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Effect of Magnetic Curvature on Density Gradient Drift Instabilities

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IPPJ-76 Errata

The stabilization condition of the slow Alfvén mode for $\omega_2 + \omega^* > 0$ should be read

$$\frac{b}{2} \quad \frac{|\omega_2|}{\omega_n} < \frac{2r}{|R_c|} < b,$$

because the magnitude of stabilizing curvature derived from eq. (3, 10) has to be restricted by the condition $\omega_2 + \omega^* > 0$.

The effect of magnetic curvature on density gradient drift instabilities is examined for a collisionless plasma in which β satisfies $m/M < \beta \ll 1$, where β = plasma pressure / magnetic pressure, m and M are the electron and ion masses. For simplicity, the effect of magnetic curvature is simulated by an equivalent gravitational field. The present considerations are restricted to the cases where the wavelength of perturbations is much longer than the mean ion Larmor radius and the radius of magnetic curvature is much larger than the characteristic length of the density gradient. It is found that the slow Alfvén mode which is unstable in a straight and parallel magnetic field is stabilized by magnetic curvature with a sign favorable to stabilization of the interchange instability while it remains unstable by unfavorable curvature, and that the fast Alfvén mode remains stable independently of the sign of magnetic curvature.

§ 1. Introduction

Recently the shear stabilization of drift instabilities driven by density gradient in a collisionless plasma has been studied intensively. The stabilizing effect of magnetic shear is to increase the component of the wave vector of perturbations parallel to the sheared magnetic field, resulting in the increase of the ion Landau damping. On the other hand, magnetic curvature shifts the resonance frequency for constant wave vector to either a lower value or a higher value depending on the sign of magnetic curvature. Krall and Rosenbluth[1)] have considered the curvature effect by introducing an equivalent gravitational field and have found that favorable curvature can stabilize the electrostatic drift instability with a wavelength longer than the mean ion Larmor radius. Taking into account the dispersion of the curvature drift velocity, Laval et al.[2)] have shown that favorable curvature has strong stabilizing effects even on the short wavelength perturbations because the ion Landau damping becomes effective in the direction of curvature drift as well as along the magnetic field. However, the two papers concern only with a low β limit in which case the electrostatic approximation may be permitted.

In the present paper, therefore, a plasma in which $1 \gg \beta > m/M$ is considered and the effects of magnetic curvature on the Alfvén type modes are examined by use of an equivalent gravitational field; the slow Alfvén mode is found to be stabilized by some small amount of favorable curvature, while Mikhailovskaya and Mikhailovsky[3)] have shown that the flute perturbations with $k_z \neq 0$ cannot be stabilized even by favorable curvature.

In the next section, the general dispersion relation in the presence of an equivalent gravitational field is derived by the same method as used in the previous paper by the present author.[4] In § 3, the effects of magnetic curvature on the Alfvén modes are examined in detail.

§ 2. Derivation of the Dispersion Relation

We consider a plane plasma slab in which a density gradient ∇n is in the x direction, the magnetic field $\vec{B}_o$ is in the z direction and a gravitational field equivalent to the effect of magnetic curvature is parallel to the x direction. Under the assumption $1 \gg \beta > m/M$, only the bending of the perturbed magnetic field is considered and the compression is neglected. The local dispersion relation is derived by neglecting the x dependence of the perturbations as usually done in the case with a uniform magnetic field. The method of derivation of the dispersion relation is the same as in Appendix A of ref. 4.

The equilibrium distribution function which is a function of constants of the motion may be taken as

$$f_{oj} = n_{oj}\left(\frac{m_j}{2\pi T_j}\right)^{3/2} \left\{ 1 - \varepsilon_j\left(x + v_y/\Omega_j\right)\right\} \exp\left\{ -\left(\frac{1}{2}m_j v^2 + m_j g_j x\right)/T_j \right\} \qquad (2.1)$$

in which j refers to the particle species, v, m and T are the velocity, mass and temperature, g_j is a gravitational acceleration equivalent to the effect of magnetic curvature $1/R_c$ and equal to $2T_j/m_jR_c$, $\Omega_j = e_jB_o/m_jc$, and

$$\varepsilon_j = -\frac{d\ell n n_j}{dx} + \frac{m_j g_j}{T_j} . \qquad (2.2)$$

The unperturbed distribution function f_{oj} given by eq.(2.1) satisfies the Vlasov and Maxwell equations

$$\vec{v}\nabla f_{oj} + \left\{ \frac{e_j}{m_j} (\vec{v} \times \vec{B}_o) + g e_x \right\} \cdot \nabla_v f_{oj} = 0 \tag{2.4}$$

$$\nabla \cdot \vec{E}_o = 4\pi \sum_j e_j \int f_{oj} d^3v = 0 \tag{2.5}$$

$$\nabla \times \vec{B}_o = \frac{4\pi}{c} \sum_j e_j \int f_{oj} v_z d^3v$$

From eq.(2.4) we obtain $\varepsilon_e = \varepsilon_i$ and $\frac{d\ell n n_e}{dx} = \frac{d\ell n n_i}{dx}$.

We introduce the vector potential $A\vec{e}_z$ as well as the scaler potential ϕ for the perturbed fields $\vec{E}_1$ and $\vec{B}_1$

$$\vec{E}_1 = - \nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t} \vec{e}_z ,$$

$$\vec{E}_1 = \nabla \times (A\vec{e}_z) .$$

ϕ and A satisfy the equations

$$(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) \phi = - 4\pi \sum_j e_j \int f_{1j} d^3v , \tag{2.6}$$

$$(\Delta - \frac{1}{c^2}\frac{\partial}{\partial t^2}) A = - \frac{4\pi}{c} \sum_j e_j \int f_{1j} v_z d^3v , \tag{2.7}$$

where f_{1j} is a small perturbation from the equilibrium distribution and given by solving the linearized Vlasov equation

$$\frac{\partial f_{1j}}{\partial t} + \vec{v}\cdot\nabla f_{1j} + (\frac{e_j}{m_j}\vec{v}\times\vec{B}_o + g\vec{e}_x)\cdot\nabla_v f_{1j} = -\frac{e_j}{m_j}(\vec{E}_1 + \frac{1}{c}\vec{v}\times\vec{B}_1)\cdot\nabla_v f_{o_j} \ . \tag{2.8}$$

Equation (2.8) is solved by the method of characteristics

$$f_{1_j} = -\frac{e_j}{m_j}\int_{-\infty}^{t}(\vec{E}_1 + \frac{1}{c}\vec{v}\times\vec{B}_1)\cdot\frac{\partial f_{o_j}}{\partial\vec{v}}\,dt', \tag{2.9}$$

where the integration is carried out along the particle orbit :

$$y(t') - y = \frac{v_\perp}{\Omega}\{\cos(\theta - \Omega t') - \cos\theta\} - (g/\Omega)t' \ ,$$

$$z(t') - z = v_z\ t' \ . \tag{2.10}$$

A time-space dependence of the perturbations is assumed as follows:

$$\begin{Bmatrix} f_1 \\ \phi \\ A \end{Bmatrix} = \begin{Bmatrix} \bar{f}_1 \\ \bar{\phi} \\ \bar{A} \end{Bmatrix} \exp\{\ i\ (k_y y + k_z z + \omega t\ )\} \ . \tag{2.11}$$

Treating $\bar{\phi}$, $\bar{A}$ and $\bar{f}_1$ as constants, the local dispersion relation at $x = 0$ is obtained; substitution of the perturbed distribution function eq.(2.9) into eqs.(2.6) and (2.7) gives us the homogeneous equations for $\bar{\phi}$ and $\bar{A}$

$$\sum_j k_{dj}^2 \{\ 1 - (1 - \frac{\omega_j^*}{\omega})\ \frac{\omega}{\omega_j'}\ (1 + W_j)\ e^{-b_j} I_o(b_j)\}\bar{\phi}$$

$$- \frac{\omega}{xk_z} \sum_j k_{dj}^2 \left(1 - \frac{\omega_j^*}{\omega} \right) e^{-b_j} I_o(b_j) W_j \bar{A} = 0, \qquad (2.12)$$

$$\frac{\omega}{ck_z} \sum_j k_{dj}^2 \left(1 - \frac{\omega_j^*}{\omega} \right) e^{-b_j} I_o(b_j) W_j \bar{\Phi}$$

$$+ \left\{ k^2 + \left(\frac{\omega}{ck_z}\right)^2 \sum_j k_{dj}^2 \left(1 - \frac{\omega_j^*}{\omega} \right) \frac{\omega_j'}{\omega} e^{-b_j} I_o(b_j) W_j \right\} \bar{A} = 0, \qquad (2.13)$$

where

$$k_{dj}^2 = 4\Pi n_j e_j^2 / T_j ,$$

$$\omega_j^* = \frac{k_y T_j}{m_j \Omega_j} \left(- \frac{1}{n_j} \frac{dn_j}{dx} + \frac{m_j g_j}{T_j} \right) \equiv \omega_{nj} + \omega_{gj} ,$$

$$\omega_j' = \omega - \frac{k_y g_j}{\Omega_j} \equiv \omega - \omega_{gj} ,$$

$$b_j = \frac{k_y^2 v_j^2}{2\Omega_j^2} = \frac{k_y^2 a_j^2}{2} ,$$

$$W_j = W\left(\frac{\omega_j'}{k_z v_j}\right), \quad W(z) = - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t\, e^{-t^2}}{t + z} dt .$$

Here v_j and a_j are the thermal velocity and mean Larmor radius, and we have assumed $\omega^2 << k^2c^2$, $|\omega| << \Omega_j$ and quasi-neutrality of charge.

The dispersion relation derived from eqs.(2.12) and (2.13) is arranged as follows

$$(B - C) \{ k^2 - (\omega/ck_z)^2 D \} - A \{ k^2 - (\omega/ck_z)^2 (B - C + D)\} = 0, \qquad (2.14)$$

where

$$A = \sum_j k_{dj}^2 (1 - \omega_j^*/\omega) e^{-b_j} I_o(b_j) W_j, \qquad (2.15)$$

$$B = \sum_j k_{dj}^2 \{ 1 - (1 - \omega_j^*/\omega) e^{-b_j} I_o(b_j)\}, \qquad (2.16)$$

$$C = \sum_j k_{dj}^2 (1 - \omega_j^*/\omega) (\omega_{gj}/\omega_j') (1 + W_j) e^{-b_j} I_o(b_j), \qquad (2.17)$$

$$D = \sum_j k_{dj}^2 (1 - \omega_j^*/\omega) (\omega_{gj}/\omega) e^{-b_j} I_o(b_j) W_j . \qquad (2.18)$$

When $g = 0$, we have $C = D = 0$ and eq.(2.14) is reduced to eq.(A.15) in ref. 4.

§ 3. Effect of Magnetic Curvature

In this section, we restrict our considerations only to the long wavelength perturbations compared with the mean ion Larmor radius, that is, $b_i = b << 1$. The radius of magnetic curvature R_c is assumed to be much larger than the characteristic length of the density gradient $r = |n/n'|$. We assume further $v_i << |\omega/k_z| << v_e$, $k_z << k_y$ and and $T_e = T_i = T$. Then we have approximately

$$A = k_d^2 \{(1 + \omega^*/\omega) (i\sqrt{\pi} \frac{\omega'_e}{k_z v_e} - 1)+(1 - \omega^*/\omega) e^{-b} I_o \tfrac{1}{2}(k_z v_i/\omega_j')^2\} , \qquad (3.1)$$

$$B = k_d^2 \left(1 - \omega^*/\omega \right)\left(1 - e^{-b} I_o \right) , \qquad (3.2)$$

$$C = k_d^2 \left\{ \left(1 + \omega^*/\omega \right)\left(-i\sqrt{\pi}\, \frac{\omega_g}{\omega_e'} \frac{\omega_e'}{k_z v_e} \right) + \left(1 - \omega^*/\omega \right)\left(\omega_g/\omega_i' \right) e^{-b} I_o \right\} , \qquad (3.3)$$

$$D = k_d^2 \left(1 + \omega^*/\omega \right)\left(\omega_g/\omega \right)\left(1 - i\sqrt{\pi}\, \frac{\omega_e'}{k_z v_e} \right) , \qquad (3.4)$$

where $k_{de} = k_{di} = k_d$, $\omega_i^* = -\omega_e^* = \omega^*$, $\omega_{gi} = - \omega_{ge} = \omega_g$; unlabeled parameters refer to ions. Substituting eqs.(3.1) - (3.4) into eq.(2.14), we obtain

$$\left\{ \left(1 + \frac{\omega^*}{\omega} \right)\left(1 - i\sqrt{\pi}\, \frac{\omega_e'}{k_z v_e} \right) - \left(1 - \frac{\omega^*}{\omega} \right) e^{-b} I_o \frac{k_z^2 v_i^2}{2\omega_i'^2} \right\} \times$$

$$\times \left\{ \omega(\omega-\omega^*)\left(1 - \frac{\omega_g}{\omega_i'} \frac{e^{-b} I_o}{1-e^{-b} I_o} \right) + \frac{\omega_g(\omega+\omega^*)}{1-e^{-b} I_o}\left(1 - i\sqrt{\pi}\, \frac{\omega_e'}{k_z v_e} \frac{\omega_g}{\omega_e'} \right) - \frac{b k_z^2 v_A^2}{1-e^{-b} I_o} \right\}$$

$$= b k_z^2 v_A^2 \left\{ \left(1 - \frac{\omega^*}{\omega} \right)\left(1 - \frac{\omega_{ge}^{-b} I_o}{\omega_i'(1-e^{-b} I_o)} \right) + i\sqrt{\pi}\, \frac{\omega_e'}{k_z v_e} \frac{\omega_g}{e} \frac{1+\omega^*/\omega}{1-e^{-b} I_o} \right\} \times$$

$$\times \left\{ 1 + \frac{\omega_g(\omega+\omega^*)}{b k_z^2 v_A^2} \left(i\sqrt{\pi}\, \frac{\omega_e'}{k_z v_e} - 1 \right) \right\} . \qquad (3.5)$$

Thus the frequencies of the Alfvén type modes are derived approximately from the equation

$$\omega^2 - \omega\omega^* - k_z^2 v_A^2 + 2\omega^* \omega_g / b = 0 \qquad (3.6)$$

where v_A is the Alfvěn speed. The solutions of eq.(3.6) give us two modes

$$\omega_{1,2} = \frac{\omega^*}{2} \pm \sqrt{\left(\frac{\omega^*}{2}\right)^2 + k_z^2 v_A^2 - 2\omega^*\omega_g/b} \; ; \qquad (3.7)$$

in the absence of magnetic curvature, the fast mode is always stable and for $2\omega^{*2} >> k_z^2 v_A^2$ the slow mode is unstable. Substituting $\omega = \omega_{1,2} + \delta\omega$ into eq.(3.5), the imaginary part of $\delta\omega$ is obtained after some careful calculations

$$\mathrm{Im}(\delta\omega) = \sqrt{\pi}\, \frac{\omega_{1,2} + \omega_g}{k_z v_e} \frac{b k_z^2 v_A^2}{(2\omega_{1,2} - \omega^*)(\omega_{1,2} + \omega^*)} \times$$

$$\times \left[(\omega_{1,2} - \omega^*)\left\{ 1 - \frac{\omega_g}{b(\omega_{1,2} - \omega_g)} \right\} + \frac{\omega_g(\omega_{1,2} + \omega^*)}{b(\omega_{1,2} + \omega_g)} \right]. \qquad (3.8)$$

Let us restrict our considerations to the case where $b k_z^2 v_A^2 > 2\omega_g\omega^*$, then we have Im ω = Im $\delta\omega$ independently of the sign of g or R_c. It must be noted that the assumption $b k_z^2 v_A^2 > 2|\omega_g|\omega$ is reduced to a simple relation

$$(k_z|R_c|)(k_z r) > \beta \quad . \qquad (3.9)$$

dn/dx is taken to be negative so that $\omega_n = -(k_y T/M\Omega)(d\ell nn/dx) > 0$, and $\omega^* = \omega_n + \omega_g$ is positive independently of the sign of g or R_c since we have assumed $r << |R_c|$. Thus we find $\omega_1 > 0$, $\omega_2 < 0$, $2\omega_1 - \omega^* > 0$, $2\omega_2 - \omega^* < 0$, $\omega_1 \pm \omega_g > 0$, and $\omega_2 \pm \omega_g < 0$.

(I) Effect of favorable curvature (g < 0, cusp curvature)

(i) The slow Alfvén mode

If $\omega_2 + \omega^* > 0$, the slow Alfvén mode is unstable in the absence of magnetic curvature, and the curvature stabilization is attained provided that the sum of the terms within the square bracket on the right-hand side of eq.(3.8) becomes positive

$$(\omega_2 - \omega^*)\{ 1 - \frac{\omega_g}{b(\omega_2 - \omega_g)}\} + \frac{\omega_g(\omega_2 + \omega^*)}{b(\omega_2 + \omega_g)} > 0 \ . \qquad (3.10)$$

Let us first study the effects of favorable curvature. Since ω_g is negative, the second term is positive. Thus the stabilization condition is given by

$$b \ < \ \frac{\omega_g}{\omega_2 - \omega_g} \ , \qquad (3.11)$$

which may be approximately rewritten as

$$k_z^2 v_A^2 \ - \ \frac{\omega_n \omega_g}{b} \ < \ (\frac{\omega_g}{b})^2 \ . \qquad (3.1])$$

If $k_z^2 v_A^2$ is roughly equal to $-\omega_n\omega_g/b$, the stabilization condition is further simplified

$$b < \frac{\omega_g}{\omega_n} = \frac{2r}{|R_c|} \ . \qquad (3.13)$$

Therefore this mode is easily stabilized by small favorable curvature.

In the case where $\omega_2 + \omega^* < 0$, the slow mode is known to be stable in the absence of curvature, and it remains stable after the introduction of favorable curvature provided that an unequality reverse to eq.(3.10) is satisfied. Now the second term is negative, and if the sum of the first and second terms becomes negative this mode remains surely stable. That is, if

$$k_z^2 v_A^2 > 2\omega_g^2 /b, \qquad (3.14)$$

this mode remains stable. The condition eq.(3.14) is easily shown to be reduced to

$$(k_z R_c)^2 > 2\beta \qquad (3.15)$$

and it is always satisfied under the assumption eq.(3.9).

(ii) The fast Alfvén mode

It is easily found that the fast mode remains stable even by curvature with any sign. Now we have $\omega_1 > 0$, $2\omega_1 - \omega^* > 0$, and $\omega_1 \pm \omega^* > 0$. Therefore the stability condition is given by

$$(\omega_1 - \omega^*)\{1 - \frac{\omega_g}{b(\omega_1-\omega_g)}\} + \frac{\omega_g(\omega_1+\omega^*)}{b(\omega_1+\omega_g)} > 0 \qquad (3.16)$$

The first term is positive and the second term is negative. This stability condition is found to be reduced to eq.(3.14) or eq. (3.15) and always satisfied under the present assumptions.

(II) Effect of unfavorable curvature ($g > 0$, mirror curvature)

(i) The slow Alfvén mode

The stabilization condition given by eq.(3.10) for $\omega_2 + \omega^* > 0$ cannot be satisfied by unfavorable curvature because each term on the left-hand side of eq.(3.10) is always negative for $g > 0$. The growth rate of this mode for $\omega_2 + \omega^* > 0$ increases by unfavorable curvature. On the other hand, the stabilization condition for the case

$\omega_2 + \omega^* < 0$ is an inequality reverse to eq.(3.10) and given by eq.(3.14) or eq.(3.15) which is independent of the sign of g. Thus this mode for $\omega_2 + \omega^* < 0$ remains stable by unfavorable curvature.

(ii) The fast Alfvén mode

The second term on the left-hand side of the stabilization condition eq.(3.16) is positive for $g > 0$. The stabilization condition is found to be the same as eq.(3.14) or eq.(3.15). Thus this mode remains stable by unfavorable curvature.

§ 4. Conclusion

We have examined the curvature effects on the Alfvén drift modes with long wavelength by simulating the effects of true curvature by a gravitational field. It has been shown that the stable modes in the absence of curvature remain stable independently of the sign of magnetic curvature provided that $(k_z R_c)^2 > 2\beta$ is satisfied. This condition may be satisfied for most experimental devices only if k_z is finite. The unstable Alfvén mode may be stabilized by introducing small favorable curvature which satisfies $b<2r/|R_c|$; by unfavorable curvature, however, it cannot be stabilized and its growth rate increases.

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