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Electromagnetic Shock Wave in a Transparent Medium

by

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Abstract

In this paper we consider the electromagnetic wave propagation in a simple model of a dielectirc medium which is characterized by a non-linear relation between the electric displacement and the electric field strength. The formation of shock-like discontinuity is shown by the method of characteristics, and we derive the jump relations across such a discontinuity corresponding to the Rankine-Hugoniot relation in gas dynamics. Then we shown that the evolutionary condition enables us to select a physically relevant solution among others admissible by the jum relations.

1. Introduction

In this paper we consider a transparent medium, in which the electric displacement is given as a local non-linear function of the electric field strength. Under such the dielectric property of the medium the Maxwell equations constitute a non-linear hyperbolic system of partial differential equations, and the characteristics representing the local phase velocity of light depend on the electric field strength in just the same way as in gas dynamics the local sound velocity depends on the density. Hence one may naturally expect that a discontinuity like a shock in gas dynamics will be formed. As a result the unique existence of solution of initial value problem will be lost. In regard to this non-uniqueness of discontinuous solutions, in hydrodynamics and magnetohydrodynamics the mathematical problem of selecting a physically relevant solution among others has been discussed extensively,[(1)(2)] and it has been shown recently that instead of using the entropy condition or considering structures of discontinuities the evolutionary condition may be used to select a relevant solution.[(3)] Since at the present stage of our knowledge we do not have any thermodynamical relation for the medium which would be essential for the physical aspect of such a discontinuity, a mathematical selection rule seems to be most desirable. The purpose of this paper is to examine whether or not the evolutionary condition still selects, for the present model, a physically relevant discontinuity. In § 2, the general solution representing polarized plane waves is obtained on the basis of the method of characteristics. Then in § 3 a simple wave solution is obtained under a given initial condition, and the formation of shock is demonstrated. In § 4, it is shown that the evolutionary condition enables us to select a physically relevant discontinuity. The formation of shock-like discontinuity seems to imply a conversion of electromagnetic wave energy into heat, without being caused by the ordinary atomic absorbtion of light. Though our model is too simple to be applied to actual phenomena such as obtained by the laser, such the conspicuous property in non-linear wave field might be observed experimentally.

2.

As is wellknown the Maxwell equations in a non-magnetic medium take the form

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = 0 , \qquad (1.a)$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = 0 , \qquad (1.b)$$

$$\nabla \cdot \vec{H} = 0 , \qquad (1.c)$$

$$\nabla \cdot \vec{D} = 0 , \qquad (1.d)$$

where $\vec{D}$ is the electric displacement vector, $\vec{E}$ the electric field vector, $\vec{H}$ the magnetic field vector and the Gaussian units are used.

We now assume that $\vec{D}$ is given in terms of $\vec{E}$ by the non-linear equation

$$\vec{D} = (\alpha + \beta \vec{E} \cdot \vec{E})\vec{E} \qquad (2)$$

in which α and β are positive constants and α will be assumed to be greater than unity.

Under the relation (2) eqs. (1) admit solutions representing the transverse plane wave solution, which may be specified as follows:

The components of $\vec{E}$ and $\vec{H}$ are zero except E_z and H_y, which are functions of x and t only. From eqs. (1.a,b) E_z and H_y are determined by the following equations,

$$\frac{\partial H}{\partial x} - \frac{\partial}{\partial \tau}(1 + \varepsilon E^2) E = 0 \qquad (3.a)$$

$$\frac{\partial E}{\partial x} - \frac{\partial H}{\partial \tau} = 0 \qquad (3.b)$$

in which E and H stand for E_z and $H_y/\sqrt{\alpha}$ respectively, ε is equal to β/α and τ denotes $ct/\sqrt{\alpha}$.

Introducing the column vector U through the equation

$$U = \begin{bmatrix} E \\ H \end{bmatrix} \qquad (4)$$

we can transform eqs. (3) into the matrix form,

$$\frac{\partial U}{\partial x} + A \frac{\partial U}{\partial \tau} = 0 \qquad (5)$$

in which A is the two dimensional matrix given by

$$A = \begin{bmatrix} 0 & -1 \\ -(1 + 3\varepsilon E^2) & 0 \end{bmatrix} . \qquad (6.a)$$

The matrix A has the two, distinct, real eigenvalues $\lambda^{\pm}$,

$$\lambda^{\pm} = \pm\sqrt{1 + 3\varepsilon E^2} \qquad (6.b)$$

where the $\pm$ signs on either sides correspond respectively. The corresponding left eigen vectors, $l^{(\pm)}$, take the form

$$l^{(\pm)} = (-\lambda^{(\pm)}, 1) \qquad (6.c)$$

Eqs. (7) and (8) imply that the system (6) is totally hyperbolic (note that $\varepsilon > 0$). Applying the method of characteristics results[4] immediately

$$l^{(+)} \cdot \frac{\partial U}{\partial \eta} = 0 \tag{7.a}$$

along ξ = constant,

$$l^{(-)} \cdot \frac{\partial U}{\partial \xi} = 0 \tag{7.b}$$

along η = constant,

in which ξ = constant and η = constant are the characteristics C^+ and C^- introduced through the equations

$$\frac{\partial \xi}{\partial x} + \lambda^+ \frac{\partial \xi}{\partial \tau} = 0 \tag{8.a}$$

i.e. $\frac{d\tau}{dx} = \lambda^+$ along C^+: ξ = constant

$$\frac{\partial \eta}{\partial x} + \lambda^- \frac{\partial \eta}{\partial \tau} = 0 \tag{8.b}$$

i.e. $\frac{d\tau}{dx} = \lambda^-$ along C^-: η = constant.

In view of eqs. (7), eqs. (8) become

$F(E) - H = r(\xi)$ along C^+: ξ = constant, (9.a)

$F(E) + H = s(\eta)$ along C^-: η = constant, (9.b)

where r and s are the Rieman invariants constant on each C^+ and C^- respectively and F is given by

$$F = \int (1+3\varepsilon E^2)^{\frac{1}{2}} dE$$

$$= \frac{1}{2} [E(1+3\varepsilon E^2)^{\frac{1}{2}} + (3\varepsilon)^{-\frac{1}{2}} \log\{(3\varepsilon)^{\frac{1}{2}} E + (1+3\varepsilon E^2)^{\frac{1}{2}}\}] \tag{10}$$

Since τ is proportional to t, $|\lambda^\pm|$ are the inverses of the absolute values of the phase velocities; consequently if E increases the absolute value of the phase velocity decreases. This is, of course, due to the assumption that as the intensity of light increases the medium acts to decelerate the propagation of light. This implies also that a light of small intensity is able to catch up with a forward advancing light of large intensity, leading to the formation of shock-like discontinuity.

3. Simple Wave

In order to illustrate the non-linear response involving the formation of shock, we consider a simple wave[4] specified by the following initial condition: at $\tau = 0$, $s = 0$ for all x namely the relation $F(E) + H = 0$ is valid everywhere. $E(x,0)$, which will be written as $\overline{E}(x)$, is given such that for $x < 0$ $\overline{E}(x)$ is equal to zero, for $x > 0$ it increases as x increases and $\overline{E}(x)$ is everywhere continuously differentiable with respect to x. Since $F(E) = 0$ for $E = 0$ and F is an increasing function of $\overline{E}$, $-H(x,0)$, which is equal to $F(\overline{E})$, has a behaviour similar to that of $\overline{E}$, consequently $F(\overline{E}) - H(x,0) = 2F(\overline{E}) = r(x)$ behaves in a similar way, namely for $x \leq 0$ $r(x) = 0$ and for $x > 0$ $r(x)$ increases as x increases.

In the limit $\varepsilon \to 0$, $F(E)$ reduces to E, hence the initial condition under consideration leads to the small amplitude wave progressing toward the positive x direction. Namely from (9.b) $E + H = 0$ for all x and t and from (9.a) $E - H = r(x,0)$ is constant along each c^+, $x - x_o = \tau$; solving these two equations immediately results the solution

$$E = -H = \frac{1}{2} r\ (x-\tau) = \overline{E}\ (x-\tau)$$

which represents the wave tail of the positively progressing plane wave. (c.f. Fig. 1) For finite amplitude wave in which ε is finite, the situation is entirely different.

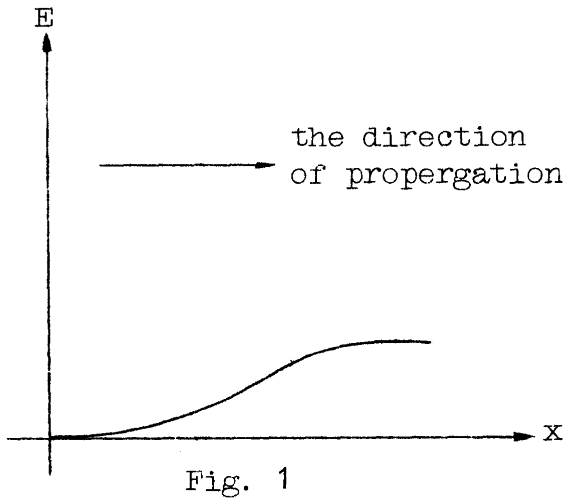

Fig. 1

As before from eqs. (10) we still have $F(E) + H = 0$ everywhere and $F(E) - H = r(x_o)$ along each c^+ characteristic issuing out of the respective initial point $x = x_o$ at $\tau = 0$;

$$\text{consequently } F(E) = -H = \frac{1}{2} r(x_o) = \text{constant along each } c^+ \qquad (11)$$

$$\text{so that} \qquad \lambda^+ = \lambda^+(x_o)(x - x_o) \qquad (12)$$

$$\text{in which} \qquad \lambda^+(x_o) = \{ 1 + 3\varepsilon \bar{E}^2(x_o) \}^{\frac{1}{2}} \qquad (13)$$

Since $\bar{E}(x_o)$ is a non-decreasing function of x_o, λ^+ increases as x_o increases. Therefore the straight c^+ characteristics cross among themselves. Thus the shock is formed. (c.f. Fig. 2) On the other hand if $|\bar{E}(x)|$ decreases as x increases, the solution represents the positively progressing wave front, which is flattened out as t increases.

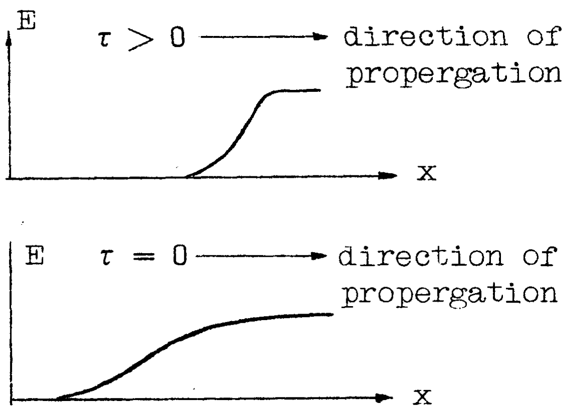

Fig. 2

4. The Propagation of Electromagnetic Shock

Introducing the column vector V through the equation

$$V = \begin{bmatrix} -H \\ \\ -(E+\varepsilon E^3) \end{bmatrix} \tag{14}$$

we may rewrite eq. (5) in the conservation form,

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial \tau} = 0 \tag{15}$$

Let us now consider a discontinuity propagating toward the positive x direction with a constant velocity, $\frac{dx}{d\tau} = \tilde{c} > 0$, by which two constant states are bounded. Beyond the dicontinuity the positive side on the x axis will be called as ahead and the quantities on this side will be specified by the suffix 0, whilst the negative side as behind and the quantities by the suffix 1. Then by means of Gaussian theorem it follows the generalized Rankine-Hugoniot relation(2):

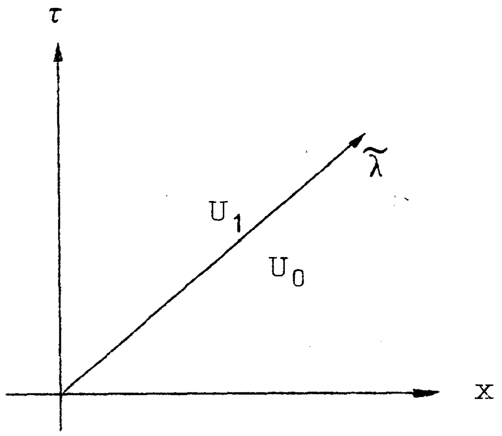

Fig. 3

$$[V] = \tilde{\lambda}[U] \tag{16}$$

in which $\tilde{\lambda} = 1/\tilde{c}$ and $[Q]$ stands for the jump, $Q_0 - Q_1$ for any quantity Q. If, for example, E_0, H_0 and E_1 are given, from (16) $\tilde{\lambda}$ and H_1 are determined, i.e.

$$\tilde{\lambda} = \pm\{1+\varepsilon(E_1^2+E_1E_0+E_0^2)\}^{\frac{1}{2}} \tag{17}$$

$$H_1 = H_0 - \tilde{\lambda}(E_0 - E_1) \tag{18}$$

If $E_1 = E_0$, then $\widetilde{\lambda}$ reduces to $\lambda^{\pm}(E_0)$; if $E_1 > E_0$,

$|\lambda^{\pm}(E_1)| > |\widetilde{\lambda}| > |\lambda^{\pm}(E_0)|$ whilst for $E_1 < E_0$, $|\lambda^{\pm}(E_0)| > |\lambda^{\pm}| > |\lambda^{\pm}(E_1)|$
From (18) we have

$$\widetilde{c}_{max} = \{ 1+3\varepsilon \text{ Min } (E_0^2,\ E_1^2) \}^{-\frac{1}{2}} \leqq 1$$

Therefore the propagation speed of discontinuity never exceeds $c/\sqrt{\alpha}$ consequently the light velocity in the vacuum. As was shown in the last section, the electric field behind shock is less than that in front when the shock is formed out of smooth wave. Hence in our terminology the shock for which $E_1 < E_0$ accordingly $\lambda^{+}(E_1)^{-1} > \widetilde{c} > \lambda^{+}(E_0)^{-1}$ seems to be physically relevant, whilst the shock for which $E_1 > E_0$ and $\widetilde{c} < \lambda^{+}(E_0)^{-1}$ does not. In order to justify this statement, we apply the evolutionary condition,[2] by means of which the unphysical shock waves can be excluded out so far as ordinary gas dynamic and hydromagnetic shocks are concerned. Since the evolutionary condition is formulated for the general conservation laws of hyperbolic type,[2] the result for the general case can directly be applied to the present case. Let us now restate the evolutionary condition[2] for the conservation law (15):

Evolutionary Condition ($\vec{E}$.1). A discontinuity is said to be evolutional if and only if the disturbances, which consist of outgoing waves, and the motion of the discontinuity, resulting from small amplitude disturbances incident upon the discontinuity, are both small and uniquely determined.

Evolutionary Condition (E.2). A discontinuity is evolutionary if and only if the number of small amplitude outgoing waves diverging from the discontinuity is equal to the number of the boundary conditions minus one, and at the same time the eigen vectors of A corresponding to these outgoing waves and the vector [V] are linearly independent provided of course that the disturbed boundary conditions resulting from eqs.(16) are independent.

The derivation of (E.2) from (E.1) for eq. (15) can be done in the same way as was done in the reference (2). The number of outgoing waves diverging from a discontinuity may be given by the phase velocity in the coordinate system moving with the discontinuity, $c_j^{\pm} - \tilde{c}$ (j=0.1), where $c^{\pm}$ is equal to $(\lambda^{\pm})^{-1}$ and the suffices j refers to the state ahead and behind the discontinuity. On the positive side of the x axis the positive value of $c_0^{\pm} - \tilde{c}$ corresponds to an outgoing wave whilst on the negative side the minus value corresponds. The number of outgoing waves may be counted by Fig. 4, which implies that

$$\tilde{c} > c_0^{+}$$

$$\tilde{c} > c_1^{+}$$

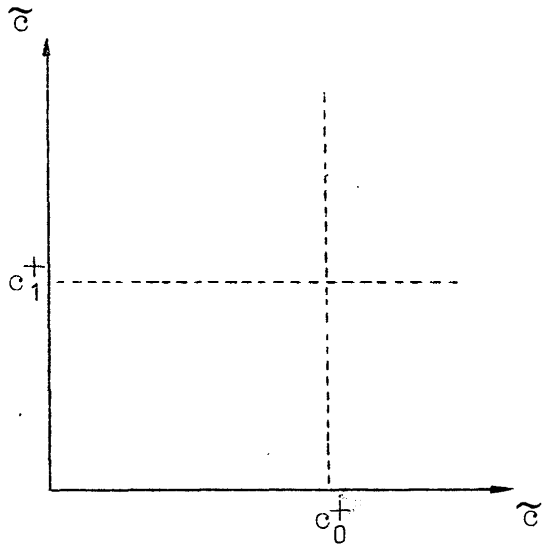

Fig. 4

consequently $E_0 > E_1$. Though the shock waves for which $E_0 < E_1$, consequently $c_1^{+} < \tilde{c} < c_0^{+}$, do satisfy the conservation laws, they are not evolutional and have to be excluded out as physically irrelevant. (The linear independence of the eigen vectors and [V] can be shown easily and also the independence of the boundary condition is obvious.)

Concluding Remarks

Though our model is too primitive to be compared with physical result, it seems to be interesting to estimate in the order of magnitude the time of shock formation, t_c which may be given roughly by the equation

$$t_c \simeq \frac{\lambda}{cE} \frac{1}{\sqrt{\beta}}$$

where λ is the characteristic wave length E the characteristic field strength. We assume, for example, $\beta \approx 10^{-16}$, $\lambda \approx 10^{-3}$cm and let the photon density be $10^{22}/\text{cm}^3$. Then $t_c \lesssim 10^{-8}$sec. consequently the corresponding distance x_c may be estimated as

$$x_c \lesssim 10^2 \text{ cm} \quad .$$

References

(1) Friedrichs, K. O., General Theory of High Speed Aerodynamics, pp 33 – 61, Oxford Univ. Press, London and New York, 1955
Lax, P. D., Communs. Pure & Appl. Math. 10 (1957) 537 – 566
Gel'fand, I. M. and Babenko, K. I., Nauch. Doklady Vysshei Shkoly, Fiz. Mat. Nauki, 1 (1958) 12 – 18

(2) Jeffrey, A. and Taniuti, T., Non-linear Wave Propagation (Academic Press) 1964, Chapter 3

(3) Akhiezer, A. I., Lubarski, G. J., and Polovin, R. V., Soviet Phys. JETP 8 (1959) 507 – 511
Syrovatskii, S. I., Soviet Phys. JETP 8 (1959) 1024 – 1027
Taniuti, T., Progr. Theor. Phys. 28 (1962) 756 –757
Jeffrey, A. and Taniuti, T., loc. cit., Chapt. 3, Chapt. 6 and Appendix E

(4) Jeffrey, A. and Taniuti, T., loc. cit. Chapter 2