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The Kinetic Theory of Stable Plasmas#

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In recent years, partly because of the attempts to achieve thermonuclear fusion in the laboratory, there have been extensive and intensive investigations on various aspects of the physics of plasmas. Underlying many of these investigations is the study of the basic equation governing the behavior of a plasma, especially its approach to equilibrium. The corresponding problem in the case of the neutral gases is a very old one. To appreciate the nature of the problem and the difficulties of the theory of plasma, let us briefly review the more familiar case of the neutral gases.

I. Kinetic theory of neutral gases: theory of Boltzmann

In dealing with a gas consisting of N particles, we are only interested in the macroscopic properties of the gas, and not in the detailed dynamical motions of the particles. To give a description of the macroscopic behavior of the gas, we introduce the concept of probability and the method of statistical mechanics. Thus we introduce the concept of an ensemble of systems and the distribution function $D\,(q_1 \cdots q_N\ ,\ p_1 \cdots p_N\ ,\ t)$ of the ensemble. Then on the basis of the equations of motion of classical dynamics, one obtains for the function D the Liouville equation

$$\frac{\partial D}{\partial t} = \{H,\ D\} \tag{1}$$

Where H is the Hamiltonian of the system of (interacting) particles and $\{\ ,\ \}$ the Poisson bracket expression. We make the basis assumption that the average value of any property, say Q, of the gas is given by the average

$$\bar{Q}(t) = \int \cdots \int Q(q_1 \cdots q_N\ ,\ p_1 \cdots p_N\ ,t)\ D\ dq_1 \cdots \cdots dq_N\ dp_1 \cdots dp_N. \tag{2}$$

In principle, if (1) has been solved as an initial value problem, then one can determine the whole history of the evolution of quantity Q in time from (2).

The equation (1) is, however, reversible in the direction of time, and the first question is how the observed irreversible processes (such as diffusion, etc.) can be understood on the basis of a reversible theory. The answer to this question was supplied by Boltzmann and by Gibbs. The theory is well known. We wish to emphasize here that the very basic concept of "course-graining" must be introduced. By this we mean the necessity of assuming a finite (not infinitesimal) volume element $\triangle$ in the phase space — the cell. This finite size is connected only with the macroscopic, instead of the microscopic, description of the gas, and is unrelated to the uncertainty principle of quantum mechanics.

It can then be shown that the state of thermodynamical equilibrium corresponds to the overwhelmingly most probable distribution of the system in the phase space. We now make the only assumption that in the macroscopic view, all (dynamical) states of the system inside this finite cell Δ are nondistinguishable (or equally probable), then, starting with a gas from any state within Δ at $q_1^o \cdots q_N^o$, $p_1^o \cdots p_N^o$ $t=t^o$, in the course of time, the phase points within Δ will be distributed, with the same volume, over almost all the available phase space — the energy shell. Since an overwhelmingly large portion of this energy shell corresponds to the state of thermodynamical equilibrium, we can understand why a gas, starting from any arbitrary state, always approaches equilibrium. This is the so-called the phase-mixing theory of Gibbs. Note that this conclusion is based on the probability argument, which is used for the macroscopic view of the gas.

The above theory, while perfectly sound, is not convenient for the purpose of application to actual calculations. The Liouville

equation in 6 N − dimensions is too complicated. It deals with the dynamical motions of the gas particles and presumes a knowledge which we do not have (i.e., we cannot specify the initial values of the coordinates and momenta anyway) and which is really of no interest to us in the macroscopic view. Thus it is most desirable to have a theory of non-equilibrium gases which has the following features: (1) It is considerably simplar than the Liouville equation in the 6N-dimensional phase space, and (2) It is explicitly irreversible such that it describes the observed macroscopic irreversible approach of the gas to equilibrium.

To achieve these, it is necessary to introduce assumptions. By a great intuition and on plausibility arguments, Boltzmann formulated the wellknown kinetic equation

$$\frac{\partial F}{\partial t} + v \cdot \frac{\partial F}{\partial r} + \frac{K}{m} \cdot \frac{\partial F}{\partial v} = \left(\frac{\partial F}{\partial t}\right)_{\text{collision}} , \tag{3}$$

$$\left(\frac{\partial F}{\partial t}\right)_{\text{collis ion}} = \iiint dv_1 \; bdb \; d\varphi \left| v - v_1 \right| \left[F(r, v', t) F(r, v_1', t) \right.$$

$$\left. - F(r, v, t) F(r, v_1, t) \right], \tag{4}$$

where b is the impact parameter and bdb the differential cross-section of the two-particle collision. We shall here only emphasize the following points in connection with the Boltzmann equation: (i) The Boltzmann equation presupposes that the behavior of the N-particle system can be sufficiently described in terms of the one-particle distribution function. (ii) The equation is irreversible and is not to be used when direction of the time t is reversed. (iii) A gas described by (3) and (4) satisfies the H-theorem $dH/dt \leq o$. This means that the Boltzmann equation describes an irreversible monotonic approach to equilibrium. (iv) Consider the time-evolution of the gas

from any arbitrary initial state. In a time of the order of the mean-free-time t_o ($t_o = \lambda/u$, where λ is the mean-free-path and u the mean speed of the gas particles), the gas will have reached a state of local equilibrium, i. e., a stage such that the average values defined by

$$
\begin{aligned}
n(r,t) &= \int F(r,p,t)\,dp \quad , \\
nu(r,t) &= \int^{\frac{p}{m}} F(r,p,t)\,dp \quad , \\
\frac{3}{2} n\theta(r,t) &= \int \frac{p^2}{2m} F(r,p,t)\,dp \quad ,
\end{aligned}
\tag{5}
$$

can be identified with the local particle density, the mean flow velocity and the local temperature. One can derive from the Boltzmann equation the equations of continuity, momentum and energy transport for n, u, θ, namely,

$$
\begin{aligned}
&\frac{\partial n}{\partial t} + \operatorname{div}(nu) = o \quad , \\
&\frac{Du_\alpha}{Dt} - \frac{1}{m} K_\alpha = -\frac{1}{mn} \frac{\partial}{\partial r_\beta} P_{\alpha\beta} \quad , \\
&\frac{D\theta}{Dt} + \frac{2}{3n}(\operatorname{div} J + P_{\alpha\beta} D_{\alpha\beta}) = o \quad ,
\end{aligned}
\tag{6}
$$

where P is the stress tensor, J the energy flux

$$
\begin{aligned}
P_{\alpha\beta} &= m\int (v-u)_\alpha (v-u)_\beta F\, dp \quad , \\
J_\alpha &= \frac{1}{2} m \int (v-u)^2 (v-u)_\alpha F\, dp \quad ,
\end{aligned}
\tag{7}
$$

and

$$
\frac{D}{Dt} = \frac{\partial}{\partial t} + u\cdot\Delta \quad ,
$$

$$
D_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha}\right) \quad .
$$

Equations (6) are the hydrodynamical equations. The second equation is in fact the Navier–Stokes equation of motion which contains the empirical concept of viscosity in the expression for $P_{\alpha\beta}$. The third equation contains the phenomenological equation of conduction of heat. The above theory furnishes not only the theoretical derivation of these transport equations, but also a theoretical expression for the transport coefficients in terms of the law of intermolecular interactions through the differential crosssection bdb in the Boltzmann collision integral (4). (v) The Boltzmann equation (3, 4) and the hydrodynamical equations (6) have been solved by Chapman and Enskog by assuming that in this hydrodynamical stage of the approach to equilibrium, the distribution function $F(r,p,t)$ depends on time only through a functional dependence on the macroscopic quantities n, u and θ of (5),

$$F(r,p,t) \rightarrow F(r,p\,|\,n,u,\theta) \tag{8}$$

This assumption has the following consequence: one does no longer solve the Boltzmann equation as an initial value problem, i. e., it is no longer possible to assign arbitrary initial values to $F(r,p,t)$. One can only prescribe the initial values of n, u, θ, and then F is already completely determined. This situation was noted early by D. Hilbert. The basis for the validity of the assumption (8) is that there exist in a gas of ordinary densities characteristic time constants, the t_0 above, $T_0 = L/u_s$, and

$$t_0 \ll T_0 \tag{9}$$

where L is length such that $\frac{1}{y}(\nabla y)\, L \simeq 1$, y being a macroscopic property such as temperature, u_s is the velocity of sound, $u_s \simeq u$. The scale of time in the "kinetic stage" of the time-development is a much faster one than that in the hydrodynamical stage. The assumption (8) is thus a change of the time scale from that of F in

the kinetic stage to that of n, u, θ in the hydrodynamical stage.

The Boltzmann equation has the satisfactory features briefly summarized above, and has been very successful in dealing with the transfer problems. But from a more basic point of view, one remembers the following limitations: (i) The starting point of the theory, namely, that an N-particle system can be sufficiently well described in terms of a one-particle function F, is heuristic. (ii) The collision integral (4) is very plausible, but is based on the assumptions (α) of binary collisions alone and (β) of the absence of correlations between the coordinates and velocities of the colliding particles. The assumption (α) has the following consequence: the theory fails to give any density dependence for the transport coefficients as one would expect by analogy with the virial expansions of the equilibrium properties of a gas such as the equation of state. The reason is of course the neglect of ternary and higher order collisions which become important for high densities. The specific nature of the assumptions (α) and (β) above is revealed most clearly when one tries to apply the Boltzmann equation to an ionized gas a plasma. When the differential cross section bdb in (4) is calculated for the Coulomb law, the collision integral (4) diverges for large values of b corresponding to distant collisions. We shall come back to this difficulty below.

It is hence of the greatest interest to have other theories describing the behavior of a gas which are based on more general assumptions than the Stosszahlansatz (4) of Boltzmann.

II. The Master Equation for Irreversible Processes

We have emphasized before that in order to have an explicitly irreversible theory, i.e., one having a definite direction of time such as in the Boltzmann equation, it is necessary to introduce some

assumptions in one form or another. There are of course many different ways by which irreversibility can be introduced other than the Stosszahlansatz (4). Just as an example, let us consider the following way of obtaining the Master Equation. Let W_k, $k = 1,2\cdots\cdots$, be the probability of a system being found in state k at time t, and let W_n' be that in state n at time $t+\triangle t$, $\triangle t > 0$.

Then $$\Sigma W_n = 1, \qquad \Sigma W_n' = 1. \tag{10}$$

Let us assume that the W_n' at $t+\triangle t$ are related to the W_k at t by the relation

$$W_n' = \sum_k A_{nk} W_k \tag{11}$$

where A_{nk} is the transition probability, in time interval $\triangle\tau$, of the state k at t passing to W_n' at $t+\triangle t$. Then the A_{nk} have the properties

$$\sum_n A_{nk} = 1 \quad , \tag{12}$$

$$0 \leq A_{nk} \leq 1 \quad , \tag{13}$$

From (11) and using (12), one readily obtains the so-called Master Equation

$$\frac{\triangle W_n}{\triangle t} = \sum_k \left(A_{nk} W_k - A_{kn} W_n \right) \tag{14}$$

where a_{nk} is the transition probability $k \to n$ per unit time

$$a_{nk} = A_{nk} / \triangle t$$

We shall now emphasize that the relation (11) and the equation (14) are not reversible in time in the following sense.

Let us assume that the matrix A_{nk} is non-singular so that A^{-1}

exist, i.e.,

$$A^{-1} A = A A^{-1} = 1 \quad , \tag{15}$$

and hence, algebraically, the inverse of (11) is

$$W_k = \Sigma A^{-1}_{kn} \; W'_n \; . \tag{16}$$

But from (15) and (12), it can easily be seen that the A^{-1}_{kn} are neither all positive, nor all less than 1. Thus the A^{-1}_{kn} do not have the meaning of transition probabilities. The Master Equation (14), which has the same from as that for Markovian processes, is irreversible in time. In fact, if we define

$$S(t) = -\sum_n W_n \ln W_n \; , \qquad S'(t+\Delta t) = -\sum_n W'_n \ln W'_n \; ,$$

then it can be shown by means of an inequality for any positive numbers a and b

$$b \int_1^{a/b} \ln x \, dx = a \, (\ln a - \ln b) + b - a \geq o$$

due to Gibbs, that

$$S' - S \geq o \; , \tag{17}$$

which can be interpreted as the law of entropy.

The Master Equation (14) is a very general irreversible equation. The Boltzmann equation (3,4) may be regarded as a special case of it if we identify W_n with $F(r, v_n, t)$ and a_{kn} with $\int\int bdb \, d\varphi \, |v_n - v_k| \, F(r, v_k, t) \, dv_k$. The apparent simplicity of (14) on account of linearity is deceptive, since the transition probabilities a_{kn} may themselves be functions of the Ws, as in the case of the Boltzmann equation. The generality of (14), i.e., the as yet unspecified

nature of the a_{kn}, makes it not directly useful for practical calculations without additional knowledge or assumptions about the a_{kn}. We shall hence pass on to a theory which is less general than (14) but more general than the Boltzmann equation.

III. Theory of Bogoliubov: Neutral Gases.

The starting point is the Liouville equation. From this equation, one obtains for the functions

$$F_s\,(q_1 \cdots q_s,\ p_1 \cdots p_s,\ t) = V^s \int \cdots \int D\,(q_1 \cdots q_N,\ p_1 \cdots p_n,\ t)\ dq_{s+1} \cdots dq_N\ dp_{s+1} \cdots dp_N \qquad (18)$$

the Bogoliubov-Born-Green-Kirkwood-Yvon system of equations

$$\frac{\partial F_s}{\partial t} = \{H_s,\ F_s\} + \frac{N-s}{V} \int\int dq_{s+1}\ dp_{s+1}\ \{\sum_{i=1}^{s} \phi\,(|q_i - q_{s+1}|)\ ,\ F_{s+1}\} \qquad (19)$$

where V is the total volume of the gas, ϕ the interparticle interaction, H_s the Hamiltonian of the s − particle subsystem, $\{\ ,\ \}$ the Poisson bracket expression. The system (19) is completely equivalent to the Liouville equation and is hence reversible in time.

One basic idea of Bogoliubov is the recognition of another widely different characteristic time τ_o (in additional to the mean-free-time $t_o = \lambda/u$ and the macroscopic time T_o mentioned before), namely, the time of a collision $\tau_o = r_o/u$, where r_o is the range of interaction. From the presence of $\phi(|q_i - q_j|)$ in H_s for $S \geq 2$ in (19), it is seen that for $S \geq 2$, F_s changes rapidly, in a time interval of order $\tau_o \simeq 10^{-12}$ second, while F_1 does not have these rapid variations. Since in the macroscopic view we are not concerned with such rapid variations, we shall make a coarse-graining in time by changing the

time scale for F_s, from the fine-grained time t to a time scale in which the slowly varying F_1 has made some changes. This is mathematically expressed by the following "functional Ansatz" that the time-dependence of F_s be expressed through a functional dependence on F_1

$$F_s(q_1 \cdots q_s,\ p_1 \cdots p_s,\ t) \to F_s(q_1 \cdots q_s,\ p_1 \cdots p_s\ F_1) \qquad (20)$$

A second basic idea of Bogoliubov is contained in the "initial condition" which is expressed as follows. Let $Z_\tau^{(s)}$ be the canonical transformation operator which transforms the dynamical state of the subsystem of s particles at time t to the state at the time $t+\tau$ in accordance with the dynamical equations of motion for the Hamiltonian H_s of the subsystem. The initial condition is then

$$\lim_{\tau \to \infty} Z_{-\tau}^{(s)} F_s(q_1 \cdots q_s,\ p_1 \cdots p_s,\ t) = \prod_{i=1}^{s} F_1(q_i, p_i) \qquad (21)$$

which postulates that when the system is traced backward in time for an interval τ much longer than τ_0 (so that the particles are farther apart than their range of interactions), the s particles become uncorrelated. Note that this condition (20) cannot be proved; it is in fact a definition of the direction of time; the "past" is that direction in which the correlation vanishes. This is of course an unphysical state, and we do not look at the gas in that direction. The theory describes only the evolution of the system in the direction of time in which the gas molecules are correlated by virtue of their interactions.

With this initial condition, the equations (19) become irreversible, in the sense of the Boltzmann equation. On account of the Ansatz (20), the equations (19) for $S \geq 2$ become functional differential equations of F_1, and the equation for F_1 is

$$\frac{\partial F_1}{\partial t} + v \cdot \frac{\partial F_1}{\partial r} + K \cdot \frac{\partial F_1}{\partial p} = \frac{1}{v_o} \iint dr' dp' \{\phi(|r-r'|),$$

$$F_2(r, r', p, p' | F_1) \}, \qquad (22)$$

where $v_o = V/N$ is the volume per particle.

The equation(22) contains F_2, and the equation for F_2 contains F_3, etc. Bogoliubov makes the virial or density expansions

$$F_s = \sum_n \left(\frac{1}{v_o}\right)^n F_s^{(n)} (q_1 \cdots q_s, p_1 \cdots p_s | F_1),$$

$$\frac{\partial F_1}{\partial t} = \sum_n \left(\frac{1}{v_o}\right)^n A_n (q, p | F_1) \quad . \qquad (23)$$

The system of equations (19) and (22), together with the Ansätze (20) and (21), can then be solved in successive orders in $(1/v_o)$. We shall not attempt any further discussions except to make the following summarizing remarks.

(i) Bogoliubov has shown that up to the first order in $1/v_o$, the theory yields for the kinetic equation (22) a generalized Boltzmann equation which reduces to the Boltzmann equation (3,4) if one makes the further simplifying assumption that F_1 can be regarded as homogeneous within the range r_o of the intermolecular interactions.

(ii) A transition from the kinetic stage, described by (22), to the hydrodynamical stage can be made in a way analogous to that in the case of the Boltzmann equation. The theory has been worked out by Choh and Uhlenbeck to the second order in $1/v_o$ so that the effect of "3-body collisions" is included. The transport coefficients for momentum and heat are then density dependent, in contrast to the result from the Boltzmann equation.

Thus we may regard the Bogoliubov theory as a more general theory than Boltzmann's. It gives the theoretical basis for working with F_1 as in Boltzmann's equation, and allows for the possibility of

including higher order correlations ("n-typle collisions") for gases of high densities, although numerical calculation cannot be carried out on account of the difficult dynamical problem of many bodies.

IV. Kinetic Theory of Plasmas: Basic Consideration

For a fully ionized gas, or a plasma, it would seem as if the theories that have been developed for neutral gases could be taken over with the provision that the intermolecular interactions be replaced by the Coulomb interactions. But if we apply the Boltzmann equation to a plasma with the differential cross-section bdb for the Coulomb law, it is found that the collision integral (4) diverges for large values of the impact parameter b. This is because the total scattering cross-section for Coulomb interactions is infinite.

This long range nature of the Coulomb interaction in plasmas constitutes a rather basic difference from neutral gases. On account of the long range, it is no longer meaningful to consider only binary collisions; also the effect of two- and many- particle correlations can no longer be neglected as in Boltzmann's equation. To include all orders of collisions, it is not sufficient to stop at the first, or the second, order in the density $1/v_0$ in the expansion (23) of Bogoliubov. Many theories have been suggested for plasmas. We shall discuss briefly the most familiar ones and the more recent attempts.

(A) Boltzmann equation with a "cutoff"

The most obvious and natural attempt at a kinetic equation of plasmas is to modify the Boltzmann equation so as to get rid of the divergence difficulty. One argues as follows. In the integration over the impact parameter b in the collision integral (4), one can divide the range of b into two regions, namely, a region I of small distances smaller than the mean distance $\sqrt[3]{v}$ between particles, and a region II greater than $\sqrt[3]{v}$. In region I, two colliding particles

do not see other particles between them and the scattering is governed by the Coulomb law. In region II, especially $b \gg \sqrt[3]{v}$, a given particle may be regarded either as being acted on simultaneously, independently and with random relative velocities, by a large number of distant particles, or as undergoing a rapid series of random, small-angle scatterings due to collisions with many particles at large distances.

Only for the binary collisions in region I, one employs the Stosszahlansatz (4), so that one integrates (4) to an upper limit b_m. There is of course no sharply defined value for b_m, but it is very plausible to assume for b_m the Debye-Hückel length

$$r_D = \left(\frac{v\theta}{4\pi e^2}\right)^{1/2} \tag{24}$$

This procedure of cutting off the range of integration in (4) can be made mathematically more convenient by replacing the Coulomb law in calculating bdb in (4) by the Debye-Hückel screened potential

$$\phi(r_{12}) \rightarrow \frac{z_1 z_2 e^2}{r_{12}} \exp(-r_{12}/r_D). \tag{25}$$

The total scattering cross-section becomes then finite.

A kinetic equation obtained in this way is irreversible and has no divergence from distant collisions. But we may emphasize that the procedure either of cutting off the integration at r_D or of replacing the Coulomb by the Debye-Hückel potential cannot be regarded so completely satisfying from a basic point of view. Such a theory does not contain within itself the statistical feature represented by (25) but has to bring it into the theory from a separate theory, namely, that of Debye-Hückel. Also the procedure of using the screened potential (25) cannot be readily identified with a systematic apporoximation method based on an expansion in powers of a parameter

of smallness.

(B) Fokker–Planck equation with a cutoff

On the same argument above of regarding a particle as being subject to a large number of independent, small angle scattering, one may employ the theory of stochastic processes in treating the rate of change of momentum of a particle in going through a medium. The theory leads to the well-known Fokker–Planck equation in which the effect of the random collisions is represented by a diffusion term in momentum space. To obtain the "Fokker–Planck" kinetic equation usually used in the literature, one regards the particle, whose distribution function F_1 is being considered, as moving in a medium described by an equilibrium distribution, thereby leading to a linear theory. It is necessary also to remove the divergence at large impact parameters by the same procedure as described in the preceeding section.

In the literature, a Fokker–Planck kinetic equation with a cutoff has been used very often both in general discussions and in actual calculations. We shall not go further into this theory, but shall make the following remarks. (i) The theory is based on the theory of stochastic processes. It must be accepted as a theory on its own, and not as the most general theory.
(ii) From the way the divergence is avoided, it is seen that the theory is subject to the same criticism mentioned at the end of the preceding section on the Boltzmann equation with a cutoff.
(iii) It has been found that the kinetic equations that have been obtained on other, more general assumptions (see the following sections) do not reduce to the usual Fokker–Planck equation unless an additional assumptions and approximations are introduced. This makes it clear that the Fokker–Planck type kinetic equation for plasma is a plausible theory, but not one to which other formulations can all be reduced.

(C) Vlasov equation

Another kinetic equation that has been used and discussed a great deal is the Vlasov equation. There are many ways of arriving at this equation which for a plasma consisting of particles of type σ (z_σ e, m_σ for the charge and mass) is

$$\frac{\partial F}{\partial t} + \frac{1}{m_\sigma} p \cdot \nabla F = z_\sigma e^2 \frac{\partial F}{\partial p} \cdot \left(\sum_{\sigma'} z_{\sigma'} \frac{\partial}{\partial r} \int \frac{1}{|r-r'|} F(r', p', \sigma', t) dr' dp' \right). \quad (26)$$

The righthand side is the scalar product of $\frac{\partial F}{\partial p}$ and the force on the particle $z_\sigma e$ due to all the other particles of the plasma. This equation is reversible. That it does not describe the irreversible approach to thermodynamic equilibrium can be seen from the following considerations. (i) For a spatially homogeneous plasma, $\nabla F = o$, and the righthand side of (26) vanishes on account of symmetry (and also of neutrality $\sum_{\sigma'} z_{\sigma'} e \int F(r', p', \sigma', t) dp' =$ charge deusity $= o$.) Hence $\frac{\partial F}{\partial t} = o$, i.e., any velocity distribution F remains stationary and does not approach the Maxwellian distribution. (ii) From (26) one obtains the equation of continuity

$$\frac{\partial}{\partial t} \int F dp + \frac{1}{m} \nabla \cdot \int pF dp = o.$$

It follows that if any instant t the F is isotropic in velocity, then the density $\int F dp$ remains constant in time and does not approach a uniform distribution. (iii) If F is assumed to vanish as $r, p \to \infty$, then one finds that the H function of Boltzmann is a constant. But this cannot mean thermodynamic equilibrium since F is an arbitrary state.

If we linearize the Vlasov equation (26) by writing

$$F = F_o(p^2) + f(r,p,t), \qquad |f| \ll F_o \quad (27)$$

where f is the deviation from the equilibrium distribution $F_o(p^2)$, we

can solve the equation

$$\frac{\partial f}{\partial t}+\frac{1}{m_\sigma}\, p\cdot\nabla f = z_\sigma\, e^2\frac{\partial F_o}{\partial p}\cdot\Big(\sum_{\sigma'} z_{\sigma'}\frac{\partial}{\partial r}\int\frac{1}{|r-r'|}$$

$$f(r',p',\sigma',t)\, dr'\, dp'\Big) \qquad (28)$$

by expanding the Fourier component f in a series of orthogonal functions of the momentum

$$f(k,p,t)=\sum_n f_n(k,t)\, \exp\Big(-\frac{p^2}{2m\theta}\Big)\, H_n\Big(\frac{p}{\sqrt{2m\theta}}\Big). \qquad (29)$$

To obtain the coefficients $f_n(k,t)$ we set

$$f_n(k,t)=a_n(k)\, e^{\lambda t}. \qquad (30)$$

The λ are given by the eigenvalues of the matrix formed by the coefficients of $a_n(k)$ in the system of linear equations from (28). The results of such a study can be summarized as follows. (i) for any finite number of terms in (29), the eigenvalues λ are all pure imaginary. (ii) for a plasma consisting of electrons of mass ω and positive ions of mass M, an n-term expansion (29) leads to 2n frequencies $\omega_j^2=-\lambda_j$ such that one frequency is given by

$$\omega^2=\Big(1+\frac{m}{M}\Big)\Big(\omega_p^2\ \frac{3\theta}{m}\,k^2\Big)+O(k^4/\omega_p^2), \qquad (31)$$

$$\omega_p=\Big(\frac{4\pi e^2}{mv_o}\Big)^{1/2}=\text{eleotron plasma frequency},$$

$$n-1 \text{ frequencies of order } \frac{3\theta}{m}\,k^2, \qquad (32)$$

and n frequencies of order $\dfrac{3\theta}{M}\,k^2$. (33)

The last 2n-1 frequencies are those of streaming and are not dependent

on the Coulomb interactions.

The general solution of $f_n(k,t)$ is then the superposition of all the "normal modes"

$$f_n(k,t) = \Sigma\, C^{(j)}_{(k)}\, a_n^{(j)}{}_{(k)}\, e^{i\omega_j t} + \text{complex conjugate},$$

where the constants $C^{(j)}_{(k)}$ are to be determined from the initial condition on $f(k,p,t)$. Thus in principle one has solved the linearized Vlasov equation as an initial value problem for F.

The Fourier component of the particle density variations is now

$$\Delta n(k,t) = \int f(k,p,t)\, dp$$

$$= \sum_j C^{(j)}_{(k)}\, a_0^{(j)}{}_{(k)}\, e^{i\omega_j t} + \text{cowplex conjugate}, \qquad (34)$$

which is a "wave packet". Whether the wave packet shows a damping (Landau damping) or a growth in time will depend on the initial conditions (i,e., on the $C^{(j)}_{(k)}$). This is essentially the result of the work of Van Kampen in 1955; but the method indicated above is more elementary.

As we have emphasized above, the Vlasov equation does not describe any irreversible approach to equilibrium. For the discussion, however, of phenomena having characteristic times very much shorter than the genuine relaxation time (i. e., the damping time due to the collision, or correlation, effects which are not included in the Vlasov equation), it is perhaps a sufficient approximation to use the Vlasov equation. Otherwise, it is necessary to use a kinetic equation that takes into account the collision, or correlation, effect. For this purpose, we have the Boltzmann equation and the Fokker–Planck equation, with the cutoff, already discussed in Sections (A) and (B) above. But it is desirable to look for more general theories.

V. Kinetic Equation of Plasmas

To formulate a kinetic theory of plasmas from a more general starting point than the Boltzmann or the Fokker–Planck equation, one now starts from the B–B–G–K–Y system of equations (19), as in Bogoliubov's theory for neutral gases. Here again it is necessary to introduce some definite assumptions or make some mathematical steps to destroy the symmetry of the Liouville equation in the two directions of time. One may again assume the initial condition (21) of Bogoliubov to distinguish the "past" from the "future". Or one may make a Laplace transformation

$$\overline{F}(r,v,p)=\int_0^\infty F(r,v,t)\ e^{-pt}\,dt \tag{35}$$

with the real part of $p>0$. This procedure excludes negative values of t from the theory. These two methods of introducing a definite direction of time are not exactly identical, but are similar to each other.

The next step from the B–B–G–K–Y system is to make some assumptions or approximations such that the chain of equation can be solved in finite terms. To do this, two procedures have been suggested which we shall now briefly describe.

(A) Theory of Ichikawa, Rosenbluth and Rostoker and of Guernsey.

If one writes the two- and three- particle distribution functions F_2, F_3 in (20) in the form

$$F_2(1,2)=F_1(1)\,F_1(2)+G(1,2), \tag{36}$$

$$F_3(1,2,3)=\prod_{i=1}^{3}F_1(i)+\sum_{i\lessdot j\lessdot k}F_1(i)\,G(j,k)+g(1,2,3), \tag{37}$$

and make the approximation of neglecting $g(1,2,3)$, then the equations for F_1, F_2 in the system (22) become a pair of coupled non-linear

integro-differential equations for F_1 and G, which are decoupled from the rest of the system. The equation for G has been formally solved by Guernsey in 1962. It is a very involved solution and it seems that very little has been discerned from it. The substitution of this solution G (in terms of F_1) into the equation for F_1 gives the kinetic equation, which is of course a very complicated non-linear integro-differentical equation. The equation is irreversible since in integrating the equation for G, the Laplace transform (35) has been made. The equation has not been studied much in its exact form. Only certain cases have been examined on its basis, namely, the equilibrium case, the spatially homogeneous, and the case of small deviations from homogeneity (linear equation). Some approximate calculations have been carried out, with many simplifying approximations, to study the effect of the two- and three- particles correlations on the frequency of the plasma oscillations and the relaxation times (damping constants). I shall not describe this theory further here since Dr. Ichikawa is here and he is certainly more qualified to discuss his own work. I shall therefore describe another approach which is somewhat different form this theory.

(B) Theory based on Bogoliubov's theory: Guernsey, Rosenberg, Sandareson and Wu.

As in Bogoliubov's theory for neutral gases, the "functional Ansatz" (2) is also made for the plasma. This presupposes that F_s, $s \geq 2$, change faster in time than F_1, a supposition which, on account of the long range nature of Coulomb interactions, may no longer be justifiable as in the case of the short-range van der Waals forces for neutral gases. This assumption also carries with it the limitation that one no longer obtains the most general solutions since only the initial value of F_1 can be specified while those of F_2, F_3..... are already determined as functionals of F_1 and no longer

arbitrarily assignable. The advantage of this functional Ansatz is that, once it is made, no further assumptions (except the expansion in powers of e^2) are necessary in the rest of the development of the theory.

We do not want to stop at a finite order in the density expansion (23) on account of the importance of many-body correlations and "collisions" in the case of the Coulomb interaction. Instead we make expansions in powers of the strength of the interaction ($4\pi e^2$), namely

$$F_S = \sum_{m=0} (4\pi e^2)^m \; F_S^{(m)} ,$$

$$\frac{\partial F_1}{\partial t} = \sum_{m=0} (4\pi e^2)^m \; A_m (r, p, \sigma \,|F_1) . \qquad (38)$$

It is then found that, with the initial condition (21), the equations for $\frac{\partial F_1}{\partial t}$ and $\frac{\partial F_2^{(1)}}{\partial t}$ are decoupled from the rest of the B-B-G-K-Y system without having to make the approximation corresponding to the step of neglecting $g(1, 2, 3)$ in (37).

For m=0, the kinetic equation

$$\frac{\partial F_1}{\partial t} = A_0 (r, p \,|F_1)$$

turns out to be precisely the Vlasov equation (26). A_0 consists of the stream term and the self-consistent field term of Vlasov.

Up to m=1, the kinetic equation is

$$\frac{\partial F_1}{\partial t} = A_0 + 4\pi e^2 A_1 \qquad (39)$$

where $A_1 (r, p, \sigma \,|F_1)$ depends on $F_2^{(1)}$. $F_2^{(1)}$ is itself given by an non-linear integro-differential equation which contains F_1. The coupled equations for F_1, $F_2^{(1)}$ are complicated in the general case.

For homogeneous plasmas, however, the integral equation for $F_2^{(1)}$ has been solved by Guernsey and by Lenard, and the kinetic equation has been obtained by these authors and also by Belescu in a somewhat different approach. This equation is

$$\frac{\partial F_1}{\partial t} = \frac{2z^2_\sigma e^4}{v} \int dk \frac{k}{k} \cdot \frac{\partial}{\partial p} \left\{ \frac{\int d\eta' z^2_{\sigma'} \delta(u-u') \, k \cdot \left(\frac{\partial}{\partial p} - \frac{\partial}{\partial p'}\right) F_1(\eta) F_1(\eta')}{\rho^2} \right\}, \tag{40}$$

$$\rho(k,u) = k^2 - \frac{2\pi i}{k} \frac{4\pi e^2}{v} \int d\eta' \, z^2_{\sigma'} \, \delta_+ (u-u'^1) \, k \cdot \frac{\partial F_1(\eta')}{\partial p'}, \tag{41}$$

where

$$u = (k \cdot p)/km_\sigma, \qquad u' = (k \cdot p')/km_{\sigma'},$$

$$\eta \equiv p, \sigma, \qquad \eta' \equiv p', \sigma',$$

and the integration over η' includes a summation over all particle types. k is the wave vector in the Fourier transform of the Coulomb potential

$$\frac{1}{4\pi} \int \frac{1}{r} e^{ik \cdot r} \, dr = \frac{1}{k^2}.$$

This equation (40) has the following properties:

(i) It is irreversible, in the same sense as the Boltzmann equation.

(ii) It satisfies the H-theorem and leads to the Maxwellian distribution as the equilibrium distribution.

(iii) The integral in (40) is convergent for small k, i.e., for large distances, in contrast to the Boltzmann equation. This convergence is brought about by the factor $1/|\rho|^2$. The second

term in ρ in (41) arises from the correlation effect represented by $F_2^{(1)}$ which appears in this approximation (to the first order in the interaction strengh $4\pi e^2$).

(iv) The two-particle correlation function $F_2^{(1)}$ in the equilibrium case turns out to be precisely the Debye-Huckel result

$$F_2^{(1)}(r_{12}) = -\frac{z_1 z_2 e^2}{kT}\frac{1}{r_{12}}\exp(-r_{12}/r_D) \qquad (42)$$

where r_D is given in (24). Thus the Debye-Huckel result comes out of the present theory instead of having to be introduced into the theory as in the theories based on the idea of a cutoff.

(v) E_q. (40) is not exactly reducible to a Boltzmann equation with a static screened potential replacing the Coulomb potential.

(vi) E_q. (40) is also not reducible to the usual Fokker-Planck equation. Even in the linearized approximation, the equation is still different from the Fokker-Planck equation. For example, the rate of change of momentum at value p depends on all other values p' instead of on p alone a kind of "non-local" effect.

(vii) The equation (40) for homogeneous plasmas obtained on the basis of the functional Ansatz (20) is equivalent to that obtained by the method based on (36, 37).

(viii) The equation (40) in the linearized approximation has recently been solved by my colleague Professor R. L. Rosenberg for the spectrum of relaxation times (or damping constants).

The method is to make an expansion (29) for the F_1 of each type of particles. For example, for a plasma consisting of electrons of mass m and positive ions of mass M=2500 m, with a 6-term expansion in (29), there are 11 distinct relaxation times. For densities and temperatures such that $r_D^3/v \simeq 10^3$, the relaxation time τ, in units of $(r_D^3/v\omega_p)$, where ω_p is given in (31), are as follows

electron–ion	ion–ion	electron–electron
$\tau_1 = 0.95 \times 10^4$	$\tau_2 = 2.31 \times 10^3$,	$\tau_7 = 6.7 \times 10$
	$\tau_3 = 1.80 \times 10^3$,	$\tau_8 = 4.5 \times 10$
	$\tau_4 = 1.16 \times 10^3$,	$\tau_9 = 2.5 \times 10$ (43)
	$\tau_5 = 0.54 \times 10^3$,	$\tau_{10} = 1.06 \times 10$
	$\tau_6 = 0.19 \times 10^3$,	$\tau_{11} = 0.38 \times 10$

The grouping of the relaxation times under the different headings is guided by an examination of the eigenvectors (i.e., the amplitudes) corresponding to the various eigenvalues (the relaxation times). It is seen that τ_6 / τ_{11} and τ_5 / τ_{10} are $\simeq 51$, which is approximately the value of $(M/m)^{1/2} = 50$. The electron–electron collisions and the ion–ion collisions are intrinsically equally efficient in the redistribution of their velocities. That τ_5 / τ_{10}, $\tau_6 / \tau_{11} \simeq (M/m)^{1/2}$ is simply due to the smaller velocities of the ions than those of the electrons. The longest relaxation time τ_1, is associated with about equal amplitudes of the electron and the ion functions; it corresponds to the slow velocity redistributions due to the large difference in the masses of the electron and the ion in their collisions.
All these are just the results one expects on elementary considerations, but it is satisfying that they come out from the kinetic equation (40) without any further physical arguments.

For the case of plasmas having only small spatial inhomogeneities, it is possible to formulate a kinetic equation along the line of the Bogoliubov method (initial condition and functional Ansatz) in the linear approximation. The equation for F_2 and hence the kinetic equation for F_1 are much more complicated than (40). In the kinetic equation up to m=1 for $\frac{\partial F_1}{\partial t}$, there are the streaming and the Vlasov terms which are reversible; then there are the corrections to both the streaming and the Vlasov terms which arise from the correlation effect represented by $F_2^{(1)}$ and which now render the equation irreversible.

Then there is the "main collision" terms coming from $F_2^{(1)}$, which is irreversible. The equation has not been solved, but one can expect the spectrum of frequencies or relaxation times τ in (30) to be

$$\lambda_j = i\omega_j - \gamma_j \tag{44}$$

where γ_j are positive (corresponding to the relaxation similar to (43)), and ω_j are similar to the values given by (31)–(33), with corrections arising from the correlation effect $F_2^{(1)}$ not present in the Vlasov equation (which is the m=0 approximation in (38) here).

Recently my colleague Professor M. K. Sundaresan has made a calculation of the thermal conductivity of a plasma on the basis of the theory described above. The formulation of the theory for the hydrodynamical stage follows the same line of Chapman and Enskog indicated in Section I above, the kinetic equation and the hydrodynamical equations are solved for the case of small deviations from equilibrium. I shall only report the result here. For a plasma of $r_D^3/v \simeq 10^3$, the conductivity coefficient is

$$\chi = \frac{\theta^{5/2}k}{m\ e^4}\ (1.05), \qquad \theta = kT, \tag{45}$$

k being the Boltzmann constant. This result is to be compared with the expression

$$\chi = \frac{\theta^{5/2}k}{m\ e^4}\ \frac{10.88}{\ln \Lambda} \tag{46}$$

obtained by Spitzer and Härm in 1953 with the use of a Fokker–Planck equation with a cutoff by means of the Debye screened potential. In (46), $\Lambda = \frac{3}{\sqrt{2}} 4\pi r_D^3/v$. For the same value $r_D^3/v = 10^3$ as used in (45), $\ln \Lambda = 9.7$ so that the two results (45), (46) are in good agreement with each other. Thus our theory seems to have furnished some justification for the usual procedure of using the Fokker–Planck

equation with a cutoff.

We have also formulated the kinetic equation of a plasma in a strong, static magnetic field on the basis of the Bogoliubov method, and we are calculating the transport properties.

I shall conclude this review with the following remark. The method (A) above employed by Ichikawa, Guernsey, et al without the use of the functional Ansatz (20) and with the use of the Laplace transform (35), shows some direct time dependence of G (1,2) in addition to the dependence on F_1. This is called the non-Markovian effect. On the other hand, the method (B) based on the Bogoliubov theory first used by Gnernsey and followed by my colleagues and myself does not have this non-Markovian behavior, since by assumption, F_2 depends on time only through its functional dependence on F_1. The significance of this difference between the two methods is not very clear. My feeling is that since we are aiming essentially at a statistical theory, the presence of some non-Markovian effect, which reflects some dynamical rather than stochastic features, may or may not be really desirable. Further clarification of this seems necessary.

References

I. A brief discussion of the Boltzmann equation can be found in the recent book "Lectures in Statistical Mechanics" by G. E. Uhlenbeck and G. W. Ford, published by the American Mathematical Society, Providence, R. I., (1962).

II. The treatment in this section together with the corresponding quantum-mechanical case can be found in an article by T. Y. wu and D. Rivier, Helv. Phys. Acta 34, 661 (1961).

III. N. N. Bogoliubov. "Problems of a Dynamical Theory in Statistical Mechanics" translated into English by E. K. Gora and published in Studies in Statistical Mechanics, ed. by J. de Boer and G. E. Uhlenbeck, North-Holland Publ. (1962).

IV. For a general discussion of the Boltzmann equation with a cutoff, see article by H. Grad in the Proc. Fifth International Conference on Ionization Phenomena in Gasses, North-Holland, p. 1630, (1962)

The treatment of the Vlasov equation given here is given by M. K. Sundaresan and T. Y. Wu, in a paper to appear in the Canadian Jour. Phys. 41 (1963).

V. (A) Yoshi H. Ichikawa, Prog. Theor. Phys. (Kyoto), 24, 1085 (1960);
M. N. Rosenbluth and N. Rostoker, Phys. of Fluids 3, 1 (1960);
R. L. Guernsey, Phys. of Fluids 5, 622 (1962).

(B) R. L. Guernsey, The kinetic Theory of Fully Ionized Gases, Thesis, Univ. of Michigan (1960).
R. Balescu, Phys. of Fluids 3, 52 (1960);
A. Lenard, Annals of Phys. 10, 390 (1960);
T. Y. Wu and R. L. Rosenberg, Can. J. Phys. 40, 463; 1251, (1962); 41, 1193 (1963), (inhomogeneous plasmas).
M. K. Sundaresan and T. Y. Wu, Can. J. Phys. 40, 1499; 1686 (1962) (plasma in magnetic fields).
R. L. Rosenberg and T. Y. Wu, Can. J. Phys. 42 (1964), (relaxation times of homogeneous plasma from Guernsey-Balescu-Lenard equation).
M. K. Sundareson and T. Y. Wu to appear in Can. J. Phys. (Thermal conductivity of plasmas).