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Unified Theory of Relaxations in Plasmas

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Abstract

The Coulomb interaction in a plasma must be considered both from the point of view of binary collisions and from the point of view of collective interactions. Either in the collision theory on the basis of binary collisions or in the wave theory based upon collective interactions one must cut-off divergent integrals and hence the numerical factors in the arguments of Coulomb logarithms remain undetermined. These two complementary theories have been unified by the present authors into a divergence – free theory, the results giving exact arguments of the Coulomb logarithms. The present article is its review.

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Introduction

The kinetic theory of gases of neutral molecules has been well established. The macroscopic properties of gases can be calculated from the velocity distribution functions. These functions are determined from the Boltzmann equation which is derived on the basis of binary collisions between the molecules. In plasmas, however, the collective interaction between charged particles is also important.

The collective interaction has been treated by the so-called wave theory, which is based on the Liouville equation or on the fact that the plasma can be regarded as a dielectric medium. Kinetic equations and several examples have been investigated[1-10]. In the wave theory, however, the curvature of orbits of particles at close collisions is not included. Hence the theory is not superior but complementary to the collision theory.

It was impossible until recently to consider both the binary collisions and the collective interactions simultaneously and exactly; the kinetic theory of irreversible processes in plasmas was, therefore, definitely inferior to that in neutral gasses. Now exists an exact theory of plasma kinetics unifying the collision and wave theories[11-18]; its review is the objective of the present article.

Besides the wave theory mentioned above, there is an attempt to apply the method of quantum field theory to the treatment of the collective interaction. This method is successful only when the temperature of the plasma is so high that the effect of orbital curvature is completely covered by the quantum mechanical diffraction effects[19-24]. A rate of relaxation derived by this method can be obtained as the high temperature limit in the present unified theory in a more systematic and elementary manner.

Irreversible processes are caused by relaxations of distribution functions due to interactions between constituent particles. The rates of these relaxations are the main subject of the present article.

I. FOUNDATIONS

1. Connection formula

In this section the merits and demerits of the collision and wave theories are pointed out; and the two theories are unified into an exact theory.

The potential energy between the particles in a hot plasma is, on the average, much smaller than their kinetic energy. In this respect a hot plasma is similar to a rarefied gas. In fact, irreversible processes in plasmas were treated mostly within the framework of the molecular theory of gases, which was based on binary collisions between the constituent particles.

The rate of a relaxation will then be expressed, in classical mechanics, in an integral with respect to the impact parameter b. For Coulomb forces this integral is of the form

$$\int_{b>0} \frac{b\,db}{b^2 + (\text{collision radius})^2}$$

in which and throughout this paper "collision radius" means a distance at which the potential energy is comparable to the kinetic energy. This integral diverges logarithmically when we let the upper limit tend to infinity. The divergence arises from the fact that collective interactions between charged particles are not included in the "collision theory". The integrand is valid only for b sufficiently shorter than the Debye screening radius. When we cut off the range of the b-integration near the Debye radius, then we obtain a finite result containing a "Coulomb logarithm" whose argument is the ratio of the cut-off limit to the collision radius. This result, however, is only of logarithmic accuracy, i.e., valid only for large values of the logarithm itself and not in general for large values of its argument.

We are now interested in an exact Coulomb logarithm which is valid so far as its argument is sufficiently large. Let us take up a simile. The sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$$

has an asymptotic form $\ln(\gamma N)$ for $N \gg 1$, where $\ln \gamma = 0.57722$ is Euler's constant. What we are interested in corresponds to this numerical factor γ in the logarithm.

In a hot plasma the number of particles which screen the electric field around a charge is very large. Hence it should be possible to ascribe the screening to the macroscopic dielectric properties of the plasma and to treat the collective interactions in terms of the dielectric permeability[6-9]. Since the screening in a kinetic problem depends on the rate of temporal and spatial variations, the dielectric permeability must be considered as a function $\varepsilon(\mathbf{k},\omega)$ of the wave number $\mathbf{k}$ and frequency ω.

The rate of a relaxation is then given by an integral with respect to the wave number k. This integral is of the form

$$\int_{k>0} \frac{k^3 dk}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2} .$$

The fact that $|\mathbf{k}\cdot\varepsilon\cdot\mathbf{k}|^2$ is different from k^4 reveals the collective interaction. This integral diverges logarithmically when we let the upper limit tend to infinity. This divergence arises from the fact that the curvature of orbits at close collisions is not included in the "wave theory". The integrand is valid only for wavelengths sufficiently longer than the collision radius. When we cut off the range of the k-integration near the inverse of the collision radius, then we obtain a finite result containing a Coulomb logarithm whose argument is the ratio of the Debye radius to the cut-off wavelength. Thus the result is again of logarithmic accuracy.

For a hot plasma the region of validity of the impact theory,

$$b \ll \text{Debye radius},$$

and the region of validity of the wave theory,

$$\text{collision radius} \ll k^{-1} ,$$

greatly overlap. In this overlapping region both the collective interaction and the orbital curvature are negligible. By virtue of this fact we can combine the collision and wave theories to a unified theory.

Let us denote by X a relaxation rate and by X_{wave} the rate calculated in the wave theory. Let us further denote by $X_{wave,\kappa}$ the quantity to which X_{wave} is reduced when ε in $|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^{-2}$ is replaced by a static dielectric permeability $1 + \kappa^2/k^2$. Here the constant κ is to be taken such that

$$\text{collision radius} \ll \kappa^{-1} \lesssim \text{Debye radius} \qquad (1.1)$$

the sign $\lesssim$ indicating "smaller than or of the order of ". The potential around a unit point charge in a medium with the dielectric permeability $1 + \kappa^2/k^2$ is the screened Coulomb potential $r^{-1} \exp(-\kappa r)$, r being the distance from the charge. Let us denote by X_κ the rate X calculated in the collision theory with use of this screened potential. The difference between X_κ and $X_{wave,\kappa}$ lies in the fact that the orbital curvature is included in the former but excluded in the latter.

We further let

$$X_{wave} = \int_{k>0} K(k)\, dk \quad ,$$

$$X_{wave,\kappa} = \int_{k>0} K_\kappa(k)\, dk \quad .$$

Both the integrals diverge logarithmically when the upper limits of the integrals go to infinity. The difference

$$X_{wave} - X_{wave,\kappa} = \int_0^\infty [K(k) - K_\kappa(k)]\,dk, \qquad (1.2)$$

however, converges. In fact, at wave lengths sufficiently short as compared with the screening radius κ^{-1}, both the collective interaction in $K(k)$ and the screening in $K_\kappa(k)$ vanish and two terms in the integrand cancel each other.

Now, we consider the combination $X_\kappa + X_{wave} - X_{wave,\kappa}$. The range of lengths considerably short in comparison with the screening radius and the range of lengths much longer than the collision radius overlap greatly. The contribution of the first range to X_κ is exact and that to $X_{wave} - X_{wave,\kappa}$ vanishes; the contribution of the second range to X_{wave} is exact and that to $X_\kappa - X_{wave,\kappa}$ vanishes. Hence we obtain the formula

$$X = X_\kappa + X_{wave} - X_{wave,\kappa} \qquad (1.3a)$$

or

$$X = X_\kappa + \int_0^\infty [K(k) - K_\kappa(k)]dk\,. \qquad (1.3b)$$

The final results are independent of the constant κ.

The term $K_\kappa(k)$ is simply given by

$$K_\kappa(k) = \frac{k^3}{(k^2+\kappa^2)^2} \lim_{k\to\infty} k K(k)\,. \qquad (1.4)$$

Because $K_\kappa(k)$ is proportional to $k^3(k^2 + \kappa^2)^{-2}$ and the constant of proportionality is determined by the relation

$$\lim_{k\to\infty} k\ [K(k) - K_{\kappa}(k)] = 0 \quad .$$

It is sometimes useful to define the "effective screening constant" κ^{o} by

$$X_{wave} - X_{wave,\kappa^{o}} = 0 \quad . \tag{1.5a}$$

By use of this constant, X is expressed simply in the form

$$X = X_{\kappa^{o}} \quad . \tag{1.5b}$$

The collision radius is inversely proportional to the kinetic energy. The de Broglie wavelength is, on the other hand, inversely proportional to the momentum. Hence at high particle energies the de Broglie wavelength is comparable with or even longer than the collision radius. When the quantum effects are appreciable we must let κ^{-1} be also

$$\text{de Broglie wavelength} \ll \kappa^{-1} \tag{1.6}$$

and calculate the collision integral X_{κ} on the basis of quantum mechanics. The quantum effects will be taken into consideration in §17.

For an oscillatory phenomenon with high frequency ω we let κ^{-1} be so short that the duration of interaction in the collision theory becomes much shorter than ω^{-1}, the duration of macroscopic state. This problem will be discussed in §11.

In the presence of a strong magnetic field we let κ^{-1} be also

$$\kappa^{-1} \ll \text{gyroradius}, \qquad (1.7)$$

so that the collision theory is not influenced by the magnetic field. In the present article relaxations in a strong magnetic field are not discussed in detail because of their complexity; an exceptionally simple example is given in the last section.

A clue to the unification of collision and wave theories was first given by Hubbard[25]. His connection formula was, in the present notations,

$$X = X_0 + X_{wave} - X_{wave,0}$$

(i.e. $\kappa = 0$ in (1.3)). Though the three terms on the right side are all diverging integrals, he did not show an exact procedure of cancelling the divergences. Baldwin[26] derived on the basis of the Liouville equation a unified kinetic equation which is of the form (1.3) and is the same as used in Chapter V. Not a single problem or example was solved by him with use of his kinetic equation.

2. Moments of the change in particle velocities

The concept of the moments of the change in particle velocities is very useful when the macroscopic state of the plasma is not highly oscillatory. We namely consider the following kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{F}\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\Delta f}{\Delta t} \tag{2.1}$$

with the rate of relaxation in f,

$$\frac{\Delta f}{\Delta t} = -\frac{\partial}{\partial \mathbf{v}}\cdot\left(\frac{\langle\Delta\mathbf{v}\rangle}{\Delta t}f\right) + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{v}}:\left(\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle}{\Delta t}f\right) - \cdots$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{\partial}{\partial \mathbf{v}}\right)^{n}\cdot\left(\frac{\langle(\Delta\mathbf{v})^{n}\rangle}{\Delta t}f\right) . \tag{2.2}$$

Here f($\mathbf{v}$, $\mathbf{r}$, t), or f($\mathbf{v}$, t) for short, is the velocity distribution function of one component at position $\mathbf{r}$ and time t; $\mathbf{F}$ is the macroscopic force per unit mass acting on the particle; $\Delta\mathbf{v}$ is the change in particle velocity due to the interactions in a small time interval Δt; and the bracket notation $\langle\ \rangle$ indicates the average over all the kinds of interactions. It is assumed that f is quasi-homogeneous. The coefficients $\langle\Delta\mathbf{v}\rangle/\Delta t$, $\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle/\Delta t$,... are called first, second,... moments.

The change in the distribution function during Δt must be infinitesimal. But Δt has to be sufficiently longer than the time of continuance of the force correlation; otherwise the moments would depend on Δt. We therefore assume for the time being that the time interval during which the macroscopic state changes appreciably is much longer than the time of continuance of the force correlation (cf. §11). In the presence of a magnetic field we also assume that the periods of gyration of plasma particles are much longer than the time of force correlation.

The equation (2.1) can be derived as follows. The time of continuance of force correlation is the time of relaxation of the binary distribution function. Hence, under our assumption the binary distribution function is in an "equilibrium" corresponding to $f(\mathbf{v}, t)$, and the rate of change in $f(\mathbf{v}, t)$ is governed by f itself at the instant t; in other words the process is Markoffian. We let

$$f(\mathbf{v}, t+\Delta t) = \int d\mathbf{u}\, f(\mathbf{u}, t)\, P(\mathbf{u}|\mathbf{v}, \Delta t). \tag{2.3}$$

Here $P(\mathbf{u}|\mathbf{v}, \Delta t)$ indicates the probability of transfer from $\mathbf{u}$ to $\mathbf{v}$ in a time Δt due to interactions. Following Wang and Uhlenbeck[27], let us consider the integral

$$\int d\mathbf{v}\, R(\mathbf{v}) \frac{\Delta f}{\Delta t}, \tag{2.4}$$

where $R(\mathbf{v})$ is an arbitrary function which goes to zero for $|\mathbf{v}| \to \infty$ sufficiently fast. Using (2.3) we have

$$\begin{aligned}
&\int d\mathbf{v}\, R(\mathbf{v}) \frac{\Delta f(\mathbf{v}, t)}{\Delta t} \\
&= \int d\mathbf{v}\, R(\mathbf{v}) [f(\mathbf{v}, t+\Delta t) - f(\mathbf{v}, t)]/\Delta t \\
&= \int d\mathbf{v}\, f(\mathbf{v}, t) \int d\mathbf{u}\, [R(\mathbf{u}) - R(\mathbf{v})] P(\mathbf{v}|\mathbf{u}, \Delta t)/\Delta t.
\end{aligned} \tag{2.5}$$

Developing $R(\mathbf{u})$ in a Taylor series in $\mathbf{u}-\mathbf{v}$,

$$R(\mathbf{u}) = R(\mathbf{v}) + \sum_{n=1}^{\infty} \frac{1}{n!} (\mathbf{u}-\mathbf{v})^n \cdot \left(\frac{\partial}{\partial \mathbf{v}}\right)^n R(\mathbf{v}),$$

and using the moments of the change in velocity,

$$\frac{1}{\Delta t} \int (\mathbf{u}-\mathbf{v})^n P(\mathbf{v}|\mathbf{u}, \Delta t)\, d\mathbf{u} = \frac{\langle (\Delta \mathbf{v})^n \rangle}{\Delta t},$$

we obtain

$$\int d\mathbf{v}\, R(\mathbf{v}) \frac{\Delta f}{\Delta t} = \int d\mathbf{v}\, f(\mathbf{v},t) \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\langle (\Delta \mathbf{v})^n \rangle}{\Delta t} \cdot \left(\frac{\partial}{\partial \mathbf{v}}\right)^n R(\mathbf{v})$$

$$= \int d\mathbf{v}\, R(\mathbf{v}) \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial \mathbf{v}}\right)^n \cdot \left(\frac{\langle (\Delta \mathbf{v})^n \rangle}{\Delta t} f(\mathbf{v},t) \right).$$

Since this must hold for any function $R(\mathbf{v})$, the equation (2.2) follows.

For the pure Coulomb potential the first and second moments in the collision theory diverge. The third and higher moments, however, can be accurately calculated on the basis of binary collisions. In other words, for

$$X = \frac{\langle (\Delta \mathbf{v})^n \rangle}{\Delta t}, \qquad n \geq 3,$$

X_κ is independent of κ and $X_{\text{wave}} - X_{\text{wave},\kappa}$ vanishes. In the following two sections the wave and collision theories for the first and second moments are developed.

3. Moments in the wave theory

Dielectric polarization produced by a moving charge

We consider a charged particle moving through a plasma. The charge will cause dielectric polarization in the plasma; and this polarization will produce an electric field at the position where the charge is located.

Let us determine the field produced by a point charge q moving through a plasma with a non-relativistic velocity $\mathbf{v}$, the charge density at a position $\mathbf{r}$ and time t being given by $q\,\delta\,(\mathbf{r}-\mathbf{v}t)$. The electric field produced by the charge will be denoted by $\mathbb{E}(\mathbf{r},t)$. We define the Fourier transform $\mathbb{E}(\mathbf{k},\omega)$ of the electric field by

$$\mathbb{E}(\mathbf{r},t) = \iint \mathbb{E}(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\, d\mathbf{k}d\omega \qquad (3.1)$$

and the Fourier transform $\rho(\mathbf{k},\omega)$ of the charge density by a similar relation. Then

$$\rho(\mathbf{k},\omega) = \frac{q}{(2\pi)^3}\,\delta(\omega - \mathbf{k}\cdot\mathbf{v})\ . \qquad (3.2)$$

The electric field produced by a non-relativistic charge is rotation-free, or $\mathbf{E}(\mathbf{k},\omega)$ is parallel to $\mathbf{k}$. Hence, by use of the Poisson equation,

$$i\,\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\,\mathbf{E}(\mathbf{k},\omega) = 4\pi\rho(\mathbf{k},\omega), \qquad (3.3)$$

we obtain

$$\mathbf{E}(\mathbf{k},\omega) = -i\,\frac{q\mathbf{k}}{2\pi^2}\,\frac{\delta(\omega-\mathbf{k}\cdot\mathbf{v})}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}}\ . \qquad (3.4)$$

Substituting (3.4) into (3.1) we obtain the field, which takes at the position of the moving charge the form

$$\mathbf{E}^{pol} = \mathbf{E}(\mathbf{v}t,t) = -i\,\frac{q}{2\pi^2}\iint\frac{\mathbf{k}\,\delta(\omega-\mathbf{k}\cdot\mathbf{v})}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}}\,d\mathbf{k}\,d\omega \qquad (3.5)$$

It is sometimes useful to transform (3.5) into a real integral

$$\mathbf{E}^{pol} = -\frac{q}{2\pi^2}\iint\frac{\mathbf{k}\cdot\varepsilon''(\mathbf{k},\omega)\cdot\mathbf{k}}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\,\mathbf{k}\,\delta(\omega-\mathbf{k}\cdot\mathbf{v})\,d\mathbf{k}d\omega. \qquad (3.6)$$

Here ε'' is the imaginary part of ε and use has been made of the relation (see Appendix)

$$\varepsilon^*(\mathbf{k},\omega) = \varepsilon(-\mathbf{k},-\omega) \tag{3.7}$$

between ε and its complex conjugate ε^*.

Correlation of fluctuations of the electric field

In a homogeneous plasma in thermal equilibrium we consider fluctuations of the electric field $\mathbf{E}(\mathbf{r},t)$ as a function of the position $\mathbf{r}$ and time t. A role is played by the correlation function

$$\langle \mathbf{E}(\mathbf{r},t)\,\mathbf{E}(\mathbf{r}+\mathbf{s},\,t+\tau)\rangle \tag{3.8}$$

which is the averaged value of the tensor product of two field vectors at different positions and times.

We define the Fourier transform of the electric field by the same relation as (3.1). Inserting this relation in (3.8), we obtain

$$\langle \mathbf{E}(\mathbf{r},t)\,\mathbf{E}(\mathbf{r}+\mathbf{s},\,t+\tau)\rangle$$

$$= \iiiint \langle \mathbf{E}(\mathbf{k}',\omega')\,\mathbf{E}(\mathbf{k},\omega)\rangle\, e^{i\mathbf{k}'\cdot\mathbf{r}+i\mathbf{k}\cdot(\mathbf{r}+\mathbf{s})-i\omega' t-i\omega(t+\tau)}\, d\mathbf{k}\,d\omega\, d\mathbf{k}'d\omega'.$$

The integral on the right side will be a function of the differences $(\mathbf{r}+\mathbf{s})-\mathbf{r}$ and $(t+\tau)-t$ only if the integrand contains δ-functions of $\mathbf{k}+\mathbf{k}'$ and $\omega+\omega'$. This requires that

$$\langle \mathbf{E}(\mathbf{k}',\omega')\,\mathbf{E}(\mathbf{k},\omega)\rangle = (\mathbf{E}\,\mathbf{E})_{\mathbf{k}\omega}\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega') \tag{3.9}$$

and therefore

$$\langle \mathbf{E}(\mathbf{r},t)\,\mathbf{E}(\mathbf{r}+\mathbf{s},t+\tau)\rangle = \iint (\mathbf{E}\mathbf{E})_{\mathbf{k}\omega}\, e^{i\mathbf{k}\cdot\mathbf{s}-i\omega\tau}\,d\mathbf{k}\,d\omega\,. \tag{3.10}$$

The relation (3.9) has to be regarded as a definition of the quantity $(\mathbf{E}\mathbf{E})_{\mathbf{k}\omega}$. It is the Fourier transform of the correlation function (3.9); or it is the spectral density of the mean square fluctuations since

$$\langle \mathbf{E}\;\mathbf{E}\rangle = \iint (\mathbf{E}\,\mathbf{E})_{\mathbf{k}\omega}\,d\mathbf{k}\,d\omega$$

for $\mathbf{s} = 0$, $\tau = 0$.

The Fourier transform $\mathbf{E}(\mathbf{k},\omega)$ of the fluctuating electric field is related to the Fourier transform $\rho(\mathbf{k},\omega)$ of the fluctuating charge density. In a non-relativistic plasma the fluctuating field due to moving charges is rotation-free, or $\mathbf{E}(\mathbf{k},\omega)$ is parallel to $\mathbf{k}$. Hence, by use of the Poisson equation, we obtain

$$\mathbf{E}(\mathbf{k},\omega) = -i\,\frac{4\pi\rho(\mathbf{k},\omega)}{\mathbf{k}\cdot\boldsymbol{\varepsilon}(\mathbf{k},\omega)\cdot\mathbf{k}}\;\mathbf{k}, \tag{3.11}$$

from which follows

$$\langle \mathbf{E}(\mathbf{k},\omega)\,\mathbf{E}(\mathbf{k}',\omega')\rangle = \frac{(4\pi)^2\,\mathbf{k}\,\mathbf{k}}{|\mathbf{k}\cdot\boldsymbol{\varepsilon}(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\,\langle\rho(\mathbf{k},\omega)\,\rho(\mathbf{k}',\omega')\rangle\,. \tag{3.12}$$

Here we have made use of the relation (3.7). On defining $(\rho^2)_{\mathbf{k}\omega}$ similarly to (3.9), we can rewrite (3.12) in the form

$$(\mathbf{E}\,\mathbf{E})_{\mathbf{k}\omega} = \frac{(4\pi)^2\,\mathbf{k}\,\mathbf{k}}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\,(\rho^2)_{\mathbf{k}\omega}\,. \tag{3.13}$$

The first and second moments in the wave theory

In the wave theory the second moment $\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle/\Delta t$ for a charged particle moving with a velocity $\mathbf{v}$ is simply related to the correlation of fluctuations of the electric field. Here $\Delta\mathbf{v}$ is a change in velocity in a time interval Δt, which is taken to be much longer than the continuance of force correlation but still so short that $\Delta\mathbf{v}$ is small. It is given by an integral of the fluctuating electric field at the position of the charged particle

$$\Delta\mathbf{v} = \frac{q}{M}\int_0^{\Delta t} \mathbf{E}(\mathbf{v}\tau,\tau)\,d\tau, \tag{3.14}$$

q and M being the charge and mass of the particle. Hence

$$\left[\frac{\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle}{\Delta t}\right]_{wave} = \frac{q^2}{M^2}\int_{-\infty}^{\infty}\langle\mathbf{E}(0,0)\,\mathbf{E}(\mathbf{v}\tau,\tau)\rangle\,d\tau, \tag{3.15}$$

where use has been made of the assumption that Δt is much longer than the continuance of correlation. This relation can be transformed, by use of (3.10), into

$$\left[\frac{\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle}{\Delta t}\right]_{wave} = \frac{2\pi q^2}{M^2}\iint(\mathbf{E}\,\mathbf{E})_{\mathbf{k}\omega}\,\delta(\omega-\mathbf{k}\cdot\mathbf{v})\,d\mathbf{k}\,d\omega \tag{3.16}$$

and further, by use of (3.13), to

$$\left[\frac{\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle}{\Delta t}\right]_{wave} = 4\,\frac{(2\pi)^3 q^2}{M^2}\iint\frac{\mathbf{k}\,\mathbf{k}}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\,(\rho^2)_{\mathbf{k}\omega}\,\delta(\omega-\mathbf{k}\cdot\mathbf{v})\,d\mathbf{k}\,d\omega \tag{3.17}$$

The first moment $\langle \Delta \mathbf{v} \rangle / \Delta t$ is composed of $(q/M)\,\mathbf{E}^{pol}$, which is due to the polarization of the medium, and of a part due to fluctuations of the electric field:

$$\left[\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave} = \frac{q}{M}\mathbf{E}^{pol} + \frac{q}{M}\frac{1}{\Delta t}\int_0^{\Delta t} \langle \mathbf{E}(\mathbf{r}(\tau),\tau) \rangle \, d\tau . \qquad (3.18)$$

Here the quantity $\mathbf{E}$ in the integral is the fluctuating electric field at the position $\mathbf{r}(\tau)$ of the moving charge. In (3.18) $\mathbf{r}(\tau)$ can not be approximated by $\mathbf{v}\tau$ as in (3.14), but the first-order effect of the fluctuating field on the trajectory must be taken into account:

$$\mathbf{r}(\tau) = \mathbf{v}\tau + \frac{q}{M}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \, \mathbf{E}(\mathbf{v}\tau'', \tau''),$$

$$\mathbf{E}(\mathbf{r}(\tau),\tau) = \mathbf{E}(\mathbf{v}\tau,\tau) + \frac{q}{M}\int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \, \mathbf{E}(\mathbf{v}\tau'', \tau'')\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{E}(\mathbf{v}\tau,\tau),$$

where a Taylor expansion has been made. Since $\langle \mathbf{E}(\mathbf{v}\tau,\tau)\rangle = 0$, we have

$$\left[\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave} = \frac{q}{M}\mathbf{E}^{pol} + \frac{q^2}{M^2}\frac{1}{\Delta t}\int_0^{\Delta t} d\tau \int_0^{\tau} d\tau' \int_0^{\tau'} d\tau'' \langle \mathbf{E}(\mathbf{v}\tau'', \tau'')\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{E}(\mathbf{v}\tau,\tau)\rangle .$$

Changing the sequence of integration and taking account of the facts that the integrand is a function of the difference $\tau - \tau'$ only and that Δt is much longer than the continuance of force correlation, we finally obtain

$$\left[\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave} = \frac{q}{M}\mathbf{E}^{pol} + \frac{q^2}{M^2}\int_0^{\infty} \tau \langle \mathbf{E}(0,0)\cdot\frac{\partial}{\partial \mathbf{r}}\mathbf{E}(\mathbf{v}\tau,\tau)\rangle \, d\tau .$$

The last relation can be expressed, by use of (3.15), in the form

$$\left[\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave} = \frac{q}{M}\mathbf{E}^{pol} + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\frac{\langle \Delta \mathbf{v}\Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave} \qquad (3.19)$$

or, by virtue of the relation

$$\mathbf{k}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\delta(\omega - \mathbf{k}\cdot\mathbf{v}) = k^2 \frac{\partial}{\partial \mathbf{v}}\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

in the form

$$\left[\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave} = \frac{q}{M}\mathbf{E}^{pol} + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}}\left[\frac{\langle \Delta \mathbf{v}\cdot\Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave}. \qquad (3.20)$$

The relations (3.19) and (3.20) were found by Hubbard[8].

Expressions in terms of the velocity distribution functions

Let us assume that the plasma under consideration is composed of several species of charged particles. Let e_s, m_s and $f_s(\mathbf{v}_s)$ be the particle charge, the particle mass and the velocity distribution function of type s.

In the absence of any strong magnetic field it is known that (see Appendix)

$$\mathbf{k}\cdot\varepsilon''(\mathbf{k},\omega)\cdot\mathbf{k} = -4\pi^2 \sum_s \frac{e_s^2}{m_s}\int \delta(\omega - \mathbf{k}\cdot\mathbf{v}_s)\,\mathbf{k}\cdot\frac{\partial f_s(\mathbf{v}_s)}{\partial \mathbf{v}_s}\,d\mathbf{v}_s \,. \qquad (3.21)$$

Substitution of this expression into (3.6) leads to

$$\mathbf{E}^{pol} = 2q\sum_s \frac{e_s^2}{m_s}\int \frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}_s)}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\,\mathbf{k}\mathbf{k}\cdot\frac{\partial f_s(\mathbf{v}_s)}{\partial \mathbf{v}_s}\,d\mathbf{v}_s\,d\mathbf{k}\,d\omega\,. \qquad (3.22)$$

Further it can be shown that (see Appendix)

$$(\rho^2)_{\mathbf{k}\omega} = (2\pi)^{-3} \sum_s e_s^2 \int \delta(\omega - \mathbf{k}\cdot\mathbf{v}_s) f_s(\mathbf{v}_s)\, d\mathbf{v}_s \ . \tag{3.23}$$

Substitution of this expression into (3.17) gives

$$\left[\frac{\langle \Delta\mathbf{v}\,\Delta\mathbf{v}\rangle}{\Delta t}\right]_{wave} = \frac{4q^2}{M^2} \sum_s e_s^2 \iiint \frac{\mathbf{k}\mathbf{k}\,\delta(\omega - \mathbf{k}\cdot\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v}_s)}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2} f_s(\mathbf{v}_s)\, d\mathbf{v}_s\, d\mathbf{k}\, d\omega . \tag{3.24}$$

From (3.19) follows

$$\left[\frac{\langle \Delta\mathbf{v}\rangle}{\Delta t}\right]_{wave}$$

$$= \frac{2q^2}{M} \sum_s e_s^2 \iiint \frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v}_s)}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\, \mathbf{k}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}_s} + \frac{1}{M}\frac{\partial}{\partial\mathbf{v}}\right)\delta(\omega - \mathbf{k}\cdot\mathbf{v}) f_s(\mathbf{v}_s)\, d\mathbf{v}_s\, d\mathbf{k}\, d\omega . \tag{3.25}$$

4. Collision cross sections in classical mechanics

The change in velocity $\mathbf{v}$ of a test particle, which has the mass M and charge q, in one collision with a particle of type s is given by

$$\frac{m_s}{M + m_s}\, g \left[(1 - \cos\theta)\frac{\mathbf{g}}{g} + \frac{\mathbf{b}}{b}\sin\theta \right] . \tag{4.1}$$

Here $\mathbf{g}$ is the initial velocity of the particle of type s relative to the test particle; $g = |\mathbf{g}|$; $\mathbf{b}$ is a vector whose length is the impact parameter b and whose direction lies in the plane of orbit and is perpendicular to $\mathbf{g}$; and θ is the angle through which the relative velocity is deflected.

The first moment in the collision theory with the screened potential $(qe_s/r)e^{-\kappa r}$ between the test and field particles is the following:

$$\left[\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t}\right]_\kappa = \sum_s \frac{m_s}{M+m_s} \int Q_\kappa^{(1)} \, g \, f_s(\mathbf{v}+\mathbf{g}) \, \mathbf{g} \, d\mathbf{g} . \qquad (4.2)$$

Here f_s is the velocity distribution function of particles of type s and

$$Q_\kappa^{(l)} = \int_0^\infty (1 - \cos^l \theta) \, 2\pi b \, db, \qquad (l = 1,2). \qquad (4.3)$$

In the second moment

$$\left[\frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle}{\Delta t}\right]_\kappa = \sum_s \left(\frac{m_s}{M+m_s}\right)^2 \iint \Big[(1-\cos\theta)^2 \frac{\mathbf{g}\mathbf{g}}{g^2}$$

$$+ \sin^2\theta \, \frac{\mathbf{b}\mathbf{b}}{b^2} \Big] g^3 f_s(\mathbf{v}+\mathbf{g}) \, d\mathbf{g} \, d\mathbf{b}$$

the tensor $\mathbf{b}\mathbf{b}/b^2$ averaged over the directions of $\mathbf{b}$ around $\mathbf{g}$ is $\frac{1}{2}(\mathbf{1} - \mathbf{g}\mathbf{g}/g^2)$, $\mathbf{1}$ being a unit tensor. Thus we obtain

$$\left[\frac{\langle \Delta \mathbf{v} \, \Delta \mathbf{v} \rangle}{\Delta t}\right]_\kappa = \sum_s \left(\frac{m_s}{M+m_s}\right)^2 \int \Big[\frac{\mathbf{g}\mathbf{g}}{g^2} (2 Q_\kappa^{(1)} - Q_\kappa^{(2)})$$

$$+ \frac{1}{2} \left(\mathbf{1} - \frac{\mathbf{g}\mathbf{g}}{g^2}\right) Q_\kappa^{(2)} \Big] g^3 f_s(\mathbf{v}+\mathbf{g}) \, d\mathbf{g} . \qquad (4.4)$$

In particular the relation

$$\left[\frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle}{\Delta t}\right]_\kappa = 2 \sum_s \left(\frac{m_s}{M+m_s}\right)^2 \int Q_\kappa^{(1)} \, g^3 f_s(\mathbf{v}+\mathbf{g}) \, d\mathbf{g} \qquad (4.5)$$

holds for the trace of the second moment.

In classical mechanics it is known that[28]

$$Q_{r}^{(1)} = 4\pi \left(\frac{g e_s}{\mu g^2}\right)^2 \left(\ln \frac{2\mu g^2}{\gamma |g e_s| r} - \frac{1}{2}\right) ,$$

$$Q_{r}^{(2)} = 8\pi \left(\frac{g e_s}{\mu g^2}\right)^2 \left(\ln \frac{2\mu g^2}{\gamma |g e_s| r} - 1\right) , \tag{4.6}$$

where $\mu = m_s M/(M + m_s)$ is the reduced mass and $\ln\gamma = \ln 1.78107 = 0.57722$ is Euler's constant. As regards the Eular's constant the following formulas will often be used in the present paper:

$$\int_0^\infty e^{-x} \ln x \, dx = -\ln \gamma , \tag{4.7}$$

$$\int_0^\infty x e^{-x} \ln x \, dx = -\ln \gamma + 1 . \tag{4.8}$$

It should be noted that the second moment can be written in the form

$$\left[\frac{\langle \Delta v \, \Delta v \rangle}{\Delta t}\right]_r = \frac{1}{2}\left(\mathbf{1} - \frac{g g}{g^2}\right)\left[\frac{\langle \Delta v \cdot \Delta v \rangle}{\Delta t}\right]_r$$

$$+ \text{ trace-free tensor,} \tag{4.9}$$

in which the trace-free tensor is independent of r and can be accurately calculated on the basis of binary collisions.

II TEST PARTICLE PROBLEMS

5. Relaxation of a slow heavy particle

As simplest examples we consider, in this and the next sections, systems composed of a plasma and a test particle. The mass, charge and velocity of the test particle are denoted by M, q and $\mathbf{v}$, respectively. We calculate the rate of diffusion of the particle velocity, which is defined by

$$D \equiv \frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle}{\Delta t}, \tag{5.1}$$

and the rate of relaxation in the particle energy, which is given by

$$\begin{aligned} \frac{dw}{dt} &= \frac{M}{2} \frac{1}{\Delta t} \left[\langle (\mathbf{v}+\Delta \mathbf{v}) \cdot (\mathbf{v}+\Delta \mathbf{v}) \rangle - v^2 \right] \\ &= \frac{M}{2} \left[2 \mathbf{v} \cdot \frac{\langle \Delta \mathbf{v} \rangle}{\Delta t} + \frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle}{\Delta t} \right] . \end{aligned} \tag{5.2}$$

Here the bracket notation $\langle \ \ \rangle$ indicates the average over all interactions with the plasma constituents whose velocity distributions are assumed to be Maxwellian

$$f_s(\mathbf{v}_s) = n_s \left(\frac{m_s}{2\pi T}\right)^{3/2} \exp\left(-\tfrac{1}{2} m_s v_s^2 / T\right) . \tag{5.3}$$

In this section we treat the case where the mass M of the test particle is large and its velocity v is much lower than the thermal velocities of the ions:

$M \gg m_s \ , \quad v \ll (T/m_s)^{\frac{1}{2}}$ for all s, T being the plasma temperature. In this case $\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle / \Delta t$ is independent of $\mathbf{v}$ and $-\mathbf{v} \cdot \langle \Delta \mathbf{v} \rangle / \Delta t$ is proportional to v^2. Furthermore, dw/dt must vanish when the kinetic energy $\frac{1}{2}Mv^2$ is equal to $\frac{3}{2}$ T. Therefore holds the relation

$$\frac{dw}{dt} = \left(1 - \frac{Mv^2}{3T}\right)\frac{M}{2} D. \tag{5.4}$$

In the wave theory (3.24) gives

$$D_{wave} = \frac{4q^2}{M^2} \sum_s e_s^2 \iint \frac{k^2}{|\mathbf{k} \cdot \varepsilon(\mathbf{k},0) \cdot \mathbf{k}|^2} \delta(\mathbf{k} \cdot \mathbf{v}_s) f_s(\mathbf{v}_s) d\mathbf{v}_s d\mathbf{k} .$$

Making use of the integration

$$\int \delta(\mathbf{k} \cdot \mathbf{v}_s) f_s(\mathbf{v}_s) d\mathbf{v}_s = k^{-1} n_s (m_s / 2\pi T)^{\frac{1}{2}}$$

and taking account of the relationship

$$\mathbf{k} \cdot \varepsilon(\mathbf{k}, 0) \cdot \mathbf{k} = k^2 + k_D^2, \tag{5.5}$$

in which

$$k_D^2 = \sum_s \frac{4\pi n_s e_s^2}{T} \tag{5.6}$$

is the square of Debye-Hückel's static screening constant, we obtain

$$D_{wave} = \frac{16\pi q^2}{M^2} \sum_s e_s^2 n_s \left(\frac{m_s}{2\pi T}\right)^{\frac{1}{2}} \int \frac{k^3 dk}{(k^2 + k_D^2)^2} . \tag{5.7}$$

For the diffusion coefficient (5.7), the effective screening constant κ° defined by (1.5) is equal to the static Debye constant; $\kappa^\circ = k_D$, as it should. Thus we obtain, by use of (4.5), (4.6) and (4.7),

$$D = \frac{8g^2}{M^2}\sum_s n_s e_s^2 \left(\frac{2\pi m_s}{T}\right)^{\frac{1}{2}}\left(\ln \frac{4TM}{\gamma^2 |g e_s| k_D (M+m_s)} - \frac{1}{2}\right). \quad (5.8)$$

6. Relaxation of a fast particle

We now consider the case where the speed v of the test particle is much higher than the thermal velocities:

$$v \gg (T/m_s)^{\frac{1}{2}} \qquad \text{for all s.} \qquad (6.1)$$

On replacing the relative velocity $\mathbf{g}$ in (4.2) and (4.3) by $-\mathbf{v}$ we obtain in the collision theory with screening κ

$$D_\kappa = \frac{8\pi g^2}{M^2 v}\sum_s n_s e_s^2 \left(\ln \frac{2\mu_s v^2}{\gamma |g e_s| \kappa} - \frac{1}{2}\right) \qquad (6.2)$$

and

$$-\left[\frac{dv}{dt}\right]_\kappa = \frac{4\pi g^2}{v}\sum_s \frac{n_s e_s^2}{m_s}\left(\ln \frac{2\mu_s v^2}{\gamma |g e_s| \kappa} - \frac{1}{2}\right) \qquad (6.3)$$

where $\mu_s = m_s M/(m_s+M)$.

From (3.24) we have in the wave theory

$$D_{wave} = \frac{4g^2}{M^2}\sum_s e_s^2 \iiint \frac{k^2\,\delta(\mathbf{k}\cdot\mathbf{v})}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_s)\,f_s(\mathbf{v}_s)\,d\mathbf{v}_s\,d\mathbf{k}\,d\omega\,,$$

which can be transformed, by use of (3.21) and (5.3), as follows:

$$D_{wave} = \frac{T q^2}{\pi M^2}\int\left(\frac{i}{\pi} P\int_{-\infty}^{\infty} \frac{k^2 \delta(k\cdot v)}{k\cdot \varepsilon(k,\omega)\cdot k}\,\frac{d\omega}{\omega}\right) dk$$

$$= \frac{2 T q^2}{M^2 v}\int\left(\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{k^3}{k\cdot \varepsilon(k,\omega)\cdot k}\,\frac{d\omega}{\omega}\right) dk. \qquad (6.4)$$

(P stands for "principal value".)

The dielectric permeability has no zero-points in the upper half-plane of complex ω. The integral with respect to ω can, therefore, be transformed into a sum of integrals along a very small semicircle C_1 and a very large semicircle C_2 in the upper half-plane as shown in Fig. 1. On the small semicircle C_1 we have

$$\varepsilon(k,\omega) = \varepsilon(k,0) = 1 + k_D^2/k^2$$

and on the large semicircle C_2 holds the relation (cf. Appendix)

$$\varepsilon(k,\omega) = 1 - \omega_p^2/\omega^2 \qquad \text{where } \omega_p^2 = \sum_s 4\pi n_s e_s^2/m_s. \qquad (6.5)$$

The path C_2 of the integral $\int (\omega^2 - \omega_p^2)^{-1}\omega\, d\omega$ can be shifted to the real axis. Hence

$$\frac{i}{\pi} P\int_{-\infty}^{\infty} \frac{k^2}{k\cdot\varepsilon(k,\omega)\cdot k}\,\frac{d\omega}{\omega} = \frac{-k^2}{k^2+k_D^2} + 1$$

$$= \frac{k_D^2}{k^2+k_D^2}.$$

Fig. 1

Thus we obtain finally

$$D_{wave} = \frac{2 T q^2 k_D^2}{M^2 v}\int_{k>0} \frac{k\, dk}{k^2 + k_D^2} \qquad (6.6)$$

On substituting (3.20) into (5.2) we have

$$-\left[\frac{dw}{dt}\right]_{wave} = -q\mathbf{v}\cdot\mathbf{E}^{pol} - \frac{M}{2}\left(\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}}\left[\frac{\langle\Delta\mathbf{v}\cdot\Delta\mathbf{v}\rangle}{\Delta t}\right]_{wave} + \left[\frac{\langle\Delta\mathbf{v}\cdot\Delta\mathbf{v}\rangle}{\Delta t}\right]_{wave}\right).$$

The sum in the brackets, however, vanishes since $[\langle\Delta\mathbf{v}.\Delta\mathbf{v}\rangle/\Delta t]_{wave}$ is proportional to v^{-1}. Hence we have

$$-\left[\frac{dw}{dt}\right]_{wave} = -q\mathbf{v}\cdot\mathbf{E}^{pol} \tag{6.7}$$

or, by use of (3.5),

$$-\left[\frac{dw}{dt}\right]_{wave} = q^2\frac{i}{2\pi^2}\iint\frac{\omega\,\delta(\omega-\mathbf{k}\cdot\mathbf{v})}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}}\,d\mathbf{k}\,d\omega, \tag{6.8}$$

which is calculated to be

$$\begin{aligned}-\left[\frac{dw}{dt}\right]_{wave} &= \frac{q^2}{v}\,\frac{i}{\pi}\iint_{-kv}^{kv}\frac{\omega\,d\omega}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}}\,k\,dk\\ &= \frac{q^2\omega_p^2}{v}\int_{k>\omega_p/v}\frac{dk}{k}.\end{aligned} \tag{6.9}$$

From (6.6) and (6.9) we have

$$\begin{aligned}D_{wave} - D_{wave,\kappa} &= \frac{2Tq^2k_D^2}{M^2v}\int_0^\infty\left[\frac{k}{k^2+k_D^2} - \frac{k^3}{(k^2+\kappa^2)^2}\right]dk\\ &= \frac{2Tq^2k_D^2}{M^2v}\left(\ln\frac{\kappa}{k_D}+\frac{1}{2}\right),\end{aligned} \tag{6.10}$$

$$-\left[\frac{dw}{dt}\right]_{wave} + \left[\frac{dw}{dt}\right]_{wave,\kappa} = \frac{g^2\omega_p^2}{v} \lim_{a\to\infty} \left[\int_{\omega_p/v}^{a} \frac{dk}{k} - \int_0^a \frac{k^3 dk}{(k^2+\kappa^2)^2} \right]$$

$$= \frac{g^2\omega_p^2}{v}\left(\ln\frac{\kappa v}{\omega_p} + \frac{1}{2}\right). \qquad (6.11)$$

Substituting (6.2),(6.3),(6.10) and (6.11) into the connection formula (1.3) we finally obtain

$$D = \frac{8\pi g^2}{M^2 v} \sum_s n_s e_s^2 \ln \frac{2\mu_s v^2}{\gamma |g e_s| k_D}, \qquad (6.12)$$

$$-\frac{dw}{dt} = \frac{g^2}{v} \sum_s \frac{4\pi n_s e_s^2}{m_s} \ln \frac{2\mu_s v^3}{\gamma |g e_s| \omega_p}. \qquad (6.13)$$

III ION-ELECTRON INTERACTION IN HOMOGENEOUS PLASMAS

7. Two-component plasmas

In the following we consider two-component plasmas which are composed of electrons and one type of ions. The mass, charge, number density, and velocity distribution function of the ion are denoted by m_1, Ze, n_1, and f_1, respectively; those of the electron by m_2, $-e$, n_2, and f_2. The electron-to-ion mass ratio, m_2/m_1, will be neglected as compared with unity.

For the dielectric permeability of a two-component plasma (in equilibrium), we have, in terms of the constants

$$k_1^2 = 4\pi n_1 Z^2 e^2/T \qquad \text{and} \qquad k_2^2 = 4\pi n_2 e^2/T, \qquad (7.1)$$

the expression

$$\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k} - k^2 = \sum_{s=1}^{2}\left(\mathbf{k}\cdot\varepsilon_s(\mathbf{k},\omega)\cdot\mathbf{k} - k^2\right) \tag{7.2}$$

where (see Appendix)

$$\mathbf{k}\cdot\varepsilon_s(\mathbf{k},\omega)\cdot\mathbf{k} - k^2 = k_s^2 F(x_s), \quad x_s \equiv \frac{\omega}{k}\left(\frac{m_s}{2T}\right)^{\frac{1}{2}}, \tag{7.3}$$

$$F(x) = 1 - 2xe^{-x^2}\int_0^x e^{t^2}dt + i\sqrt{\pi}\,xe^{-x^2}. \tag{7.4}$$

Since the mean speed of ions is sufficiently lower than the mean electron speed, the electronic part of the screening around an ion is of static character. Namely, the dielectric permeability effective to ion-electron interaction has the form

$$\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k} = k_2^{\,2} + \mathbf{k}\cdot\varepsilon_1(\mathbf{k},\omega)\cdot\mathbf{k}, \tag{7.5}$$

where $\mathbf{k}\cdot\varepsilon_1(\mathbf{k},\omega)\cdot\mathbf{k}$ refers to the ions only. For the imaginary part ε''_1 holds the relation (cf. (3.21))

$$\mathbf{k}\cdot\varepsilon_1''(\mathbf{k},\omega)\cdot\mathbf{k} = -4\pi^2\frac{Z^2e^2}{m_1}\int\delta(\omega-\mathbf{k}\cdot\mathbf{v})\,\mathbf{k}\cdot\frac{\partial f_1(\mathbf{v})}{\partial\mathbf{v}}\,d\mathbf{v} \tag{7.6}$$

The ε effective to ion-ion interaction has also a similar form. For electron-electron interaction, on the other hand, the ions do not play any role in the screening:

$$\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k} = \mathbf{k}\cdot\varepsilon_2(\mathbf{k},\omega)\cdot\mathbf{k}. \tag{7.7}$$

8. Relaxation between the ion and electron temperatures

Let us first consider the relaxation between the ion and electron temperatures. Since the mass ratio is far from unity, the energy transfer between the ion and electron systems may be much smaller than among the ions or among the electrons. Hence the ion temperature T_1 and the electron temperature T_2 can often be different even when the velocity distributions are both Maxwellian:

$$f_s(\mathbf{v}) = n_s \left(\frac{m_s}{2\pi T_s}\right)^{\frac{3}{2}} \exp\left(-\frac{m_s v^2}{2T_s}\right) \qquad \text{for } s = 1,2. \tag{8.1}$$

Here the difference $|T_1 - T_2|$ is not supposed to be small; but we assume that the mean ion speed is much smaller than the mean electron speed.

In an elementary manner, or on multiplying by $\frac{1}{2}m_1 v^2$ the both sides of the kinetic equation

$$\frac{\partial f_1(\mathbf{v})}{\partial t} = -\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t} f_1(\mathbf{v})\right) + \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \frac{\partial}{\partial \mathbf{v}} : \left(\frac{\langle \Delta \mathbf{v}\, \Delta \mathbf{v} \rangle}{\Delta t} f_1(\mathbf{v})\right) - \cdots$$

and integrating with respect to $\mathbf{v}$, we obtain the time rate of change in the ion temperature T_1 in the form

$$\frac{3n_1}{2} \frac{dT_1}{dt} = \frac{d}{dt} \int \frac{m_1}{2} v^2 f_1(\mathbf{v})\, d\mathbf{v}$$

$$= \frac{m_1}{2} \int \left(2\mathbf{v} \cdot \frac{\langle \Delta \mathbf{v} \rangle}{\Delta t} + \frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle}{\Delta t} \right) f_1(\mathbf{v})\, d\mathbf{v}. \tag{8.2}$$

It is easy to see, in the collision theory, that the ion-ion interactions are cancelled on the right side of (8.2), as it should; we need only to consider the ion-electron interactions:

$$\left[\frac{dT_1}{dt}\right]_\kappa = \frac{m_1}{3n_1}\int\left(2\mathbf{v}_1\cdot\left[\frac{\langle\Delta\mathbf{v}_1\rangle_2}{\Delta t}\right]_\kappa + \left[\frac{\langle\Delta\mathbf{v}_1\cdot\Delta\mathbf{v}_1\rangle_2}{\Delta t}\right]_\kappa\right) f_1(\mathbf{v}_1)\,d\mathbf{v}_1, \qquad (8.3)$$

where the notation $\langle\ \ \rangle_2$ indicates the part originated in the interaction with electrons. Further it can be shown that

$$\left[\frac{dT_1}{dt}\right]_\kappa = \frac{T_2 - T_1}{T_2}\,\frac{m_1}{3n_1}\int\left[\frac{\langle\Delta\mathbf{v}_1\cdot\Delta\mathbf{v}_1\rangle_2}{\Delta t}\right]_\kappa f_1(\mathbf{v}_1)\,d\mathbf{v}_1. \qquad (8.4)$$

(The quantity $[\langle\Delta\mathbf{v}_1\cdot\Delta\mathbf{v}_1\rangle_2/\Delta t]_\kappa$ in (8.3) is independent of $\mathbf{v}_1$ but $[\langle\Delta\mathbf{v}_1\rangle_2/\Delta t]_\kappa$ depends linearly upon $\mathbf{v}_1$. For constant T_2, therefore, $\int[\langle\Delta\mathbf{v}_1\cdot\Delta\mathbf{v}_1\rangle_2/\Delta t]_\kappa f_1(\mathbf{v}_1)d\mathbf{v}_1$ is independent of T_1 and $\int 2\mathbf{v}_1\cdot[\langle\Delta\mathbf{v}_1\rangle_2/\Delta t]_\kappa f_1(\mathbf{v}_1)d\mathbf{v}_1$ is proportional to T_1. On taking account of the fact that dT_1/dt vanishes for $T_1=T_2$, we obtain (8.4).) Hence, by use of (4.5) and (4.6),

$$\left[\frac{dT_1}{dt}\right]_\kappa = \frac{T_2-T_1}{T_2}\,\frac{8\pi Z^2e^4}{3m_1}\int d\mathbf{v}_2\,\frac{f_2(\mathbf{v}_2)}{v_2}\left(\ln\frac{2m_2v_2^2}{\gamma Ze^2\kappa} - \frac{1}{2}\right)$$

$$= \frac{T_2-T_1}{T_2}\,\frac{8\pi Z^2e^4}{3m_1}\left(\frac{2m_2}{\pi T_2}\right)^{\frac{1}{2}} n_2\left(\ln\frac{4T_2}{\gamma^2Ze^2\kappa} - \frac{1}{2}\right). \qquad (8.5)$$

Here g and μ_2 have been replaced by $\mathbf{v}_2$ and m_2, because the mass ratio m_1/m_2 is far from unity.

In the wave theory we substitute (3.24) and (3.25) into (8.2) and integrate by parts. Then we obtain

$$\left[\frac{dT_1}{dt}\right]_{wave} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\frac{4Z^2e^4}{3n_1}\iiiint \frac{\omega^2}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}$$

$$\times\ \delta(\omega-\mathbf{k}\cdot\mathbf{v}_1)\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_2)\,f_1(\mathbf{v}_1)\,f_2(\mathbf{v}_2)\,d\mathbf{v}_1\,d\mathbf{v}_2\,d\mathbf{k}\,d\omega\,,$$

the ion-ion interaction again vanishing.

Since the mean electron velocity is much larger than the mean ion velcoity, $\delta(\omega - \mathbf{k}\cdot\mathbf{v}_2)$ can be replaced by $\delta(\mathbf{k}\cdot\mathbf{v}_2)$ and the integration over $\mathbf{v}_2$ can be performed,

$$\left[\frac{dT_1}{dt}\right]_{wave} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\frac{4Z^2e^4}{3n_1}\, n_2\left(\frac{m_2}{2\pi T_2}\right)^{\frac{1}{2}}\iiint \frac{\omega^2\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_1)}{k\,|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\, f_1(\mathbf{v}_1)\,d\mathbf{v}_1\,d\mathbf{k}\,d\omega\,.$$

The dielectric permeability and its imaginary part are given by (7.5) and (7.6). Thus

$$\frac{4\pi Z^2e^4}{T_1}\iint \frac{\omega^2\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_1)}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\, f_1(\mathbf{v}_1)\,d\mathbf{v}_1\,d\omega$$

$$= \frac{1}{\pi}\int \frac{\omega\,\mathbf{k}\cdot\varepsilon_1''(\mathbf{k},\omega)\cdot\mathbf{k}}{|k_2^2 + \mathbf{k}\cdot\varepsilon_1(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\,d\omega$$

$$= \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\omega\,d\omega}{k_2^2 + \mathbf{k}\cdot\varepsilon_1(\mathbf{k},\omega)\cdot\mathbf{k}}\,.$$

Since the dielectric permeability has no zero-points in the upper half-plane of complex ω, and the relation

$$\varepsilon_1(\mathbf{k},\omega) = 1 - \omega_1^2/\omega^2 \qquad \text{where } \omega_1^2 = 4\pi Z^2e^2 n_1/m_1 \quad (8.6)$$

holds true on a large semicircle in the upper half-plane, the last integral can be transformed into

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\omega^3\, d\omega}{(k^2+k_2^2)\,\omega^2 - k^2\omega_1^2} = \frac{k^2\omega_1^2}{(k^2+k_2^2)^2} .$$

Thus we obtain

$$\left[\frac{dT_1}{dt}\right]_{wave} = \frac{T_2 - T_1}{T_2}\,\frac{16\pi Z^2 e^4}{3 m_1}\left(\frac{m_2}{2\pi T_2}\right)^{\frac{1}{2}} n_2 \int_{k>0} \frac{k^3\, dk}{(k^2+k_2^2)^2} \qquad (8.7)$$

Since

$$\left[\frac{dT_1}{dt}\right]_{wave} - \left[\frac{dT_1}{dt}\right]_{wave,\,\kappa} = \text{const.}\int_0^{\infty}\left[\frac{k^3}{(k^2+k_2^2)^2} - \frac{k^3}{(k^2+\kappa^2)^2}\right] dk$$

the effective screening constant defined by (1.5) is k_2; the final result can be obtained by substituting k_2 for κ in (8.5),

$$\frac{dT_1}{dt} = \frac{T_2 - T_1}{T_2}\,\frac{8\pi Z^2 e^4}{3 m_1}\left(\frac{2 m_1}{\pi T_1}\right)^{\frac{1}{2}} n_2 \left(\ln\frac{4T_2}{\gamma^2 Z e^2 k_2} - \frac{1}{2}\right). \qquad (8.8)$$

The relaxation constant R defined by

$$\frac{d}{dt}(T_2 - T_1) = -\,(T_2 - T_1)\,R , \qquad (8.9)$$

$n_1 T_1 + n_2 T_2$ being kept constant, is

$$R = R^{\circ}\left(\ln\frac{4T_2}{\gamma^2 Z e^2 k_2} - \frac{1}{2}\right), \qquad (8.10)$$

where

$$R^{\circ} = \frac{8}{3}(n_1+n_2)\frac{m_2}{m_1}\left(\frac{Ze^2}{T_2}\right)^2\left(\frac{2\pi T_2}{m_2}\right)^{\frac{1}{2}} . \qquad (8.11)$$

It is a remarkable fact that the ionic screening does not play any role in the relaxation caused by energy transfers between the ion and electron systems. In the case of relaxations caused by momentum transfers between the ion and electron systems some fraction of the ionic screening contributes as shown in the following sections.

9. Effective screening constant for ion-electron scattering

This section is devoted to the following useful theorem. The effective screening constant for the first moment, $\langle \Delta \mathbf{v} \rangle_1 / \Delta t$ and the trace of the second moment, $\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle_1 / \Delta t$, of the change in electron velocity due to interactions with the ions is equal to k_{12} defined by

$$\ln \frac{k_{12}^2}{k_2^2} = \frac{k_D^2}{k_1^2} \ln \frac{k_D^2}{k_2^2} - 1 , \qquad (9.1)$$

where $k_D^2 = k_1^2 + k_2^2$. The constant k_{12} is between k_2 and k_D as it should; more precisely

$$k_2^2 + \exp(-1)\, k_1^2 < k_{12}^2 < k_2^2 + \tfrac{1}{2} k_1^2 .$$

In the wave theory we have from (3.24)

$$\left[\frac{\langle \Delta v \cdot \Delta v\rangle_1}{\Delta t}\right]_{wave}$$

$$= \frac{4Z^2 e^4}{m_2^2}\iiint \frac{k^2}{|k\cdot\varepsilon(k,\omega)\cdot k|^2}\,\delta(k\cdot v)\,\delta(\omega - k\cdot v_1)\, f_1(v_1)\,dv_1\,dk\,d\omega. \quad (9.2)$$

The dielectric permeability is again given by (7.5) and (7.6). Thus we proceed as follows:

$$\left[\frac{\langle \Delta v \cdot \Delta v\rangle_1}{\Delta t}\right]_{wave} = \frac{e^2 T}{\pi m_2^2}\int I(k)\,\delta(k\cdot v)\,dk \qquad (9.3)$$

where

$$I(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{k^2\, k\cdot\varepsilon_1''(k,\omega)\cdot k}{\omega\,|k_2^2 + k\cdot\varepsilon_1(k,\omega)\cdot k|^2}\,d\omega$$

$$= \frac{i}{\pi}\,P\int_{-\infty}^{\infty} \frac{k^2\,d\omega}{\omega[k_2^2 + k\cdot\varepsilon_1(k,\omega)\cdot k]}$$

(P indicating the principal value at $\omega = 0$).

The integral with respect to ω can be transformed into a sum of integrals along a small semicircle C_1 and a large semicircle C_2 in the upper half-plane of complex ω(Fig. 1). On C_1, $k_2^2 + k\cdot\varepsilon_1\cdot k$ is equal to $k_2^2 + k_1^2$ or k_D^2; on C_2 we have the relation (8.6). The path C_2 of the integral $\int[(k^2 + k_2^2)\omega^2 - k^2\omega_1^2]^{-1}\omega d\omega$ can be shifted to the real axis. Hence

$$I(k) = \frac{-k^2}{k^2 + k_D^2} + \frac{k^2}{k^2 + k_2^2}$$

$$= \frac{k_1^2 k^2}{(k^2 + k_D^2)(k^2 + k_2^2)} . \qquad (9.4)$$

Thus we obtain

$$\left[\frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle}{\Delta t}\right]_{wave} = \frac{2e^2 T}{m_2^2 v} k_1^2 \int_{k>0} \frac{k^3 dk}{(k^2 + k_D^2)(k^2 + k_2^2)} . \qquad (9.5)$$

We have therefore, by use of (1.2) and (1.4),

$$\left[\frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle_1}{\Delta t}\right]_{wave} - \left[\frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle_1}{\Delta t}\right]_{wave, \kappa}$$

$$= \frac{2e^2 T}{m_2^2 v} k_1^2 \int_0^\infty \left[\frac{k^3}{(k^2 + k_D^2)(k^2 + k_2^2)} - \frac{k^3}{(k^2 + \kappa^2)^2}\right] dk . \qquad (9.6)$$

From this expression and the definition (1.5a) the theorem follows.

As regards the first moment $\langle \Delta \mathbf{v} \rangle_1 / \Delta t$, the vector is parallel to $-\mathbf{v}$. Moreover the relation

$$-\mathbf{v} \cdot \frac{\langle \Delta \mathbf{v} \rangle_1}{\Delta t} = \frac{1}{2} \frac{\langle \Delta \mathbf{v} \cdot \Delta \mathbf{v} \rangle_1}{\Delta t} \qquad (9.7)$$

holds since the ion-electron energy transfer can be neglected in comparison with the momentum transfer.

It is to be noted that the contribution of the ions to the screening comes from the integral along a small semicircle at the origin in the ω-plane. When a relaxation is caused by energy transfer between the electron and ion systems, (9.7) does not hold and interactions corresponding to $\omega = 0$ or $\mathbf{k}\cdot\mathbf{v}_1 = 0$ do not play any role. This is the reason why the rate of temperature relaxation discussed in §8 does not depend on the ionic screening.

The third and higher moments and the trace-free part of the second moment do not diverge for a pure Coulomb potential (see the last paragraphs in sections 2 and 4). Hence the above-mentioned theorem can be generalized to the following:

The effective screening constant for any moment of the change in electron velocity due to the ion-electron scattering is k_{12} defined by (9.1).

10. Electric conduction in an oscillating field

The electric conductivity will be considered under the condition that the frequency of the applied field $\mathbf{E}_0 e^{-i\omega t}$ is much lower than the plasma frequency but sufficiently high as compared with the collision frequency.

By virtue of the second part of the above-mentioned condition the right side of the kinetic equation (2.1),

$$\frac{\partial f_2}{\partial t} - \frac{e}{m_2} e^{-i\omega t}\,\mathbf{E}_0 \cdot \frac{\partial f_2}{\partial \mathbf{v}} = -\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t} f_2 \right) + \cdots , \qquad (10.1)$$

is very small as compared with the first term on the left side.

Thus, to lowest order, f_2 is simply given by

$$f_2(v) = f_2^\circ(v)\left(1 - \frac{ie}{\omega T} v \cdot E_0 e^{-i\omega t}\right), \tag{10.2}$$

where $f_2^\circ(v)$ denotes the Maxwellian distribution. The corresponding function $f_1(v)$ of the ions remains Maxwellian.

Multiplying both sides of (10.1) by $-ev$ and integrating with respect to v, we obtain

$$-i\omega j e^{-i\omega t} - \frac{n_2 e^2}{m_2} E_0 e^{-i\omega t} = -e\int \frac{\langle\Delta v\rangle}{\Delta t} f_2 \, dv, \tag{10.3}$$

where $je^{-i\omega t} = -e\int v f_2(v)dv$ is the electric current density.

We substitute (10.2) and (4.2) and make use of (4.6) in which κ is replaced by the effective screening constant k_{12} given by (9.1). Thus we have

$$\int \frac{\langle\Delta v\rangle}{\Delta t} f_2(v)dv = \frac{4\pi}{3}\frac{n_1(Ze^2)^2}{m_2^2}\frac{ie}{\omega T} E_0 e^{-i\omega t}\int \frac{1}{v}\left(\ln\frac{2m_2v^2}{\gamma Ze^2 k_{12}} - \frac{1}{2}\right) f_2^\circ(v)dv.$$

Here we have taken into account the facts that the electron-electron interactions do not play any role and that the ion velocity is negligible in comparison with the electron velocity.

The electric conductivity σ, which is defined by

$$j e^{-i\omega t} = \left(\frac{i n_2 e^2}{\omega m_2} + \sigma\right) E_0 e^{-i\omega t} \tag{10.4}$$

is found to be

$$\sigma = \sigma^\circ\left(\ln\frac{4T}{\gamma^2 Ze^2 k_{12}} - \frac{1}{2}\right), \tag{10.5}$$

where

$$\sigma^{\circ} = \frac{n_1}{3\pi} \frac{\omega_o^2}{\omega^2} \left(\frac{2e^2}{T}\right)^2 \left(\frac{2\pi T}{m_2}\right)^{\frac{1}{2}}, \tag{10.6}$$

$\omega_o = (4\pi n_2 e^2/m_2)^{\frac{1}{2}}$ being the plasma frequency.

11. High frequency conductivity

We are developing our theory under the assumption that the time interval during which the macroscopic state changes appreciably is much longer than the time of continuance of the force correlation (see the second paragraph of section 2). Thus it is assumed in the preceeding section that the frequency of the applied electric field is much lower than the plasma frequency. The case where the applied frequency is comparable with the plasma frequency is treated in this section. For the present case the concept of moments of the change in particle velocity cannot be used in the wave theory; but a somewhat similar procedure can be followed.

The collision theory is the same as in the preceeding section; the conductivity σ defined by (10.4) is of the form (10.5)

$$\sigma_{\kappa} = \sigma^{\circ} \left(\ln \frac{4T}{\gamma^2 2e^2 \kappa} - \frac{1}{2} \right). \tag{11.1}$$

Here κ is the screening constant which is taken in the range

$$\text{collision radius} \ll \kappa^{-1} \ll (2t/m_2)^{\frac{1}{2}}/\omega$$

and should be eliminated by unification of the collision and wave theories.

In the wave theory the system of electrons can be treated as a medium with the dielectric permeability $\varepsilon(\mathbf{k},\omega)$; and the ions, as particles in this medium. In the electric field $\mathbf{E}_0 e^{-i\omega t}$ the medium oscillates with the amplitude $\mathbf{a} = (e/m_2\omega^2)\mathbf{E}_0$, while the ions are standing still. The force between the ions and the medium is due to the polarization which is induced by the ions. We denotes by $Ze\,\mathbf{F}\,e^{-i\omega t}$ the polarization force acting on one ion. The current density is then given in the form

$$j = \frac{ie}{m_2\omega}\left(n_2 e\mathbf{E}_0 + n_1 Ze\mathbf{F}\right). \tag{11.2}$$

For the purpose of calculating the field $\mathbf{F}$ it is convenient to introduce a coordinate system which makes the same oscillatory motion as the motion of the medium. In this coordinate system the position of the i-th ion is given by

$$\mathbf{r}_i = \mathbf{r}_{io} - \mathbf{a}e^{-i\omega t}, \tag{11.3}$$

$\mathbf{r}_{io}$ being the time average. The electric charge density of the ion $\rho(\mathbf{r}_i,t) = Ze\delta(\mathbf{r}_i - \mathbf{r}_{io} + \mathbf{a}e^{-i\omega t})$ has the Fourier transform

$$\rho(\mathbf{k},\omega') = (2\pi)^{-3} Ze\left[\delta(\omega') + i\mathbf{k}\cdot\mathbf{a}\,\delta(\omega'-\omega)\right]\exp(-i\mathbf{k}\cdot\mathbf{r}_{io}) \tag{11.4}$$

to the first order in $\mathbf{a}$. The Fourier component $\mathbf{E}(\mathbf{k},\omega')$ of the field induced by this charge density is determined by the Poisson equation:

$$\mathbf{E}(\mathbf{k},\omega') = -i\mathbf{k} \; \frac{4\pi\rho(\mathbf{k},\omega')}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega')\cdot\mathbf{k}} \; . \tag{11.5}$$

The corresponding field at the position of the j-th ion is

$$\mathbf{E}(\mathbf{r}_j, t) = \iint \mathbf{E}(\mathbf{k},\omega')\exp(i\mathbf{k}\cdot\mathbf{r}_j - i\omega' t)\,d\omega' d\mathbf{k} \, . \tag{11.6}$$

The quantity $\mathbf{F}$ is calculated to be

$$\mathbf{F}\,e^{-i\omega t} = \sum_i \mathbf{E}(\mathbf{r}_j, t)$$

$$= -\sum_i \frac{Ze}{2\pi^2}\int \mathbf{k}\mathbf{k}\cdot\mathbf{a}\left[\frac{1}{\mathbf{k}\cdot\varepsilon(\mathbf{k},0)\cdot\mathbf{k}} - \frac{1}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}}\right]$$

$$\times \exp(i\mathbf{k}\cdot(\mathbf{r}_{jo}-\mathbf{r}_{io}) - i\omega t)\,d\mathbf{k} \, . \tag{11.7}$$

This expression is independent of $\mathbf{r}_{jo}$.

We shall evaluate the sum

$$\sum_i \exp i\mathbf{k}\cdot(\mathbf{r}_{jo} - \mathbf{r}_{io}) \, .$$

The number $N(\mathbf{r})$ of ions at a distance $\mathbf{r}$ from the j-th ion is known as

$$N(\mathbf{r}) = n_1 - \frac{k_1^2}{(2\pi)^3}\int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2 + k_D^2}\,d\mathbf{k} \, ,$$

where $k_D^2 = k_1^2 + k_2^2$, k_1^2 and k_2^2 being biven by (7.1). Hence we have

$$\sum_i \exp i\mathbf{k}\cdot(\mathbf{r}_{jo}-\mathbf{r}_{io}) = 1 + \int N(\mathbf{r})\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r}$$

$$= 1 - \frac{k_1^2}{k^2 + k_D^2} + (2\pi)^3 n_1 \delta(\mathbf{k}) \, . \tag{11.8}$$

Substituting (11.8) into (11.7) and referring to (11.2) and (10.4) we have

$$\sigma_{wave}$$
$$= -\frac{ie}{m_2\omega}\, n_1 Ze\, \frac{Ze}{2\pi^2}\, \frac{e}{m_2\omega^2} \int \left[\frac{1}{\mathbf{k}\cdot\varepsilon(\mathbf{k},0)\cdot\mathbf{k}} - \frac{1}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}} \right] \frac{k^2+k_2^2}{k^2+k_D^2}\, \frac{4\pi}{3} k^4 dk, \qquad (11.9)$$

where the third term in (11.8) does not contribute because the charge neutrality is assumed. An expression for $\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}$ is

$$\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k} = k^2 + k_2^2\left(1 - 2xe^{-x^2}\int_0^x e^{t^2}dt + i\sqrt{\pi}\, x e^{-x^2}\right), \qquad (11.10)$$

where $x = (\omega/k)(m_2/2T)^{\frac{1}{2}}$. Hence

$$\sigma_{wave} = \sigma^0 \int \frac{2\pi^{-\frac{1}{2}}\, i\, e^{-x^2}\int_0^x e^{t^2}dt + e^{-x^2}}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}\;(k^2+k_D^2)}\, k^3 dk, \qquad (11.11)$$

σ^0 being given in (10.6). This expression in the wave theory was first derived by Oberman et al.[10] The real part of σ_{wave}

$$\mathrm{Re}\ \sigma_{wave} = \sigma^0 \int_{k>0} \frac{e^{-x^2}}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2}\, \frac{k^2+k_2^2}{k^2+k_D^2}\, k^3 dk \qquad (11.12)$$

diverges logarithmically as expected. In the following we remove this divergence by the aid of the connection formula. On the other hand the imaginary part of σ_{wave} converges: the imaginary part of σ is peculiar to the collective interaction and can be accurately calculated on the basis of the wave theory. A graph of $\mathrm{Im}\,\sigma/\sigma^0$ is shown in Fig. 2 in case $Z = 1$.

Fig. 2

The connection formula unifies σ_{κ} and Re σ_{wave} into

$$Re\,\sigma/\sigma^{0}$$

$$= \ln\frac{4T}{\gamma^{2}Ze^{2}\kappa} - \frac{1}{2} + \int_{0}^{\infty}\left[\frac{\exp\left(-\frac{\omega^{2}}{k^{2}}\frac{m_{2}}{2T}\right)(k^{2}+k_{2}^{2})}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^{2}(k^{2}+k_{D}^{2})} - \frac{1}{(k^{2}+\kappa^{2})^{2}}\right]k^{3}dk. \quad (11.13)$$

In some limiting cases the integral can be performed analytically.

When $\omega \ll \omega_{0}$, we have

$$Re\,\sigma/\sigma^{0} = \ln\frac{4T}{\gamma^{2}Ze^{2}\kappa} - \frac{1}{2} + \int_{0}^{\infty}\left[\frac{1}{(k^{2}+k_{2}^{2})(k^{2}+k_{D}^{2})} - \frac{1}{(k^{2}+\kappa^{2})^{2}}\right]k^{3}dk$$

$$= \ln\frac{4T}{\gamma^{2}Ze^{2}k_{12}} - \frac{1}{2}, \quad (11.14)$$

k_{12} being defined by (9.1). This result is the same as (10.5). obtained on the basis of the quasi-static theory.

When $\omega \gg \omega_{0}$, we have

$$Re\,\sigma/\sigma^{0} = \ln\frac{4T}{\gamma^{2}Ze^{2}\kappa} - \frac{1}{2} + \int_{0}^{\infty}\left[\frac{\exp\left(-\frac{\omega^{2}}{k^{2}}\frac{m_{2}}{2T}\right)}{k} - \frac{k^{3}}{(k^{2}+\kappa^{2})^{2}}\right]dk$$

$$= \ln\frac{4T}{\gamma^{2}Ze^{2}}\left(\frac{2T}{\gamma m_{2}\omega^{2}}\right)^{\frac{1}{2}}, \quad (11.15)$$

where use has been made of the formula

$$\int_{x}^{\infty} e^{-x}/x \; dx = -\ln(\gamma x) \quad , \text{ for } x \ll 1.$$

In this case the distance covered by the electron in one period of the electric field plays qualitatively the role of the effective screening length. This distance is much shorter than the Debye radius; and hence the Debye screening does not play any role. Thus it is possible to derive the relation (11.15) merely on the basis of binary collisions without using a connection formula. In fact, the relation (11.15) was first obtained from this point of view by Elwert[29] in 1948. Careful discussions on Elwert's work are given by Scheuer[30].

A graph of $\mathrm{Re}\,\sigma/\sigma^0$ in case Z = 1 is shown in Fig. 3.

Fig. 3

Now that σ is obtained we can investigate the plasma oscillation. The complex frequency of the plasma oscillation is determined by

$$1 - \frac{\omega_0^2}{\omega^2} + \frac{4\pi i \sigma}{\omega} = 0 . \tag{11.16}$$

Since the third term is small, we have, in case Z = 1,

$$\mathrm{Re}\,\omega - \omega_0 = 2\pi \sigma^0_{\omega=\omega_0} \times 0.319 , \tag{11.17}$$

$$\mathrm{Im}\,\omega = -2\pi \sigma^0_{\omega=\omega_0} \left(\ln \frac{2T}{\gamma^2 e^2 k_2} - 0.0103 \right) . \tag{11.18}$$

The first expression shows that the frequency of the plasma oscillation increases when the interactions are introduced. The second gives the damping of the plasma oscillation.

IV ION-ION AND ELECTRON-ELECTRON INTERACTIONS

12. Relaxation of an anisotropic distribution of ion velocity

We have treated examples of the ion-electron interaction. In this section we discuss relaxation phenomenon which is caused by the interaction between ions and in which the ion-electron interaction is effective only through the dielectric permeability whose electronic part corresponds to a perfect screening.

A plasma in a rapidly varying magnetic field has in general an anisotropic distribution of ion velocities, the longitudianl temperature $T_{\parallel}$ being different from the transverse temperature $T_{\perp}$. The relaxation of this velocity distribution plays a role in heating a plasma by means of the so-called magnetic pumping. We consider the mean transfer of ion energy between longitudinal and transverse directions when the velocity distribution is as follows:

$$f_1(v) = n_1 \left(\frac{m_1}{2\pi T_{\parallel}}\right)^{\frac{1}{2}} \left(\frac{m_1}{2\pi T_{\perp}}\right) \exp\left(-\frac{m_1 v_z^2}{2 T_{\parallel}} - \frac{m_1 (v_x^2 + v_y^2)}{2 T_{\perp}}\right) . \qquad (12.1)$$

The effects of the magnetic field on the rate of transfer are usually negligible.

The time rate of change in the mean longitudinal energy of an ion is given by

$$\frac{d w_z}{dt} \equiv \frac{d}{dt} \int \frac{m_1}{2} v_z^2 \frac{f_1(v)}{n_1} dv$$

$$= \frac{m_1}{2 n_1} \int \left(2 v_z \frac{\langle \Delta v_z \rangle}{\Delta t} + \frac{\langle (\Delta v_z)^2 \rangle}{\Delta t}\right) f_1(v) dv . \qquad (12.2)$$

The results of calculations in the collision and wave theories are the following:

$$\left[\frac{dw_2}{dt}\right]_r = \frac{2\pi Z^4 e^4}{n_1 m_1}\left(\ln \frac{m_1 g^2}{\gamma Z^2 e^2 r} - 1\right)$$

$$\times \frac{1}{g}\left(1 - \frac{3g_z^2}{g^2}\right) f_1(v+g) f_1(v)\, dv\, dg \,, \qquad (12.3)$$

$$\left[\frac{dw_2}{dt}\right]_{wave} = \left(\frac{1}{T_\parallel} - \frac{1}{T_\perp}\right) \frac{2Z^4 e^4}{n_1} \iiiint \frac{k_z^2 v_z (v_z - v_z')}{|k\cdot\varepsilon(k,\omega)\cdot k|^2}$$

$$\times\, \delta(\omega - k\cdot v)\delta(\omega - k\cdot v') f_1(v) f_1(v')\, dv\, dv'\, dk\, d\omega \,. \qquad (12.4)$$

In order to proceed further, let us assume that

$$|T_\perp - T_\parallel|/T \ll 1 \,, \qquad \text{where } T = \tfrac{2}{3}T_\perp + \tfrac{1}{3}T_\parallel \,,$$

and consider only terms of the first order of this small quantity. Then (12.3) becomes

$$\left[\frac{dw_2}{dt}\right]_r = \frac{T_\perp - T_\parallel}{T} \frac{8n_1}{15} Z^4 e^4 \left(\frac{\pi}{m_1 T}\right)^{\frac{1}{2}} \ln \frac{4T}{\gamma^2 Z^2 e^2 r} \,. \qquad (12.5)$$

Here we have used the formula (4.8). In (12.4) the velocity distribution and dielectric permeability can be replaced by their isotropic approximations. After performing intergrations with respect to the ion velocities we obtain

$$\left[\frac{dw_2}{dt}\right]_{wave} = \frac{T_\perp - T_\parallel}{T} \frac{8n_1}{15} \frac{Z^4 e^4}{T} \int_{k>0} \int_{-\infty}^{\infty} \frac{k^2}{|k\cdot\varepsilon(k,\omega)\cdot k|^2}$$

$$\times \exp\left(-\frac{m_1 \omega^2}{T k^2}\right) d\omega\, dk \,. \qquad (12.6)$$

Since ω/k is very small compared with the electron speed, the dielectric permeability is given by(7.5),

$$k \cdot \varepsilon \cdot k = k^2 + k_3^2 + k_1^2 \left(1 - 2x e^{-x^2}\int_0^x e^{t^2}dt + i\sqrt{\pi}x e^{-x^2}\right), \qquad (12.7)$$

where

$$x = \frac{\omega}{k}\left(\frac{m_1}{2T}\right)^{\frac{1}{2}}$$

From (12.4) follows

$$\left[\frac{dw_2}{dt}\right]_{wave} - \left[\frac{dw_2}{dt}\right]_{wave,\kappa}$$

$$= \text{const.}\int_0^\infty \left[\left(\frac{m_1}{\pi T}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{k^2 \exp(-m_1\omega^2/Tk^2)}{|k \cdot \varepsilon(k,\omega)\ k|^2}d\omega - \frac{k^3}{(k^2+\kappa^2)^2}\right] dk.$$

The effective screening constant κ^* defined by (1.5) can be obtained by letting both sides equal to zero. The result is of the form

$$\kappa^* = [\,k_2^2 + \eta(Z)\,k_1^2\,]^{\frac{1}{2}} \qquad (12.8)$$

in which the effective fraction η of the ionic screening in the present process is calculated to be

$\eta(1) = 0.66147$

$\eta(2) = 0.66098$

$\eta(3) = 0.66151$

$\eta(4) = 0.66228$

$\eta(5) = 0.66306$

$\eta(\infty) = 0.67137.$

On replacing $\varkappa$ in (12.5) by $\varkappa^\circ$ we obtain the final expression for the rate:

$$\frac{dw_2}{dt} = \frac{T_\perp - T_\parallel}{T} \frac{8n_1}{15} Z^\kappa e^\kappa \left(\frac{\pi}{m_1 T}\right)^{\frac{1}{2}} \ln \frac{4T}{\gamma^2 Z^2 e^2 \varkappa^\circ} \qquad (12.9)$$

It is to be noted that the effective fraction of the ionic screening in another process is in general different.

13. Relaxation of an anisotropic distribution of electron velocities

This section is devoted to the study of relaxation of an anisotropic distribution

$$f_2(v) = n_2 \left(\frac{m_2}{2\pi T_\parallel}\right)^{\frac{1}{2}} \left(\frac{m_2}{2\pi T_\perp}\right) \exp\left(-\frac{m_2 v_z^2}{2T_\parallel} - \frac{m_2(v_x^2+v_y^2)}{2T_\perp}\right)$$

of electron velocities. We assume that both $T_\parallel$ and $T_\perp$ are not far from the ion temperature T:

$$|T_\parallel - T| \ll T, \qquad |T_\perp - T| \ll T.$$

The time rate of change in the mean longitudinal energy of an electron is given by

$$\frac{dw_3}{dt} \equiv \frac{d}{dt}\int \frac{m_2}{2} v_3^2 \frac{f_2(v)}{n_2} dv$$
$$= \left(\frac{dw_3}{dt}\right)_1 + \left(\frac{dw_3}{dt}\right)_2 ,$$

where

$$\left(\frac{dw_3}{dt}\right)_s = \frac{m_2}{2n_2}\int \left(2v_3 \frac{\langle \Delta v_3 \rangle_s}{\Delta t} + \frac{\langle (\Delta v_3)^2 \rangle_s}{\Delta t}\right) f_2(v)\, dv ,$$

s = 1 corresponding to the interaction with ions, s = 2 with electrons.

The part due to the ion-electron interaction is calculated to be

$$\left(\frac{dw_2}{dt}\right)_1 = \frac{T_\perp - T_\parallel}{T} \frac{8n_1}{15} Z^2 e^4 \left(\frac{2\pi}{m_2 T}\right)^{\frac{1}{2}} \ln \frac{4T}{\gamma^2 Z e^2 k_{12}} .$$

Replacing, in section 12, m_1, n_1, Ze, $k_2^2 + k\cdot\varepsilon_1\cdot k$ by m_2, n_2, $-e$, $k\cdot\varepsilon_2\cdot k$, respectively, we obtain

$$\left(\frac{dw_2}{dt}\right)_2 = \frac{T_\perp - T_\parallel}{T} \frac{8n_2}{15} e^4 \left(\frac{\pi}{m_2 T}\right)^{\frac{1}{2}} \ln \frac{4T}{\gamma^2 e^2 (0.67137\, k_2^2)^{\frac{1}{2}}} ,$$

$(0.67137k_2^2)^{\frac{1}{2}}$ or $(0.67137\times 4\pi n_2 e^2/T)^{\frac{1}{2}}$ being the effective screening constant for the electron-electron interaction in this particular process.

V DIRECT CURRENT PROBLEMS

14. Kinetic equation

Although the rates of relaxation in simple cases can be obtained by calculating one or two moments, it is necessary in general to solve the kinetic equation (2.1) with

$$\frac{\Delta f(v)}{\Delta t} = \left[\frac{\Delta f(v)}{\Delta t}\right]_{\kappa} + \left[\frac{\Delta f(v)}{\Delta t}\right]_{wave} - \left[\frac{\Delta f(v)}{\Delta t}\right]_{wave,\kappa} . \qquad (14.1)$$

In the collision theory the $\underline{n}$th moment is given by

$$\left[\frac{\langle(\Delta v)^n\rangle}{\Delta t}\right]_{\kappa} = \sum_s \iint \Delta^n f_s(v+g)\, g\, dg\, db, \qquad (14.2)$$

where Δ^n is a tensor of $\underline{n}$th rank composed of the change (4.1) in the velocity v per one collision.

Substituting (14.2) into (2.2) and performing the summation, we obtain

$$\left[\frac{\Delta f(\mathbf{v})}{\Delta t}\right]_{\kappa} = \sum_s \iint [f(\mathbf{v}-\boldsymbol{\Delta}) f_s(\mathbf{v}+\mathbf{g}-\boldsymbol{\Delta}) - f(\mathbf{v}) f_s(\mathbf{v}+\mathbf{g})]\, g\, d\mathbf{g}\, d\mathbf{b}\,.$$

This is Boltzmann's equation, which is usually expressed, in terms of inverse collisions for the terms $f(\mathbf{v}-\boldsymbol{\Delta})f_s(\mathbf{v}+\mathbf{g}-\boldsymbol{\Delta})$, in the form

$$\left[\frac{\Delta f(\mathbf{v})}{\Delta t}\right]_{\kappa} = \sum_s \iint [f(\mathbf{v}') f_s(\mathbf{v}_s') - f(\mathbf{v}) f_s(\mathbf{v}_s)]\, |\mathbf{v}_s - \mathbf{v}|\, d\mathbf{v}_s\, d\mathbf{b}\,, \qquad (14.3)$$

where $\mathbf{v}'$ and $\mathbf{v}_s'$ indicate the velocities after the collision.

In the wave theory, formulas (2.2), (3.24), (3.25) and the relation (1.4) lead to

$$\left[\frac{\Delta f(\mathbf{v})}{\Delta t}\right]_{wave} - \left[\frac{\Delta f(\mathbf{v})}{\Delta t}\right]_{wave,\kappa}$$

$$= \frac{2g^2}{M} \sum_s e_s^2 \iiint \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{|\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}|^2} - \frac{1}{(k^2+\kappa^2)^2}\right) \delta(\omega-\mathbf{k}\cdot\mathbf{v})\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_s)$$

$$\times\ \mathbf{k}\cdot\left(\frac{1}{M}\frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}_s}\right) f(\mathbf{v}) f_s(\mathbf{v}_s)\, d\mathbf{v}_s\, d\mathbf{k}\, d\omega\,, \qquad (14.4)$$

the contribution from the third and higher moments vanishing. Such a unified kinetic equation was independently found by Baldwin[26] and the present authors[14].

15. Electric conduction in a Lorentz plasma

As an example let us consider steady electric conduction in a Lorentz plasma. (The Lorentz plasma is a plasma in which electron-electron interaction is neglected, or, what is the same thing, the ion-electron charge ratio Z is very large.)

Since the effective screening constant for the ion-electron scattering in this case is known (see the last paragraph of section 9), it is unnecessary to perform calculations in the wave theory.

In the collision theory the velocity distribution function $f_2(\mathbf{v})$ of electrons is determined by the Boltzmann equation with the interaction term (14.3):

$$-\frac{e}{m_2}\mathbf{E}\cdot\frac{\partial f_2(\mathbf{v})}{\partial \mathbf{v}}$$

$$= \iint [\, f_2(\mathbf{v}')\, f_1(\mathbf{v}_1') - f_2(\mathbf{v}) f_1(\mathbf{v}_1)]\, |\mathbf{v}-\mathbf{v}_1|\, d\mathbf{v}_1\, d\mathbf{b}\,. \qquad (15.1)$$

We expand as follows

$$f_2(\mathbf{v}) = f_2^{\circ}(\mathbf{v})\,[\,1 + \mathbf{v}\cdot\boldsymbol{\varphi}(v)\,], \qquad (15.2)$$

$f_2^{\,o}$ being Maxwellian. On the left side of (15.1), f_2 can be approximated by $f_2^{\,o}$; and on the right the relative speed can be approximated by the electron speed v, which remains unchanged during a collision. Hence

$$eT^{-1}\,\mathbf{E}\cdot\mathbf{v}\, f_2^{\circ} = n_1 \iint (\mathbf{v}'-\mathbf{v})\cdot\boldsymbol{\varphi}(v)\, f_2^{\circ}\, v\, d\mathbf{b}$$

$$= -\,n_1\, \mathbf{v}\cdot\boldsymbol{\varphi}(v)\, f_2^{\circ}\, v \int_0^{\infty} (1-\cos\theta)\, 2\pi b\, db\,,$$

where θ is the angle through which the electron velocity is deflected. Substituting (4.3) and (4.6) and taking account of the theorem in section 9, we obtain

$$\frac{e}{T}\mathbf{E}\cdot\mathbf{v}\, f_2^{\circ} = -\,4\pi\, n_1 \left(\frac{Ze^2}{m_2}\right)^2 \frac{\mathbf{v}\cdot\boldsymbol{\varphi}(v)\, f_2^{\circ}}{v^3}\left(\ln\frac{2m_2 v^2}{\gamma Z e^2 k_{12}} - \frac{1}{2}\right), \qquad (15.3)$$

where k_{12} is defined by (9.1).

The electric conductivity σ is given by

$$\sigma E = -e\int v\, f_2(v)\, dv = -e\int v\, v\cdot\varphi(v)\, f_2^{\,0}\, dv .$$

We have therefore

$$\sigma = \frac{1}{4\pi n_1 T}\left(\frac{m_2}{Ze}\right)^2 \int \frac{\frac{1}{3} v^5 f_2^{\,0}(v)\; dv}{\ln(2 m_2 v^2/\gamma Z e^2 k_{12}) - 1/2}$$

$$= \frac{2 m_2 n_2}{3 n_1 Z^2 e^2}\left(\frac{2T}{m_2}\right)^{\frac{3}{2}} \frac{1}{\pi^{\frac{1}{2}}} \int \frac{e^{-w^2}\; w^7\, dw}{L + \ln w^2}$$

$$= \frac{4 n_2 e^2}{3\pi n_1 m_2 (Ze^2/T)^2 (2\pi T/m_2)^{\frac{1}{2}}}\left[\frac{\Gamma(4)}{L} - \frac{\Gamma'(4)}{L^2} + \frac{\Gamma''(4)}{L^3} - \cdots\right], \quad (15.4)$$

where

$$L \equiv \ln(4T/\gamma Z e^2 k_{12}) - \frac{1}{2},$$

$$\Gamma(4) = 6,$$

$$\Gamma'(4) = 11 - 6\ln\gamma = 7.537,$$

$$\Gamma''(4) = \pi^2 - \frac{49}{6} + \frac{(11 - 6\ln\gamma)^2}{6} = 11.17 .$$

16. Conduction of electricity and heat in general

In a plasma with electric field and temperature gradient the electricity and heat are carried only by the electrons in the absence of any magnetic field, the velocity distribution of ions remaining Maxwellian.

The velocity distribution of electrons can be obtained by solving the kinetic equation

$$\mathbf{v}\cdot\frac{\partial f_2}{\partial \mathbf{r}} - \frac{e}{m_2}\mathbf{E}\cdot\frac{\partial f_2}{\partial \mathbf{v}} = \frac{\Delta f_2}{\Delta t} . \qquad (16.1)$$

On the left the distribution function f_2 can be approximated by its unperturbed Maxwellian distribution $f_2^{\,o}$:

$$\left[\frac{\partial \ln(n_2T)}{\partial \mathbf{r}} + \frac{e\mathbf{E}}{T} - \left(\frac{5}{2} - \frac{m_2}{2T}v^2\right)\frac{\partial \ln T}{\partial \mathbf{r}}\right]\cdot \mathbf{v}\, f_2^{\,o} = \frac{\Delta f_2}{\Delta t} . \qquad (16.2)$$

On the right we let again

$$f_2(\mathbf{v}) = f_2^{\,o}(\mathbf{v})\,[1 + \mathbf{v}\cdot\boldsymbol{\varphi}(v)]$$

and expand $\boldsymbol{\varphi}(v)$ by use of the Laguerre-Sonine polynomials into

$$\boldsymbol{\varphi}(v) = \sum_{r=0}^{\infty} \mathbf{P}_r\, L_r^{3/2}\left(\frac{m_2}{2T}v^2\right) .$$

Multiplying both sides of (16.2) by

$$\frac{2}{n_2}\left(\frac{m_2}{2T}\right)\mathbf{v}\, L_s^{3/2}\left(\frac{m_2}{2T}v^2\right)$$

and integrating over $\mathbf{v}$, we have the simultaneous equations for $\mathbf{P}_r$ of the form

$$\left(\frac{\partial \ln(n_2T)}{\partial \mathbf{r}} + \frac{e\mathbf{E}}{T}\right)\delta_{s0} - \frac{5}{2}\frac{\partial \ln T}{\partial \mathbf{r}}\delta_{s1} = -\sum_{r=0}^{\infty}\left(H_{sr}^{(1)} + H_{sr}^{(2)}\right)\mathbf{P}_r \qquad (16.3)$$

The matrix elements in this equation are calculated to be

$$H_{sr}^{(1)} = \frac{4}{3}\sqrt{2\pi}\,\frac{n_1 e^4 Z^2}{\sqrt{m_2 T^3}}\left[\left(\ln\frac{4T}{\gamma^2 Z e^2 k_1} - \frac{1}{2}\right)A_{sr} - B_{sr}\right],$$

$$H_{sr}^{(2)} = \frac{8\sqrt{\pi}}{3}\,\frac{n_2 e^4}{\sqrt{m_2 T^3}}\left[\left(\ln\frac{4T}{\gamma^2 e^2 k_2} + \frac{1}{2}\right)C_{sr} - D_{sr}\right],$$

where

$$A_{sr} = \begin{pmatrix} 1 & \frac{3}{2} & \frac{15}{8} & \cdots \\ \frac{3}{2} & \frac{13}{4} & \frac{69}{16} & \cdots \\ \frac{15}{8} & \frac{69}{16} & \frac{433}{64} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad B_{sr} = \begin{pmatrix} 0 & 1 & 2 & \cdots \\ 1 & 2 & \frac{35}{8} & \cdots \\ 2 & \frac{35}{8} & \frac{121}{16} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$C_{sr} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & \frac{3}{4} & \cdots \\ 0 & \frac{3}{4} & \frac{45}{16} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$D_{sr} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0.080793 & 0.920620 & \cdots \\ 0 & 0.920620 & 0.423119 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The electric current $\mathbf{j}$ is related to the coefficient $\mathbf{P}_0$ as

$$\mathbf{j} = -e\int \mathbf{v} f_2(\mathbf{v})\,d\mathbf{v} = -\frac{n_2 e T}{m_2}\mathbf{P}_0 \tag{16.4}$$

The heat flow $\mathbf{q}$ is given by

$$\mathbf{q} = \int \frac{1}{2} m_2 v^2\, \mathbf{v} f_2(\mathbf{v})\,d\mathbf{v}$$

$$= \frac{5}{2}\,\frac{n_2 T^2}{m_2}\left(\mathbf{P}_0 - \mathbf{P}_1\right).$$

Hence the reduced heat flow q* due to conduction is

$$q^* = q + \frac{5}{2}\frac{T}{e} j = -\frac{5}{2}\frac{n_2 T^2}{m_2} P_1 \quad . \tag{16.5}$$

Thus we can calculate the electric conductivity σ, thermal conductivity λ, and thermoelectric coefficient α which are defined by

$$\begin{aligned} q^* &= -\lambda T \frac{\partial \ln T}{\partial r} + \alpha j , \\ E &= \alpha \frac{\partial \ln T}{\partial r} + \sigma^{-1} j . \end{aligned} \tag{16.6}$$

The electric conductivity σ, for example, is the following:

$$\sigma = \lim_{s \to \infty} [\sigma]_s \tag{16.7}$$

in which the (s+1)-th approximation to σ is

$$[\sigma]_{s+1} = \frac{e^2 n_2}{m_2} \begin{vmatrix} H_{11} & \cdots & H_{1s} \\ \cdots & \cdots & \cdots \\ H_{s1} & \cdots & H_{ss} \end{vmatrix} \Bigg/ \begin{vmatrix} H_{00} & \cdots & H_{0s} \\ \cdots & \cdots & \cdots \\ H_{s0} & \cdots & H_{ss} \end{vmatrix} . \tag{16.8}$$

For $Z = 1$, the third approximation is calculated to be

$$\begin{aligned} [\sigma]_3 = {} & \frac{3e^2}{4m_2} \left(\frac{T}{e^2}\right)^2 \left(\frac{m_2}{2\pi T}\right)^{\frac{1}{2}} \\ & \times [21.24(\ln\Lambda)^2 + 7.30(\ln\Lambda) - 25.89] \\ & [10.89(\ln\Lambda)^3 + 11.05(\ln\Lambda)^2 - 28.49 \ln\Lambda - 9.94]^{-1} , \end{aligned} \tag{16.9}$$

where

$$\ln\Lambda \equiv \ln \frac{T}{e^2 k_2} .$$

VI FURTHER DEVELOPMENTS

17. Quantum effects

It is a characteristic feature of hot plasmas that the quantum effects become appreciable at very high temperatures, or, more precisely, at very high energies of the charged particles under consideration. The form of the connection formula (1.3) need not be altered in the presence of the quantum effects. We need only to evaluate under the condition (1.6) the terms on the right side in quantum mechanics.

The quantum effects do not play any role in the difference $X_{wave} - X_{wave,\kappa}$. Because the quantum effects in the wave theory are appreciable only when $\hbar k$ ($2\pi\hbar$=Planck's constant) is comparable with or larger than the particle momentum, i.e., when $k^{-1} \lesssim$ de Broglie wave length; waves of such small wave lengths do not contribute to the difference $X_{wave} - X_{wave,\kappa}$.

The term X_{κ} is not necessarily an explicit integral with respect to the impact parameter, which is a classical concept. It can be expressed in the form of an integral over the differential cross section. Quantum-mechanical expression for the differential cross section in a pure Coulomb field is the same as the classical Rutherford formula. A quantum expression for X_{κ}, therefore, is the same as the corresponding classical form excepting the argument of the Coulomb logarithm, where κ appears. (By virtue of this fact the connection formula (1.3) does not depend on the choice of κ in quantum mechanics as well as in classical mechanics.)

The above-mentioned facts can be expressed in the formula

$$X = X_{\kappa}^{\text{quantum}} + (X_{\text{wave}} - X_{\text{wave},\kappa})^{\text{classical}} \qquad (17.1)$$

When X is a third or higher moment, X_{κ} is independent of κ and hence its quantum and classical expressions are the same. The first and second moments are given by (4.2) and (4.4) in which the cross section $Q_{\kappa}^{(\ell)}$ should now be calculated on the basis of

$$Q_{\kappa}^{(\ell)} = \int (1-\cos^{\ell}\theta)\,d\sigma\,, \quad (\ell = 1,2), \qquad (17.2)$$

in quantum mechanics. Here $d\sigma$ denotes the differential collision cross section.

Let q be the charge of the particle under consideration and $(qe_s/r)e^{-\kappa r}$ be the screened Coulomb potential due to a field particle of type s. Let the reduced mass and the relative velocity be μ and g, respectively.

The classical expressions (4.6) hold for $\hbar g/|qe_s| \ll 1$, namely, when the de Broglie wavelength is sufficiently short in comparison with the collision radius.

First quantum corrections to the classical expressions are calculated by Kihara[14], the result being

$$Q_{\kappa}^{(1)} = 4\pi\left(\frac{qe_s}{\mu g^2}\right)^2\left[\ln\frac{2\mu g^2}{\gamma|qe_s|\kappa} - \frac{1}{2} - \frac{1}{12}\left(\frac{\hbar g}{qe_s}\right)^2\right], \qquad (17.3a)$$

$$Q_{\kappa}^{(2)} = 8\pi\left(\frac{qe_s}{\mu g^2}\right)^2\left[\ln\frac{2\mu g^2}{\gamma|qe_s|\kappa} - 1 - \frac{1}{12}\left(\frac{\hbar g}{qe_s}\right)^2\right], \qquad (17.3b)$$

up to the terms of order $(\hbar g/qe_s)^2$.

In general, (17.2) has the form

$$Q_\kappa^{(l)} = \int (1-\cos^l\theta)\,|f(\theta)|^2\, 2\pi \sin\theta\, d\theta\,, \quad (l=1,2), \tag{17.4}$$

in which f(θ) is called the scattering amplitude. At the high energy limit, where the classical collision radius is much shorter than the de Broglie wave length or $|qe_s|/\hbar g \ll 1$, Born approximation

$$f(\theta) = \frac{2\mu g e_s}{2\mu^2 g^2(1-\cos\theta) + \hbar^2\kappa^2} \tag{17.5}$$

holds true and leads to

$$Q_\kappa^{(1)} = 4\pi \left(\frac{g e_s}{\mu g^2}\right)^2 \left(\ln\frac{2\mu g}{\hbar\kappa} - \frac{1}{2}\right), \tag{17.6a}$$

$$Q_\kappa^{(2)} = 8\pi \left(\frac{g e_s}{\mu g^2}\right)^2 \left(\ln\frac{2\mu g}{\hbar\kappa} - 1\right). \tag{17.6b}$$

From (17.3) or from (17.6) follows the identity

$$2Q_\kappa^{(1)} - Q_\kappa^{(2)} = \int (1-\cos\theta)^2 d\sigma = 4\pi\left(\frac{g e_s}{\mu g^2}\right)^2, \tag{17.7}$$

in which the quantum effects vanish, as it should.

The case where two identical particles collide requires special consideration. Thus, for the collision of two electrons, (17.4) must be replaced by

$$Q_\kappa^{(l)} = \int_0^{\pi/2} (1-\cos^l\theta)\left[\tfrac{1}{4}|f(\theta)+f(\pi-\theta)|^2 + \tfrac{3}{4}|f(\theta)-f(\pi-\theta)|^2\right] 2\pi\sin\theta\, d\theta \tag{17.8}$$

For $Q_\kappa^{(2)}$ this integral from 0 to $\pi/2$ is equal to one half of the integral from 0 to π. For $Q_\kappa^{(1)}$, however, the restriction of the integration to the range $\theta \le \frac{\pi}{2}$ comes from the following convention. One of the two electrons which has after the collision a positive velocity component in the direction of the initial velocity of a test particle is named the same test particle.) The scattering amplitude

$$f(\theta) = \frac{m e^2}{\frac{1}{2} m^2 g^2 (1-\cos\theta) + \hbar^2 \kappa^2}$$

in Born approximation leads to

$$\frac{1}{4}|f(\theta)+f(\pi-\theta)|^2 + \frac{3}{4}|f(\theta)-f(\pi-\theta)|^2$$

$$= \left(\frac{2e^2}{mg^2}\right)^2 \frac{1+3\cos^2\theta}{[(1-\cos^2\theta)+(2\hbar\kappa/mg)^2]^2} ,$$

where -e and m are the charge and mass of an electron. Table I gives the results.

Table I. Collision cross sections between electrons in Born approximation

	from (17.4)	from (17.8)
$(4\pi)^{-1}(2e^2/mg^2)^{-2}Q_\kappa^{(1)}$	$\ln\frac{mg}{\hbar\kappa} - \frac{1}{2}$	$\ln\frac{mg}{\hbar\kappa} - \frac{3}{2}\ln 2$
$(8\pi)^{-1}(2e^2/mg^2)^{-2}Q_\kappa^{(2)}$	$\ln\frac{mg}{\hbar\kappa} - 1$	$\ln\frac{mg}{\hbar\kappa} - \frac{5}{4}$

The case of Born approximation will be considered further. The effect of orbital curvature in X_κ is completely covered by the quantum mechanical diffraction effect. Each of X_{wave} and $X_{wave,\kappa}$, in which the orbital curvatures are neglected, should converge by itself. Then X_{κ} and $X_{wave,\kappa}$ should be equal, and the quantity X is given by X_{wave} only. In conclusion, the rate of relaxation in Born approximation can be evaluated either from (17.1) or by use of the relation

$$X^{Born} = X_{wave}^{Born} \quad . \tag{17.9}$$

Thus the process of unification is not necessary in the Born approximation. In fact, several examples have been treated along this line[19-24].

18. Stopping power in a magnetic field

This last section is devoted to an exceptionally simple example of relaxation in a strong magnetic field. Here a strong magnetic field means a field in which the electron gyrofrequency ω_H is comparable with the plasma frequency ω_e.

We namely calculate the stopping power of a plasma against a charge which is moving fast in the direction parallel to a magnetic field. We neglect the electron-ion mass ratio compared with unity; then the ions do not contribute to the stopping power (cf. (6.13')). Let the charge, mass and number density of the plasma electron be denoted by -e, m and n, respectively; let q, M and v denote the charge, mass and velocity of the fast charge, respectively; and let μ be the reduced mass, $\mu = Mm/(M + m)$. The speed of the fast charge is assumed to be considerably high in comparison with the thermal velocity of the plasma electrons.

We denote by $X(\omega_H)$ the stopping power or the energy loss per unit time in a magnetic field. The quantity $X(\omega_H)$ in the wave theory is of the form (6.8):

$$X_{wave}(\omega_H) = \frac{e^2}{2\pi^2} \iint_{-\infty}^{\infty} \frac{i\omega\,\delta(\mathbf{k}\cdot\mathbf{v}-\omega)}{\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}}\, d\omega\, d\mathbf{k} . \tag{18.1}$$

Under the assumption that the incident particle has a high velocity, $\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}/k^2$ is given by

$$\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k}/k^2 = \left(1-\frac{\omega_0^2}{\omega^2}\right)\cos^2\alpha + \left(1-\frac{\omega_0^2}{\omega^2-\omega_H^2}\right)\sin^2\alpha , \tag{18.2}$$

where α is the angle between $\mathbf{k}$ and the magnetic field. In the present case α is equal to the angle between $\mathbf{k}$ and $\mathbf{v}$: $\cos\alpha = (\mathbf{k}\cdot\mathbf{v})/kv$. On substituting (18.2) into (18.1) we obtain

$$X_{wave}(\omega_H) = \frac{ie^2}{\pi v}\int_{k>0}\int_{-kv}^{kv} \frac{\omega^2-\omega_H^2}{\omega^2-\omega_0^2-\omega_H^2+\omega_0^2\omega_H^2/k^2v^2}\,\omega\, d\omega\, \frac{dk}{k} .$$

Here only the poles of the integrand contribute to the integral. The path near the poles should be in such a way that each contribution to X is always positive. (This comes from the fact that the original dielectric permeability has no zero-point in the upper half-plane of complex ω). The condition that the denominator becomes zero at a point in the region $0 < \omega^2 < k^2v^2$ of ω^2 is

$$\frac{\omega_0^2\omega_H^2}{\omega_0^2+\omega_H^2} < k^2v^2 < \mathrm{Min}(\omega_0^2,\omega_H^2), \quad \text{or} \quad \mathrm{Max}(\omega_0^2,\omega_H^2) < k^2v^2 ,$$

which we name first and second domains of k, respectively. Thus we obtain

$$X_{wave}(\omega_H) = \frac{g^2\omega_0^2}{v}\left[\int_{first}\left(\frac{\omega_H^2}{k^2v^2}-1\right)\frac{dk}{k} + \int_{second}\left(1-\frac{\omega_H^2}{k^2v^2}\right)\frac{dk}{k}\right]$$

$$= \frac{g^2\omega_0^2}{v}\left[\int_{k>\omega_0/v}\frac{dk}{k} - \frac{1}{2}\ln\frac{\omega_0^2+\omega_H^2}{\omega_0^2}\right]. \tag{18.3}$$

The collision theory is not influenced by the magnetic field (by virtue of (1.7)); the final result is therefore

$$X(\omega_H) = X(0) - \frac{g^2\omega_0^2}{2v}\ln\frac{\omega_0^2+\omega_H^2}{\omega_0^2} \tag{18.4}$$

or, in the case of classical mechanics ($\hbar v \ll |q|e$)

$$X(\omega_H) = \frac{g^2\omega_0^2}{2v}\ln\left[\left(\frac{2\mu v^3}{\gamma|g e|}\right)^2\frac{1}{\omega_0^2+\omega_H^2}\right]. \tag{18.5}$$

The relation (18.4) was first derived by Akhiezer[20] in Born approximation.

APPENDIX

Definition of the dielectric permeability

For a material medium the Maxwell equations take the form

$$\operatorname{div} \mathbf{E} = 4\pi(\rho + \rho_0), \qquad \operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$
$$\operatorname{rot} \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}(\mathbf{j} + \mathbf{j}_0), \qquad \operatorname{div} \mathbf{B} = 0. \tag{1}$$

Here ρ_0 and $\mathbf{j}_0$ are the charge density and current density of external sources of the field, while ρ and $\mathbf{j}$ are the corresponding densities induced in the medium. For the induced charge and current densities holds the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0. \tag{2}$$

The physical meaning of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ is determined by the expression for the force $\mathbf{F}$ acting on a point test charge q moving in the medium with velocity $\mathbf{v}$:

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}). \tag{3}$$

By virtue of the relation (2) we can eliminate ρ and $\mathbf{j}$ making use of the electric displacement which is defined by

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{j}, \tag{4}$$

the result being

$$\operatorname{div} \mathbf{D} = 4\pi\rho_o , \qquad \operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} ,$$
$$\tag{5}$$
$$\operatorname{rot} \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\, \mathbf{j}_o , \qquad \operatorname{div} \mathbf{B} = 0 .$$

The system of equations (5) must be supplemented by a material equation giving an explicit expression for $\mathbf{D}$. For a stationary and homogeneous medium the material equation has the form

$$\mathbf{D}(\mathbf{r},t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}'\, \varepsilon(\mathbf{r}-\mathbf{r}'; t-t') \cdot \mathbf{E}(\mathbf{r}', t') . \tag{6}$$

It is convenient to expand the electromagnetic field in a Fourier integral, representing it as a set of plane monochromatic waves whose dependence on the coordinates and time is given by the functions $\exp(i\mathbf{k}\mathbf{r} - i\omega t)$. For such waves the relation (6) takes the form

$$\mathbf{D}(\mathbf{k},\omega) = \varepsilon(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega), \tag{7}$$

where the dielectric permeability tensor ε is given by

$$\varepsilon(\mathbf{k},\omega) = \int_0^{\infty} dt \int d\mathbf{r}\; e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}\; \varepsilon(\mathbf{r},t) . \tag{8}$$

The quantity $\varepsilon(\mathbf{k},\omega)$ is, in general, a complex tensor function of the real variables $\mathbf{k}$ and ω. Taking account of the fact that the function $\varepsilon(\mathbf{r},t)$ is real, we obtain from (8) the relationship

$$\varepsilon^*(\mathbf{k},\omega) = \varepsilon(-\mathbf{k}, -\omega) \tag{9}$$

between ε and its complex conjugate ε^*.

When we regard ω as a complex variable, ε is a one-valued regular function in the upper half-plane of ω. Furthermore, the diagonal elements of ε has no zeros in the upper half-plane.[31]

$\varepsilon(\mathbf{k},\omega)$ in the absence of any strong magnetic field

The dielectric permeability of a stationary and homogeneous collision-free plasma composed of several types of charged particles will be calculated. We denote by $f_s^o(\mathbf{v})$ the velocity distribution function for the charged particles of type s assuming that

$$\int \mathbf{v} f_s^o(\mathbf{v})\, d\mathbf{v} = 0, \quad \text{for all s.}$$

The Boltzmann equation for a distribtion function f_s slightly deviated from f_s^o takes the form

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{r}} + \frac{e_s}{m_s}\mathbf{E}\cdot\frac{\partial f_s}{\partial \mathbf{v}} = -\frac{1}{\tau}(f_s - f_s^o). \tag{10}$$

Here $\tau > 0$ is the effective collision time, for which the limit $\tau^{-1} \to 0$ will be taken later. Substituting

$$f_s = f_s^o + f_s' \tag{11}$$

and considering a monochromatic plane wave with the factor $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$ we have the linearized equation

$$-i\omega f_s' + i\mathbf{k}\cdot\mathbf{v} f_s' + \frac{e_s}{m_s}\mathbf{E}\cdot\frac{\partial f_s^o}{\partial \mathbf{v}} = -\frac{1}{\tau} f_s',$$

from which follows

$$f_s^1 = -i \frac{e_s}{m_s} \mathbf{E} \cdot \frac{\partial f_s^0}{\partial \mathbf{v}} \left[\omega - \mathbf{k} \cdot \mathbf{v} + \frac{i}{\tau} \right]^{-1} . \tag{12}$$

From (4) or

$$\varepsilon \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \sum_s e_s \int \mathbf{v} f_s^1 \, d\mathbf{v}$$

we therefore obtain

$$\varepsilon(\mathbf{k},\omega) = 1 + \lim_{\tau\to\infty} \frac{4\pi}{\omega} \sum_s \frac{e_s^2}{m_s} \int \frac{\mathbf{v} \, \partial f_s^0/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v} + i/\tau} \, d\mathbf{v} . \tag{13}$$

Using

$$\lim_{\eta\to 0} \frac{1}{x + i\eta} = P\frac{1}{x} - i\pi\delta(x)$$

we obtain the imaginary part of ε in the form (3.21) where f_s^o are denoted by f_s.

In the case where f_s^o are Maxwellian,

$$f_s^0(\mathbf{v}) = n_s (m_s/2\pi T)^{3/2} \exp(-m_s v^2/2T) ,$$

we have

$$\mathbf{k}\cdot\varepsilon(\mathbf{k},\omega)\cdot\mathbf{k} - k^2 = -\lim_{\tau\to\infty} \frac{1}{\omega} \sum_s \left(\frac{m_s}{2\pi T}\right)^{3/2} k_s^2 \int \frac{(\mathbf{k}\cdot\mathbf{v})^2 \exp(-m_s v^2/2T)}{\omega - \mathbf{k}\cdot\mathbf{v} + i/\tau} \, d\mathbf{v}$$

in which

$$k_s^2 = 4\pi n_s e_s^2 / T .$$

In terms of

$$x_s = \frac{\omega}{k}\left(\frac{m_s}{2T}\right)^{\frac{1}{2}} , \qquad y^2 = \frac{m_s (\mathbf{k}\cdot\mathbf{v})^2}{2T k^2}$$

this expression is transformed as follows

$$k \cdot \varepsilon(k,\omega) \cdot k - k^2 = \sum_s \frac{k_s^2}{\sqrt{\pi}\, x_s} \left[P\int_{-\infty}^{\infty} \frac{y^2 e^{-y^2}}{y - x_s} dy + i\pi x_s^2 e^{-x_s^2} \right]$$

$$= \sum_s k_s^2 \left[1 + \frac{x_s}{\sqrt{\pi}} P\int_{-\infty}^{\infty} \frac{e^{-y^2}}{y - x_s} dy + i\sqrt{\pi}\, x_s e^{-x_s^2} \right].$$

Taking into account that the function

$$G(x) \equiv P\int_{-\infty}^{\infty} \frac{e^{-y^2}}{y - x} dy = P\int_{-\infty}^{\infty} \frac{\exp[-(t+x)^2]}{t} dt$$

satisfies the differential equation

$$G'(x) = -2\sqrt{\pi} - 2x\,G(x),$$

we finally obtain

$$k \cdot \varepsilon(k,\omega) \cdot k - k^2 = \sum_s k_s^2 F(x_s), \tag{14}$$

where

$$F(x) = 1 - 2x e^{-x^2} \int_0^x e^{t^2} dt + i\sqrt{\pi}\, x e^{-x^2}. \tag{15}$$

When we regard x as a complex variable, the function F(x) has the limiting form

$$F(x) = -\frac{1}{2x^2}, \qquad \text{for } |x| \gg 1$$

in the upper half-plane of x ; and hence

$$k_s^2 F(x_s) = -\frac{k^2}{\omega^2} \frac{4\pi n_s e_s^2}{m_s}, \quad \text{for } \frac{|\omega|}{k}\left(\frac{m_s}{2T}\right)^{\frac{1}{2}} \gg 1 \tag{16}$$

in the upper half-plane of ω. Another limiting form is

$$F(x) = 1 \qquad \text{for} \quad |x| \ll 1 \quad ;$$

and hence

$$k_s^2 F(x_s) = k_s^2 \qquad \text{for} \quad \frac{|\omega|}{k}\left(\frac{m_s}{2T}\right)^{\frac{1}{2}} \ll 1 \, . \tag{17}$$

Fluctuation of the charge density

The relation (3.23) can be derived as follows. The charge density ρ at a position $\mathbf{r}$ and time t is given by

$$\rho(\mathbf{r},t) = \sum_s \sum_j e_s \delta(\mathbf{r} - \mathbf{r}_{sj}(t)), \tag{18}$$

where $\mathbf{r}_{sj}(t)$ is the position at time t of jth particle of type s. In terms of the Fourier transform

$$\rho(\mathbf{k},\omega) = \sum_s \frac{e_s}{(2\pi)^4} \sum_j \int \exp[-i\mathbf{k}\cdot\mathbf{r}_{sj}(t) + i\omega t]\, dt \, ,$$

the correlation function is calculated to be

$$\langle \rho(\mathbf{k},\omega)\rho(\mathbf{k}',\omega') \rangle$$

$$= \frac{1}{(2\pi)^8} \sum_s e_s^2 \sum_j \iint \langle \exp[-i\mathbf{k}\cdot\mathbf{r}_{sj}(t') - i\mathbf{k}'\cdot\mathbf{r}_{sj}(t) + i\omega t' + i\omega' t] \rangle \, dt\, dt' ,$$

terms concerning different particles vanishing. By use of the time difference $\tau \equiv t' - t$ and the displacements $\mathbf{v}_{sj}\tau = \mathbf{r}_{sj}(t') - \mathbf{r}_{sj}(t)$ we rewrite the right side in the form

$$\frac{1}{(2\pi)^8} \sum_s e_s^2 \sum_j \iint \langle \exp[-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_{sj}(t) + i(\omega+\omega')t + i(\omega - \mathbf{k}\cdot\mathbf{v}_{sj})\tau] \rangle \, dt\, d\tau \, .$$

Since particles of the same type are equivalent, we obtain, by using the velocity distribution f_s,

$$\langle \rho(\mathbf{k},\omega)\,\rho(\mathbf{k}',\omega')\rangle$$

$$= \frac{1}{(2\pi)^3} \sum_s e_s^{\,2}\, \delta(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega') \int \delta(\omega-\mathbf{k}\cdot\mathbf{v}_s)\, f_s(\mathbf{v}_s)\, d\mathbf{v}_s \tag{19}$$

or the relation (3.23).

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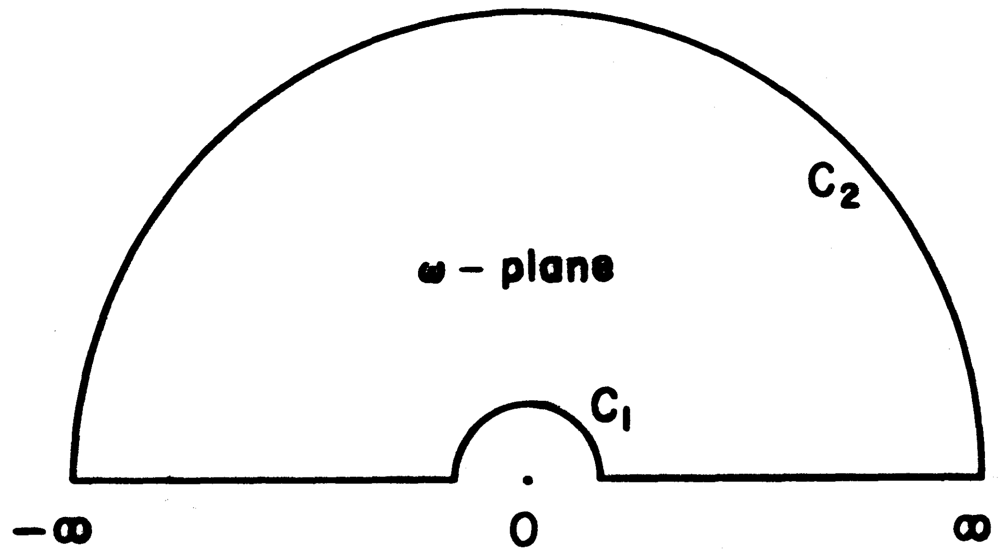

Fig. 1

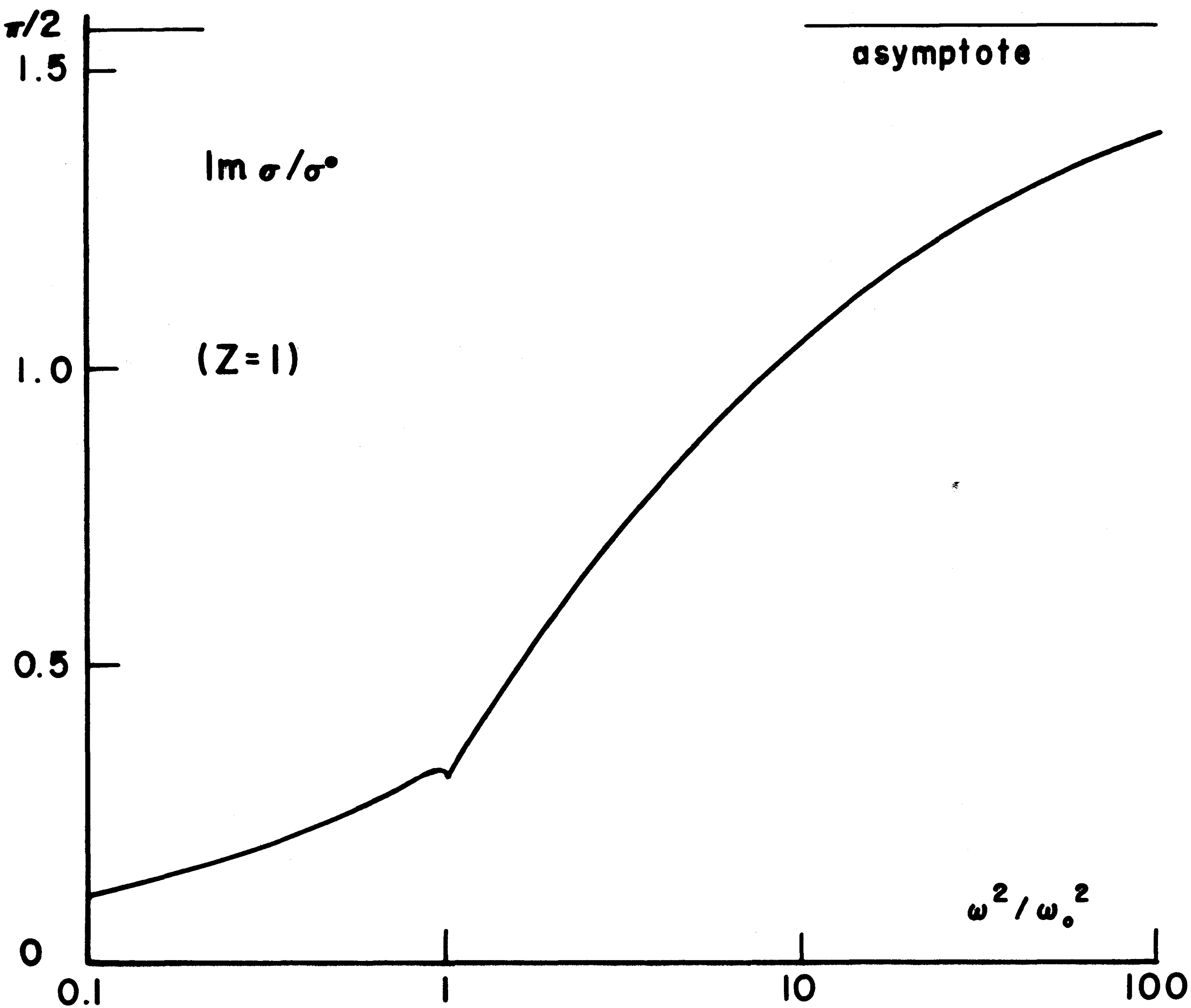

Fig. 2

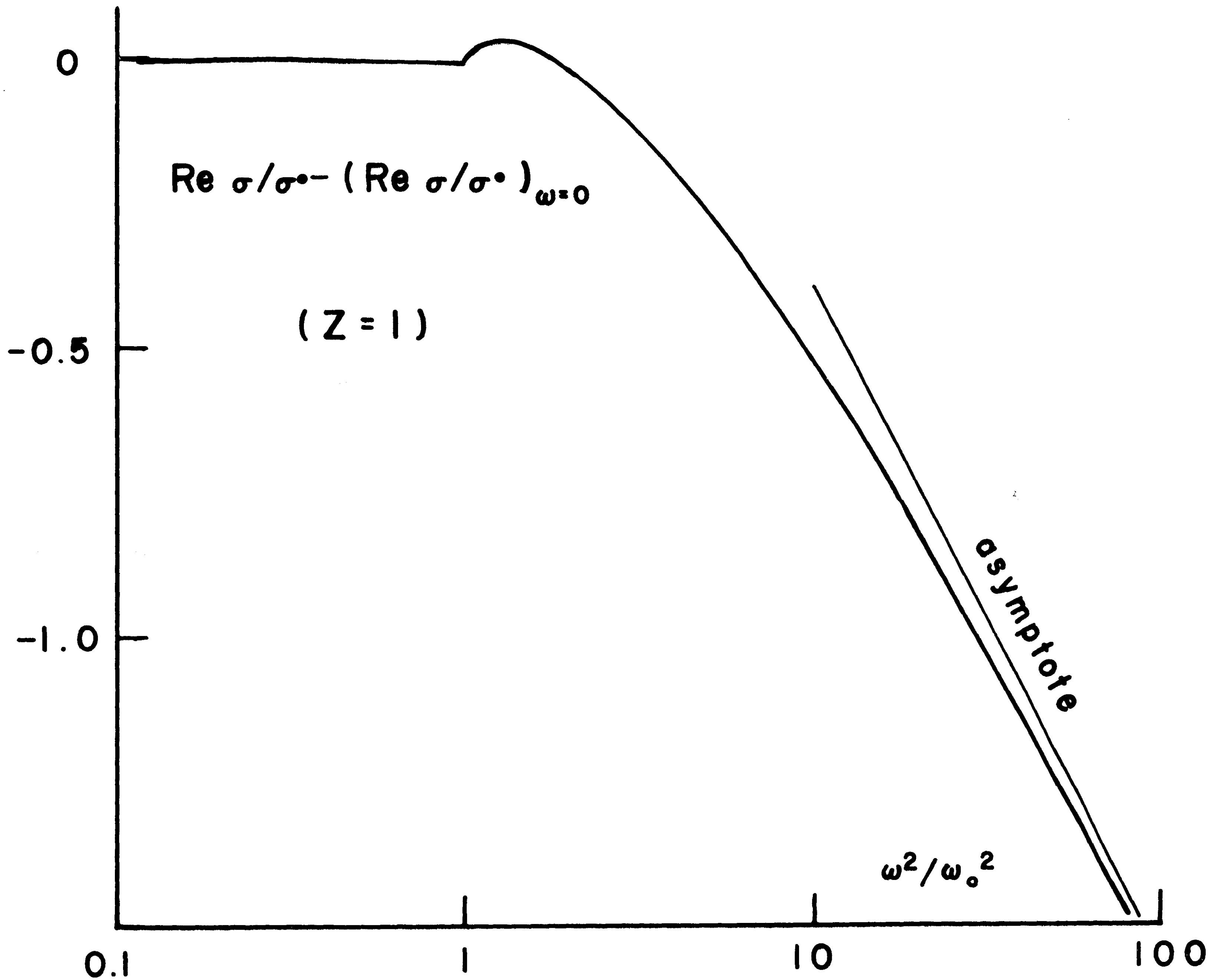

Fig. 3