S})d\vec{s} = \frac{3\sqrt{3}}{2\pi^2}\frac{\omega_p{}^4}{n_b c^3}(\varepsilon\,U^2)(\sin^2\theta\cos^2\theta\,d\vec{s})\frac{(2\pi)^7}{T}\int d\nu\,\frac{\varepsilon}{2}|\mathcal{E}_1(\vec{k}=\vec{n}'-\vec{K},\omega=\nu-\Omega)|^2 \tag{47}$$

The integral $\int d\nu(\varepsilon/2)|\mathcal{E}_1|^2$ can be expressed in the auto-correlation form and estimated approximately in (A-9) as

$$\frac{\varepsilon}{2}\int d\omega\,|\mathcal{E}_1(\vec{k},\omega)|^2 = \frac{e^2}{\varepsilon k^2}\frac{T}{2}\langle\Delta N_p^c(\vec{k},t)\,\Delta N_p^{c*}(\vec{k},t)\rangle\cong\frac{\Theta}{2}\frac{TV}{(2\pi)^7} \tag{48}$$

The fact that $\vec{k}$ does not appear in (48) comes from our neglect of the Landau damping for $\vec{\varepsilon}_1$. In this approximation the frequency spectrum of the radiation is simply the line spectrum at $\gamma = \bar{\omega} + \Omega$ where $\bar{\omega}$ is the averaged frequency of the plasma oscillations in the medium. Therefore the final results are

$$I^c(\vec{S})d\vec{s} = \frac{\sqrt{3}}{(2\pi)^2}\frac{\omega_p^{\,4}}{n_0 c^3}\left(\frac{V_T}{c}\right)^2(\varepsilon\, U^2)(\sin^2\theta\,\cos^2\theta\; d\vec{s})\int d\nu\,\delta(\nu-\bar{\omega}-\Omega), \tag{49}$$

$$Q^c = \frac{1}{V}\int I^c(\vec{S})d\vec{s} = \frac{2}{5\sqrt{3}\,\pi}\frac{\omega_p^{\,4}}{n_0 c^3}\left(\frac{V_T}{c}\right)^2(\varepsilon\, U^2)\int d\nu\,\delta(\nu-\bar{\omega}-\Omega)\ . \tag{50}$$

$Q^c(\nu \simeq 2\omega_p)$ must be compared with the Ginzburg and Zheleznıakov value and their value is to be reduced by the factor or $(V_{ph}/c)^2 = (\Omega/cK)^2$ as already stated in §1. The effective cross-sections for the box of the volume V such as defined in (39) are found to be

$$\begin{aligned}
\sigma_{pe}(\nu \simeq 2\omega_p)\,d\vec{s} &= \frac{\sqrt{3}}{\pi^2}\frac{\omega_p^{\,4}V}{n_0 c^3}\left(\frac{V_{ph}}{c^2}\right)(\sin^2\theta\,\cos^2\theta\; d\vec{s})\ ,\\
\sigma_t^{\,c}(\nu \simeq 2\omega_p) &= \frac{8}{5\sqrt{3}\,\pi}\frac{\omega_p^{\,4}V}{n_0 c^3}\left(\frac{V_{ph}}{c^2}\right)\ .
\end{aligned} \tag{51}$$

It may be remarked that the due estimation of the auto-correlation coefficient in (48) will result in a small width γ given in (A-5, 6) in the spectrum and possibly,

$$\begin{aligned}
\int d\nu\,\delta(\nu-\bar{\omega}-\Omega) &\rightarrow \frac{\gamma}{\pi}\int d\nu\,\frac{1}{(\nu-\omega-\Omega)^2+\gamma^2}\\
\gamma &\simeq (3/2)^{3/2}\sqrt{\pi}\,(\omega_p^{\,4}/K^3 V_T^{\,3})\exp[-(3/2)(\omega_p^2+V_T^{\,2}K^2)(K^2V_T^{\,2})^{-1}]\ .
\end{aligned} \tag{52}$$

§7. Comparison with non-uniform models.

It is interesting to compare our model with non-uniform plasma models. The transformation efficiencies of the plasma waves into the electromagnetic waves in the presence of inhomogeneities are estimated by many authors. We shall refer only to the Tidman and Weiss theory[3]) and the Cohen theory[5)] because their results can be easily compared with ours.

As stated briefy in §1, the radiation intensities given by these authors will become equal to our value of the radiation intensity due to mass fluctuation $Q^M(\nu \simeq \Omega)$ in (38-b) when the scales of localized inhomogeneities in non-uniform models are reduced to such submacroscopic scale as the Debye length or the wave length of plasma oscillations.

Tidman and Weiss treated the collision of an incident plasma wave with a small localized variation in density where the static non-uniform background of the ions is assumed. They derived the source function of the scattered fields and the radiation intensity to the first order in the small parameter by which the density variation is characterized. Their perturbation procedure corresponds to the Born approximation above.

The source function of the Tidman and Weiss model corresponding to $\vec{S}$ in (19) can be expressed as

$$\vec{S}(\text{T-W}) = \frac{\omega_p^2\, e\psi}{\Theta}\, \vec{\mathcal{E}}_0 + \frac{e}{m}\{2\vec{\mathcal{E}}_1(\vec{\nabla}\cdot\vec{\mathcal{E}}_0) + \vec{\mathcal{E}}_0(\vec{\nabla}\cdot\vec{\mathcal{E}}_1) + (\vec{\mathcal{E}}_0\cdot\vec{\nabla})\vec{\mathcal{E}}_1 + (\vec{\mathcal{E}}_1\cdot\vec{\nabla})\vec{\mathcal{E}}_0\},$$

$$\vec{\mathcal{E}}_1(\vec{r}) = \vec{\nabla}\psi(\vec{r}) = \int_{-\infty}^{+\infty} d\vec{r} \int_{-\infty}^{+\infty} dt\; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \left\{ \frac{\vec{K}}{ik}\, \frac{\Theta}{e\omega_p^2}\, \frac{J_K}{2\pi^2}\, \delta(\omega) \right\}, \tag{53}$$

and

$$J_K = \int_0^{\infty} r\,dr\,(-e\psi/\Theta)\, \omega_p^2 \sin kr,$$

where $\vec{\varepsilon}_1$ is the corresponding electrostatic field of the inhomogeneity, and $\vec{\varepsilon}_0$ is that of the incident wave defined in (10). The difference between $\vec{S}$ (ours) and $\vec{S}$ (T-W) comes from that

$$\frac{\partial}{\partial t}\vec{\mathcal{E}}_1(\text{T-W}) = 0, \text{ while } \frac{\partial}{\partial t}\vec{\mathcal{E}}_1(\text{ours}) \neq 0 .$$

According to their assumption for $\vec{\mathcal{E}}_1$ the first terms in $\vec{S}$ (T-W) can be rewritten in terms of the perturbed distribution of the electrons F_1^c as

$$\frac{\omega_p^2 e\psi}{\Theta}\vec{\mathcal{E}}_0 = \frac{e}{m}\left(-\frac{e}{\varepsilon}\int d\vec{v}\, F_1^{\,c}\right)\vec{\mathcal{E}}_0 \tag{54}$$

Then

$$\vec{S}(T\text{-}W) = \frac{e}{m}\left[-\frac{e}{\varepsilon}\int d\vec{v}\, F_1^{\,0}\vec{\mathcal{E}}_0 + 2\vec{\mathcal{E}}_1(\vec{\nabla}\cdot\vec{\mathcal{E}}_0) + \vec{\mathcal{E}}_0(\vec{\nabla}\cdot\vec{\mathcal{E}}_1) + (\vec{\mathcal{E}}_0\cdot\vec{\nabla})\vec{\mathcal{E}}_1 + (\vec{\mathcal{E}}_1\cdot\vec{\nabla})\vec{\mathcal{E}}_0\right] \tag{55}$$

It is seen that the electron perturbation $F_1^{\,e}$ in the Tidman and Weiss model plays the same role as the ion pertubation $F_1^{\,i}$ in our case since $F_1^{\,i} \approx F_1^{\,M}$ the mass fluctuation in $\vec{S}$ in (19) and which ascertains the coincidences between our result and the suitable version of the Tidman and Weiss result for the radiation having the same frequency as the incident plasma wave. We shall show it in the following.

From $\vec{S}$ (T−W) we obtain the scattered field of electromagnetic mode $\vec{E}$ (T-W) as

$$\vec{E}_{T,\nu}(T\text{-}W) = -\frac{e^{i\frac{\nu}{c'}r}}{r}\,\frac{\vec{n}\times(\vec{n}\times\vec{K})}{K}\left(1+\frac{1}{3}\,\frac{V_T^{\,2}}{V_{ph}^{\,2}}\right)\frac{U}{K}J_k \tag{56}$$

where the same notations in (32) are used. This coincides with $\vec{e}_T$ in Eq. (76) of the Tidman and Weiss paper.

For the model Cohen dealed with, in which the linear dimension of the localized inhomogeneity is much less than the wave-length of the incident wave and the density there deviates by Δn from the mean value n_0, the respective parameters are obtained by the substitutions

$$e\psi/\Theta \;\rightarrow\; -\Delta n/n_0 \text{ in } V_0 \text{ and zero outside } V_0.,$$

$$J_k = \frac{\omega_p^{\,2}K}{4\pi}\left(\frac{\langle\Delta n\rangle}{n_0}\right)V_0 \tag{57}$$

where $\langle\Delta n\rangle$ is the root mean square of density variation in V_0. Then $\vec{E}_{T,\nu}$ in this case is given by

$$\vec{E}_{T,\nu} = - \frac{e^{i\frac{\nu}{c'}r}}{4\pi r} \frac{\vec{n}\times(\vec{n}\times\vec{K})}{K} \frac{K\omega_p^2}{c^2} \left(\frac{<\Delta n>}{n_0}\right) U\, V_0 \tag{58}$$

This is equivalent to $\vec{E}_e^S$ in Eq. (8, 16) of the Cohen's paper.

Cohen has derived this by the continuum theory of a simple plasma model with the continuous loss-free electron fluid immersed in the ion background. Now the radiation intensity is proportional to $<\Delta n>^2$ and $<\Delta n>^2 = (1/2)(n_0/V_0)$ must be used corresponding to (A-6, 7), though the statistical theory of one fluid theory tells $<\Delta n>^2 = n_0/V_0$ at thermal equilibrium.

References

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Appendix

The statistical ensemble means of density fluctuations

We recall the definitions in (27)

$$\Delta N^M(\vec{r}, t) = \int d\vec{v}(F_1^M + \delta F_1^M),$$
$$\Delta N_T^C(\vec{r}, t) = \int d\vec{v}(F_1^C + \delta F_1^C) - \int d\vec{v}\ F_1^C = \Delta N^C(\vec{r}, t) - \Delta N_p^C(\vec{r}, t). \tag{A-1}$$

As defined in (28) their spatial Fourier transforms are

$$\Delta N^M(\vec{k}, t) = \frac{1}{(2\pi)^3}\int d\vec{r}\ \Delta N^M(\vec{r}, t)\, e^{-i\vec{k}, \vec{r}}$$
$$\Delta N_{P,T}^C(\vec{k}, t) = \frac{1}{(2\pi)^3}\int d\vec{r}\ \Delta N_{P,T}^C(\vec{r}, t)\, e^{-i\vec{k}, \vec{r}}, \tag{A-2}$$
$$\Delta N^C(\vec{k}, t) = \Delta N_P^C + \Delta N_T^C .$$

Now

$$<\Delta N(\vec{k}, t)\, \Delta N^*(\vec{k}, t)>$$
$$\cong \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' <\Delta N(k)>\delta(\omega)<\Delta N(k)>\delta(\omega')\, e^{i(\omega'-\omega)t}$$
$$= |<\Delta N(\vec{k})>|^2,$$

According to Salpeter[4)] we have for the plasma at thermal equilibrium,

$$\frac{(2\pi)^6}{N}|<\Delta N^M(\vec{k})>|^2 \cong \frac{(2\pi)^6}{N}|<\Delta N^i(\vec{k})>|^2 = \frac{1}{Z}\ \frac{1+\alpha^2}{1+(Z+1)\alpha^2} \tag{A-3}$$

$$\frac{(2\pi)^6}{N}|<\Delta N^C(\vec{k})>|^2 = \frac{Z+1}{1+(Z+1)\alpha^2} \tag{A-4}$$

$$\frac{(2\pi)^6}{N}|<\Delta N_P^C(\vec{k})>|^2 \cong \begin{cases} 0 & \text{for } \alpha \ll 1, \\ \dfrac{\alpha^2}{4\sqrt{\pi}}\, e^{-x_0^2}\displaystyle\int \frac{dx}{(x-x_0)^2+(\gamma_L/2)^2} & \end{cases} \tag{A-5}$$

where

$$
\begin{aligned}
x &= \omega/(2/3)^{1/2}\, kV_T \\
x_o^2 &= (3/2)(\omega_P^2 + V_T^2 k^2)(k^2 V_T^2)^{-1} \\
\gamma_L &= (\sqrt{\pi}/2)\, \alpha^4 \exp(-x_o^2) \\
\alpha^2 &= 1/k^2 D^2 = 3\omega_P^2/k^2 V_T^2
\end{aligned}
\qquad \text{(A-6)}
$$

If $\vec{k} = \vec{K}$ the wave vector of a plasma oscillation, then in usual cases

$$\alpha^2 = 3(\omega_P^2/K^2 V_T^2) \cong 3(V_{ph}/V_T)^2 \gg 1.$$

Therefore one obtains

$$\frac{(2\pi)^6}{N} |\langle \Delta N^M(\vec{k})\rangle|^2 \cong \frac{1}{Z(Z+1)} \qquad \text{(A-7)}$$

$$\frac{(2\pi)^6}{N} |\langle \Delta N_T^C(\vec{k})\rangle|^2 \cong \frac{1}{\alpha^2} - \frac{1}{\alpha^2} \simeq 0\left(\frac{1}{\alpha^4}\right). \qquad \text{(A-8)}$$

These values are used in (38-a) and (38-b)

Next we consider the integral in (48)

$$\frac{\varepsilon}{2}\int d\omega\, |\vec{\mathcal{E}}_1(\vec{k}, \omega)|^2 = \frac{e^2}{\varepsilon k^2} \frac{T}{2} \langle \Delta N_P^C(\vec{k}, t)\, \Delta N_P^{C*}(\vec{k}, t)\rangle$$

As seen from (A-4) and (A-5) the harmonically oscillatory part in charge fluctuation is dominant for $\alpha^2 = 1/K^2 D^2 \gg 1$, and then

$$\frac{\varepsilon}{2}\int d\omega\, |\mathcal{E}_1(\vec{K}, \omega)|^2 = \frac{e^2}{\varepsilon K^2} \frac{NT}{2(2\pi)^7} \frac{1}{\alpha^2} = \frac{\Theta}{2} \frac{NTV}{(2\pi)^7}. \qquad \text{(A-9)}$$

This is inserted in (47).