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Asymptotic Method and Nonlinear Behaviors of Waves in Plasma

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Nonlinear behaviors of monochromatic traveling waves are discussed by the use of the asymptotic method of Bogoliubov and Mitropolsky. The wave is assumed to be stationary in the first approximation and nonlinear interactions are taken into account only among the wave under consideration and its harmonics. It is shown that two cases should be distinguished according as the frequency of the wave in the first approximation is given by a simple root or a double root of the dispersion relation. The dominant corrections to the frequency and the growth rate due to nonlinearity are of the second order with respect to the amplitude in the former case, but is of the first order in the latter case. Characteristic behaviors of the wave in the latter case are studied by discussing a model equation and a special case of two beam instability.

§ 1. Introduction

Plasma is a nonlinear medium. If the amplitude of a wave propagating through a uniform plasma is infinitesimally small the wave is characterized by the dispersion relation of the linear theory. But when the amplitude becomes finite, several nonlinear effects occur.

In the linear theory, we consider usually a simple harmonic wave. But in a nonlinear medium a harmonic wave is always accompanied with its higher harmonics. Accordingly the smallest closed unit which we must consider in a nonlinear theory is the whole set of the wave under consideration and its harmonics.

Suppose that we want to study a dispersion relation of a wave and excite the wave by an external force with a prescribed wavelength. When the amplitude becomes large, we will see that higher harmonics are excited as well as the wave under consideration. Then we stand at the situation in which we must analyse the phenomena considering the whole set of waves mentioned above.

The nonlinearity brings about also the deviation of dispersion relation from that in the linear theory[1)]. When the amplitude of the wave is fairly small, we can consider the nonlinearity as a perturbation. But in this case the ordinary perturbation expansion technique is not useful, because it brings about terms secular with respect to time. This difficulty can be avoided by allowing of a small shift of the frequency and a slow change of the amplitude due to the nonlinearity. The method for such is given systematically in a book by Bogoliubov and Mitropolsky[2)], and several modified version of the method have been applied to problems in plasma physics[3) 4)].

One of the most interesting problems in the theory of nonlinear waves is the determination of the wave amplitude. Usually we expect that when the growth rate of a linearly unstable wave is small, the amplitude will be saturated at a low level owing to the nonlinearity. This is, however, not always true. If both the growth rate and the saturated amplitude are really small, we will be able to determin the amplitude by using the asymptotic method of Bogoliubov and Mitropolsky[5]. A typical example of such cases is given by the well-known Van der Pol equation:

$$\frac{d^2u}{dt^2} - (\mu - u^2) \frac{du}{dt} + u = 0 \quad ,$$

where μ is a small positive parameter[6].

In this paper we study some typical nonlinear behaviors of waves, which are traveling, simple harmonic waves in the first approximation. In § 2, we give a general formulation. There it is shown that two cases must be distinguished according as the frequency of the wave in the first approximation is determined as a simple root or a double root of the dispersion relation. In the former case, the asymptotic method can be applied straightforwardly. But in the case of a double root, the situation is a little different. The section 3 is devoted to a model equation of the latter case, in which the nonlinearity tends to suppress the growth of the amplitude. The two-stream instability gives also examples of the double-root case. In § 4 we treat a special case of two-beam instability, in which the nonlinearity is shown not to suppress but to enhance the growth of the wave. A more detailed analysis of two-stream instability in this method will be given in subsequent papers.

§ 2. General Formulation

Longitudinal waves or waves in the quasi-electrostatic approximation in a steady uniform plasma are governed by an equation of the type[7)]

$$\varepsilon (k,\omega) E_{k\omega} = (EE) + (EEE) + \cdots\cdots \quad , \qquad (2.1)$$

where $\varepsilon(k,\omega)$ is the dielectric permeability of the plasma in the linear theory, $E_{k\omega}$ is the Fourier component of the electric field and (EE), (EEE), $\cdots$ represent the nonlinear terms quadratic, cubic, $\cdots$ with respect to $E_{k\omega}$. When the wave is neither longitudinal nor quasi-electrostatic, we need only to replace $\varepsilon(k,\omega)$ in eq.(2.1) by the dispersion function $D(k,\omega)$ and a similar argument can be made. We shall neglect the quasi-linear effect, that is the temporal changes of characteristics of the plasma as a medium.

In the configuration space, eq.(2.1) takes the form

$$\varepsilon \left(\frac{1}{i}\frac{\partial}{\partial x}, \ -\frac{1}{i}\frac{\partial}{\partial t} \right) E = (EE) + (EEE) + \cdots \quad , \qquad (2.2)$$

We divide $\varepsilon(k,\omega)$ into two parts:

$$\varepsilon (k,\omega) = \varepsilon_0 (k,\omega) - \mu\varepsilon' (k,\omega) \quad , \qquad (2.3)$$

where $\varepsilon_0(k,\omega)$ is the main part, which is assumed to be a real function of k and ω, and $\mu\varepsilon'(k,\omega)$ is a small deviation from it, characterized by a small parameter μ . We take $\varepsilon_0(k,\omega)$ such that the dispersion relation

$$\varepsilon_0(\ k,\omega\ )\ =\ 0 \tag{2.4}$$

has a real root $\omega = \omega_0(k)$. This means that the plasma characterized by $\varepsilon_0(k,\omega)$ sustains a stationary wave with a prescribed k.

Now we consider a wave of a small amplitude of the order λ and replace E by λE. Then the basic equation (2.2) becomes

$$\varepsilon_0\left(\frac{1}{i}\frac{\partial}{\partial x},\ -\frac{1}{i}\frac{\partial}{\partial t} \right)E = \mu\varepsilon'\left(\frac{1}{i}\frac{\partial}{\partial x},\ -\frac{1}{i}\frac{\partial}{\partial t} \right)E$$

$$+\ \lambda(EE) + \lambda^2(EEE) + \cdots\ . \tag{2.5}$$

In the example of the Van der Pol equation, this corresponds to the arrangement

$$\frac{d^2u}{dt^2} + u = \mu\frac{du}{dt} - \lambda^2u^2\frac{du}{dt}\ . \tag{2.6}$$

The equation in the first approximation

$$\varepsilon_0\left(\frac{1}{i}\frac{\partial}{\partial x},\ -\frac{1}{i}\frac{\partial}{\partial t} \right) E = 0 \tag{2.7}$$

has a solution of the following form

$$E_0 = a \exp i(\ kx - \omega_0 t) + \text{c.c.}\ , \tag{2.8}$$

where a, k and ω_0 are real and c.c. means the complex conjugate of the first term. Our purpose of the following analysis is to study the effect of the nonlinearity on a solution of the form (2.8).

We assume the following form of solution

$$E = a \exp i(kx - \omega_0 t - \psi) + c.c. + \lambda E_1 + \lambda^2 E_2 + \cdots , \tag{2.9}$$

and substitute this into eq.(2.5). Then the righthand side of eq.(2.5) can be regarded as external forces to the unperturbed system. If we take a and ψ as constants, then the terms proportional to $\exp i(kx - \omega_0 t)$, which appear in the righthand side of eq.(2.9), resonate with the unperturbed system and bring about the appearance of terms which are secular in t. In order to avoid the appearance of secular terms, we allow slow variations of a and ψ:

$$\frac{da}{dt} = \lambda A_1(a) + \lambda^2 A_2(a) + \cdots , \tag{2.10}$$

$$\frac{d\psi}{dt} = \lambda \omega_1(a) + \lambda^2 \omega_2(a) + \cdots , \tag{2.11}$$

of which A_1, ω_1, A_2, ω_2, $\cdots$ are assumed to depend on the amplitude a only and are to be determined by the condition of non-secularity.

The next relation is an identity for a function f(x,t) which can be expressed in the form of a Fourier expansion in t:

$$\varepsilon_0\left(\frac{1}{i}\frac{\partial}{\partial x}, -\frac{1}{i}\frac{\partial}{\partial t}\right) f = e^{-i\omega_0 t}\, \varepsilon_0\left(\frac{1}{i}\frac{\partial}{\partial x}, \omega_0 + i\frac{\partial}{\partial t}\right)\left[e^{i\omega_0 t} f \right] . \tag{2.12}$$

Applying this identity to

$$f = a e^{i(kx - \omega_0 t - \psi)}$$

we have

$$\varepsilon_0 \left(\frac{1}{i}\frac{\partial}{\partial x}, -\frac{1}{i}\frac{\partial}{\partial t} \right) \left[a e^{i(kx - \omega_0 t - \psi)} \right]$$

$$= e^{i(kx - \omega_0 t)} \, \varepsilon_0\left(k, \omega_0 + i\frac{\partial}{\partial t} \right) \left[a e^{-i\psi} \right]$$

$$= e^{i(kx - \omega_0 t)} \left\{ \varepsilon_0(k, \omega_0) + \left(\frac{\partial \varepsilon_0}{\partial \omega} \right)_{\omega=\omega_0} i\frac{\partial}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 \varepsilon_0}{\partial \omega^2} \right)_{\omega=\omega_0} \left(i\frac{\partial}{\partial t} \right)^2 + \cdots \right\} \left[a e^{-i\psi} \right]$$

$$= e^{i(kx - \omega_0 t - \psi)}$$

$$\times \left\{ \left(\frac{\partial \varepsilon_0}{\partial \omega} \right)_{\omega=\omega_0} i\left[\lambda(A_1 - i a\omega_1) + \lambda^2(A - i a\omega_2) + \cdots \right] \right.$$

$$- \frac{1}{2} \left(\frac{\partial^2 \varepsilon_0}{\partial \omega^2} \right)_{\omega=\omega_0} \left[\lambda^2 \left[\left(A_1 \frac{dA_1}{da} - a{\omega_1}^2\right) - iA_1 \left(\frac{d}{da}(a\omega_1) \right] \right. \right.$$

$$\left. + \cdots \right] + \cdots \Big\} \quad . \qquad (2.13)$$

where we have used eqs.(2.4), (2.10) and (2.11)

So far, μ and λ are small parameters independent of each other. The parameter μ indicates, for example, the order of linear growth rate. An interesting choice of the amplitude λ is such that the contribution of the nonlinearity is the same order as that from the term $\mu\varepsilon' E$. It can be shown that (EE) term cannot yield the first harmonic in the order λ, that is terms proportional to $\exp i(kx - \omega_0 t - \psi)$. Thus the first harmonic, which is the origin of the change of amplitude and the frequency shift, appears first in the order λ^2. Consequently we should retain the terms of the order λ^2 in $\mu\varepsilon' E$, too. The appropriate choice is $\mu = \lambda^2$ if $\varepsilon'(k, \omega_0) \neq 0$ and $\mu = \lambda$ if $\varepsilon'(k, \omega_0) = 0$, so that we take $\mu\varepsilon'$ in the following form:

$$\mu\varepsilon'(k, \omega) = \lambda\varepsilon_1 (k, \omega) + \lambda^2\varepsilon_2(k, \omega) , \qquad (2.14)$$

where $\varepsilon_1(k, \omega_0) = 0$.

Then a similar procedure to the previous one yields

$$\mu\varepsilon'\left(\frac{1}{i}\frac{\partial}{\partial x} , - \frac{1}{i}\frac{\partial}{\partial t} \right) E$$

$$= \lambda^2 \left\{ \left(\frac{\partial \varepsilon_1}{\partial \omega} \right)_{\omega=\omega_0} \left[i(A - ia\omega_1)\, e^{i(kx - \omega_0 t - \psi)} + c.c. \right] \right.$$

$$\left. + \left[(s_r + is_i) a e^{i(kx - \omega_0 t - \psi)} + c.c.\right]\right\} + O(\lambda^3) , \qquad (2.15)$$

where

$$\varepsilon_2(k, \omega_0) = s_r + is_i \qquad (2.16)$$

Now we must treat two cases separately.

I. Normal Case, $(\partial\varepsilon_0/\partial\omega)_{\omega=\omega_0} \neq 0$.

When ω_0 is a simple root of the dispersion relation, the nonsecularity condition in the order λ gives

$$i \left(\frac{\partial\varepsilon_0}{\partial\omega}\right)_{\omega=\omega_0} (A_1 - ia\omega_1) = 0 , \tag{2.17}$$

from which we have

$$A_1 = 0 \quad \text{and} \quad \omega_1 = 0 . \tag{2.18}$$

Thus the wave suffers no effect in the order λ .

In the next order, the nonsecularity condition gives

$$\left(\frac{\partial\varepsilon_0}{\partial\omega}\right)_{\omega=\omega_0} i(A_2 - ia\omega_2)$$

$$= (s_r + is_i)\, a + (\alpha_r + i\alpha_i)\, a^3 , \tag{2.19}$$

where the second term on the righthand side comes from the nonlinear terms of eq.(2.5) and α_r and α_i are constants. It can be easily shown that this contribution should be proportional to a^3 . Consequently the growth rate A_2 and the frequency shift ω_2 are given by

$$A = \left(\frac{\partial\varepsilon_0}{\partial\omega}\right)^{-1}_{\omega=\omega_0} (s_i a + \alpha_i a^3) , \tag{2.20}$$

$$\omega_2 = \left(\frac{\partial \varepsilon_0}{\partial \omega} \right)^{-1}_{\omega=\omega_0} (s_r + \alpha_r a^2) \quad . \tag{2.21}$$

Since this case is simple and has been already treated frequently, we shall no further discuss on it here.

II Exceptional Case. $(\partial\varepsilon_0/\partial\omega)_{\omega=\omega_0} = 0$ but $(\partial^2\varepsilon_0/\partial\omega^2)_{\omega=\omega_0} \neq 0$.

This situation arises, e.g., in the case of two beam system near its marginally stable state. In this case the first harmonic in the order λ disappears automatically, and the nonsecularity condition in the order λ^2 gives

$$- \frac{1}{2}\left(\frac{\partial^2\varepsilon_0}{\partial\omega^2} \right)_{\omega=\omega_0} \left[\left(A_1\frac{dA_1}{da} - a\omega_1^2 \right) - iA_1\left(\frac{d}{da}(a\omega_1) + \omega_1 \right) \right]$$

$$= \left(\frac{\partial\varepsilon_1}{\partial\omega} \right)_{\omega=\omega_0} i(A_1 - ia\omega_1) + (s_r + is_i)a + (\alpha_r + i\alpha_i)a^3 \ . \tag{2.22}$$

The term $\varepsilon_1(k, \omega)$ arises, e.g., from the collision term, in which case $(\partial\varepsilon_1/\partial\omega)_{\omega=\omega_0}$ is pure imaginary. Then from eq. (2.22) we obtain

$$\left.\begin{aligned} A_1\frac{dA_1}{da} - a\omega_1{}^2 + \nu A_1 &= -\left(\frac{\partial^2\varepsilon_0}{\partial\omega^2} \right)^{-1}_{\omega=\omega_0} 2(s_r a + \alpha_r a^3) \ , \\ A_1\frac{d}{da}(a\omega_1) + A_1\omega_1 + \nu a\omega_1 &= \left(\frac{\partial^2\varepsilon_0}{\partial\omega^2} \right)^{-1}_{\omega=\omega_0} 2(s_i a + \alpha_i a^3) \ , \end{aligned}\right\} \tag{2.23}$$

where we have assumed that $\nu = 2i[(\partial\varepsilon_1/\partial\omega)/(\partial^2\varepsilon_0/\partial\omega^2)]_{\omega=\omega_0}$ is real, avoiding unnecessary generality.

§ 3. Investigation of Exceptional Case by a Model Equation

A typical example of the exceptional case is given by the following model equation with a small positive parameter λ :

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x} \right)^2 v - \lambda^2 \left(2 + \ell^2 \frac{\partial^2}{\partial x^2} - v^2 \right) v = 0 , \qquad (3.1)$$

where V and ℓ are constants. The linearized version of this equation has an unstable solution, the frequency of which is given by

$$\omega = kV + i\lambda(2-k^2\ell^2)^{1/2} \qquad (3.2)$$

where k is the wave number of the wave and is chosen so as to satisfy the inequality $(k\ell)^2 < 2$. The effective growth rate, i.e. an average of $\lambda(2 - k^2\ell^2 - v^2)^{1/2}$ decreases with the increase of the amplitude, and for a sufficiently large amplitude we may expect that the wave will become stable.

At first we arrange eq.(3.1) in the form of eq.(2.5)

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right)^2 v = \lambda^2 \left(2 + \ell^2 \frac{\partial^2}{\partial x^2} \right) v - \lambda^2 v^3 , \qquad (3.3)$$

and seek for a solution of the form

$$v = a\cos(kx - \omega_0 t - \psi) + \lambda v_1 + \lambda^2 v_2 + \cdots , \qquad (3.4)$$

where $\omega_0 = kV$. We choose k as ℓ^{-1}. Owing to the term $-\lambda^2\ell^2\, \partial^2 v/\partial x^2$ in eq.(3.1), this choice of k ($k\ell = 1$) guarantees that only the fundamental mode is unstable and all higher harmonics are stable. We allow slow variations of a and ψ in accordance with eqs.(2.10) and (2.11):

$$\frac{da}{dt} = \lambda A_1(a) + \lambda^2 A_2(a) + \cdots \quad , \tag{3.5}$$

$$\frac{d\psi}{dt} = \lambda \omega_1(a) + \lambda^2 \omega_2(a) + \cdots \quad , \tag{3.6}$$

Then following the procedure of the proceeding section, we find that $A_1 = 0$, $\omega_1 = 0$ and that A_2 and ω_2 are governed by a set of simultaneous equations:

$$A_1 \frac{dA_1}{da} - a\omega_1{}^2 = a - \frac{3}{4} a^3 \quad ,$$

$$A_1 \left[\frac{d}{da}(a\omega_1) + \omega_1 \right] = 0 \quad . \tag{3.7}$$

In solving this set of equations, we use the condition that A_1 should approach to a as a tends to 0 in accordance with eq.(3.2). The solution compatible with this condition is

$$\left. \begin{aligned} \omega_1 &= 0 \quad , \\ A_1 &= \sqrt{a^2 - \frac{3}{8} a^4} \quad . \end{aligned} \right\} \tag{3.8}$$

Thus we conclude that for a small amplitude, the frequency shift does not occur and the amplitude grows up according to the equation

$$\frac{da}{dt} = \lambda a \sqrt{1 - \frac{3}{8} a^2} \tag{3.9}$$

The solution of eq.(3.9) is easily shown to be

$$a(t) = \sqrt{\frac{8}{3}} \operatorname{sech} [\lambda(t - t_c)] \quad \text{for } t \leq t_c \quad , \tag{3.10}$$

where t_c is defined by

$$a_0 = \sqrt{\frac{8}{3}} \operatorname{sech} (t_c) \quad , \tag{3.11}$$

a_0 being the initial value of a at $t = 0$. Thus if we have a small amplitude wave initially, the amplitude grows up and reaches its maximum value $\sqrt{8/3}$ at $t = t_c$. No frequency shift occurs in this case.

For $t > t_c$, the behavior of the solution cannot be determined from the arguement so far made. But by solving eq.(3.1) numerically with an initial condition, we found that the amplitude seems to decrease for $t > t_c$, in accordance with the equation

$$\frac{da}{dt} = - \lambda a \sqrt{1 - \frac{3}{8} a^2} \tag{3.12}$$

so that eq.(3.10) is valid for $t > t_c$, too. The equation (3.12) arises from such a solution of eq.(3.7), that corresponds to the damping mode in the linear theory. This may mean that the growing wave changes into a damping wave (in the sense of the linear theory) when it passes the maximum of the amplitude.

Once we have known that $\omega_1 = 0$, we can obtain the result above more visually. Consulting eq.(3.5) we can write eq.(3.7) in the form:

$$\frac{d^2 a}{dt^2} = \lambda^2 \left(a - \frac{3}{4} a^3 \right) , \tag{3.13}$$

of which eq.(3.9) is one branch of the first integral. Thus we can regard the amplitude a as the coordinate of a particle which moves in a potential $\Phi(a)$: (Fig. 1)

$$\Phi(a) = \lambda^2 \left(- \frac{1}{2} a^2 + \frac{3}{16} a^4 \right) \tag{3.14}$$

The excitation of a wave from a small signal corresponds to the start of the particle from a point very near the origin 0. The particle moves to the right and reaches the point A, where the amplitude a attains its maximum value. Then the particle will return to the left (a decreases) and approaches to the origin ($a = 0$) asymptotically. The solution (3.9) has just such a character.

A system with a dissipation is simulated by an equation slightly more general than eq.(3.1) :

$$\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right)^2 v + \lambda\nu \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) v - \lambda^2 \left(2 + \ell^2 \frac{\partial^2}{\partial x^2} - v^2\right) v = 0 \tag{3.15}$$

This is an equation of the type just described in § 2. Then in place of eq.(3.7) we have

$$
\left.
\begin{aligned}
A_1 \frac{dA_1}{da} - a\omega_1{}^2 + \nu A_1 &= a - \frac{3}{4} a^3 , \\
A_1 \frac{d}{da} (a\omega_1) + A_1\omega_1 + \nu a\omega_1 &= 0 .
\end{aligned}
\right\} \qquad (3.16)
$$

For small a, we can put $\omega_1 = 0$ and eq.(3.15) reduces to

$$
\frac{d^2 a}{dt^2} + \lambda\nu \frac{da}{dt} = \lambda^2 (a - \frac{3}{4} a^3) \qquad (3.17)
$$

Thus we can visualize the behavior of $a(t)$ as a motion of a particle, which moves in the potential $\Phi(a)$ given by eq.(3.14), suffering a resistance. The particle will gradually "lose the energy", reflex at A_1, A_2, $\cdots$ and finally approach the bottom B. This corresponds to an amplitude oscillation, whose amplitude gradually damps away, and $a(t)$ approaches to a constant $2/\sqrt{3}$. (Fig. 2)

Finally we shall give a brief discussion on the solution of eq.(3.1) for the case $a^2 > 4/3$. When $a^2 < 4/3$ the expression (3.8) and the one corresponding to eq.(3.12) are the only reasonable solutions of eq.(3.7). But if $a^2 > 4/3$, another solution is possible. The solution is

$$
\left.
\begin{aligned}
\omega_1{}^2 &= \frac{3}{4} a^2 - 1 , \\
A_1 &= 0 .
\end{aligned}
\right\} \qquad (3.18)
$$

Especially, when $a^2 > 8/3$, eq.(3.18) gives the only reasonable

Solutions of eq.(3.7). Accordingly, if we start with a wave with a sufficiently large amplitude, the amplitude remains constant and the frequency shifts from that of the linear theory by ω_1 given in eq.(3.18).

§ 4. A Special Case of Two-Beam Instability

A system of two cold electron beam is governed by the following set of equations:

$$\left.\begin{aligned} &\frac{\partial N_j}{\partial t} + \frac{\partial}{\partial x} (N_j V_j) = 0 , \\ &\frac{\partial V_j}{\partial t} + V_j \frac{\partial V_j}{\partial x} = - \frac{e}{m} E , \\ &\frac{\partial E}{\partial x} = - 4\pi e (N_a + N_b - 2N_0) . \end{aligned}\right\} \qquad (4.1)$$

where the two beams are specified by suffixes a and b, and j = a or b. We have taken a special case, in which the two beams have the same density N_0. Then the dispersion equation can be solved analytically[8) and the system is shown to be unstable when the relative velocity V is smaller than a critical velocity $V_c = 2\omega_p/k$, where $\omega_p = \sqrt{8\pi e^2 N_0/m}$ and k is the prescribed wave number. We shall study the behavior of the system near the marginal state $V = V_c$.

We consider a state in which the beam a is at rest and introduce dimensionless variables as follows:

$$
\left.
\begin{aligned}
N_j &= N_0 \, (1 + \tilde{n}_j) \, , \quad j = a, b \, , \\
V_a &= (V_c/2) \, \tilde{v}_a \, , \\
V_b &= V_c \, (1 - \mu) + (V_c/2) \, \tilde{v}_b \, , \\
E &= (8\pi e N_0/k) \, \tilde{E} \\
t &= \tilde{t}/\omega_p \, , \quad x = \tilde{x}/k \, ,
\end{aligned}
\right\} \qquad (4.2)
$$

where μ is a small parameter which indicates the degree of departure of the relative velocity from the critical value; the system is unstable for $\mu > 0$ and stable for $\mu < 0$. Then eq.(4.1) reduces to

$$
\left.
\begin{aligned}
&\frac{\partial n_a}{\partial t} + \frac{\partial v_a}{\partial x} = - \frac{\partial}{\partial x} (n_a v_a) \, , \\
&\frac{\partial n_b}{\partial t} + \frac{\partial v_b}{\partial x} + 2(1 - \mu) \frac{\partial n_b}{\partial x} = - \lambda \frac{\partial}{\partial x} (n_b v_b) \, , \\
&\frac{\partial v_a}{\partial t} + E = - \lambda v_a \frac{\partial v_a}{\partial x} \, , \\
&\frac{\partial v_b}{\partial t} + 2(1 - \mu) \frac{\partial v_b}{\partial x} + E = - \lambda v_b \frac{\partial v_b}{\partial x} \, , \\
&2 \frac{\partial E}{\partial x} + (n_a + n_b) = 0 \, ,
\end{aligned}
\right\} \qquad (4.3)
$$

where we have omitted tildes over the variables and introduced a small parameter λ , which indicates the level of the wave.

The dispersion relation of the linearized system is

$$\frac{1}{\omega^2} + \frac{1}{[\ \omega - 2(1-\mu)]^2} = 2 \quad , \tag{4.4}$$

which can be solved for $|\mu| << 1$ as

$$\omega = \begin{cases} 1 - \mu \pm i\sqrt{\frac{2}{3}}\,\mu^{1/2} & \text{for} \quad \mu > 0 \quad , \\ \\ 1 + |\mu| \pm \sqrt{\frac{2}{3}}\ |\mu|^{1/2} & \text{for} \quad \mu < 0 \quad . \end{cases} \tag{4.5}$$

This is a typical example of the exceptional case.

In solving the system of equations (4.3), it is more convenient to handle eq.(4.3) as it stands, than to transform it to the form of eq.(2.1). We choose λ^2 as $|\mu|$, since then the contribution of the nonlinearity is the same order as that of μ.

We assume the solution of eq.(4.3) in the form:

$$\left.\begin{aligned} \begin{pmatrix} n_a \\ n_b \\ v_a \\ v_b \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} a\cos\phi + \lambda \begin{pmatrix} n_{a1} \\ n_{b1} \\ v_{a1} \\ v_{b1} \end{pmatrix} + \lambda^2 \begin{pmatrix} n_{a2} \\ n_{b2} \\ v_{a2} \\ v_{b2} \end{pmatrix} + \cdots \quad , \\ E &= a \sin\phi + \lambda E_1 + \lambda^2 E_2 + \cdots \end{aligned}\right\} \tag{4.6}$$

where $\phi = x - t - \psi$ and

$$\left.\begin{aligned} \frac{da}{dt} &= A_1(a) + \lambda^2 A_2(a) + \cdots \quad , \\ \frac{d\psi}{dt} &= \lambda\omega_1(a) + \lambda^2\omega_2(a) + \cdots \quad . \end{aligned}\right\} \quad (4.7)$$

We substitute these expressions into eq.(4.3) and equate the coefficients of the same power of λ on both side.

The coefficients of λ gives the following set of equations:

$$\left.\begin{aligned} &\frac{\partial n_{a1}}{\partial t} + \frac{\partial v_{a1}}{\partial x} = -A_1 \cos\phi - a\omega_1 \sin\phi + a^2 \sin 2\phi \quad , \\ &\frac{\partial n_{b1}}{\partial t} + \frac{\partial v_{b1}}{\partial x} + 2\frac{\partial n_{b1}}{\partial x} = -A_1 \cos\phi - a\omega_1 \sin\phi - a^2 \sin 2\phi , \\ &\frac{\partial v_{a1}}{\partial t} + E_1 = -A_1 \cos\phi - a\omega_1 \sin\phi + \frac{1}{2} a^2 \sin 2\phi \quad , \\ &\frac{\partial v_{b1}}{\partial t} + 2\frac{\partial v_{b1}}{\partial x} + E_1 = A_1 \cos\phi + a\omega_1 \sin\phi + \frac{1}{2} a^2 \sin 2\phi \, , \\ &2\frac{\partial E_1}{\partial x} + n_{a1} + n_{b1} = 0 \end{aligned}\right\} \quad (4.8)$$

Apparently eq.(4.8) has source terms which can resonate with the lefthand side. But at this stage the requirement of nonsecularity, i.e. the requirement that the solution should have only terms periodic in ϕ , does set no restriction on A_1 or ω_1. The solution of eq.(4.8) is obtained as follows:

$$\left.\begin{aligned}
\begin{pmatrix} n_{a1} \\ n_{b1} \\ v_{a1} \\ v_{b1} \end{pmatrix} &= \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \end{pmatrix} (A_1 \sin\phi - a\omega_1 \cos\phi) \\
&\quad + \frac{1}{2} \begin{pmatrix} 2 \\ 2 \\ 1 \\ -1 \end{pmatrix} a^2 \cos 2\phi \quad , \\
E_1 &= -\frac{1}{2} a^2 \sin 2\phi \quad ,
\end{aligned}\right\} \quad (4.9)$$

where A_1 and ω_1 are arbitrary at this stage, and the first harmonic terms of E_1 has been taken to be zero.

Using these results we proceed to the order of λ^2. The calculation is straightforward but tedius. At this stage the nonsecularity condition yields equations for A_1 and ω_1 :

$$\left.\begin{aligned}
A_1 \frac{dA_1}{da} - a\omega_1{}^2 &= \pm \frac{2}{3} a + \frac{1}{3} a^3 \quad , \\
A_1 \left[\frac{d}{da} (a\omega_1) + \omega_1 \right] &= 0
\end{aligned}\right\} \quad (4.10)$$

where the double signe corresponds to the same signes of μ . It should be noted that this set of equations has just the form predicted in § 2.

i) The case when $\mu > 0$

In this case the solution of eq.(4.10), which is compatible with the linear dispersion relation (4.5), is

$$\left.\begin{aligned} A_1 &= \sqrt{\frac{2}{3}a^2 + \frac{1}{6}a^4} \quad , \\ \omega_1 &= 0 \end{aligned}\right\} \quad (4.11)$$

The second term in the square root in the expression for A_1 is the contribution from the nonlinearity. Thus we see that the nonlinearity in this case does not suppress but enhance the instability and that we must seek for other mechanism to get a finite-amplitude, stationary wave.

ii) The case when $\mu < 0$.

In this case the solution of eq.(4.10) is

$$\left.\begin{aligned} A_1 &= 0 \quad , \\ \omega_1 &= \pm\sqrt{\frac{2}{3} - \frac{1}{3}a^2} \quad , \end{aligned}\right\} \quad (4.12)$$

for small a. Accordingly when a is small, we have a constant-amplitude solution with a frequency shift, which decreases in magnitude as the amplitude increases. When $a \geq \sqrt{2}$, the constant-amplitude solution (4.12) is not possible. The solution should be determined by equations

$$\left.\begin{aligned} A_1 \frac{dA_1}{da} &= -\frac{3}{2} a + \frac{1}{3} a^3 \quad , \\ \omega_1 &= 0 \quad , \end{aligned}\right\} \qquad (4.13)$$

whence we have

$$A_1 = \sqrt{C - \frac{2}{3} a^2 + \frac{1}{6} a^4} \qquad (4.14)$$

As for the integration constant C, we have no rule to determin it. A reasonable choice of C may be 2/3, which makes $A_1 = 0$ at $a = \sqrt{2}$.

We can summarize the results in this case as follows. When the amplitude is small, there is a stable oscillation with the frequency shift due to nonlinearity. The frequency shift decreases as the amplitude increases until the shift becomes zero at the amplitude $| 2\mu |^{1/2}$. Further increase of the amplitude makes the wave unstable, and the amplitude grows up according to the equation

$$\frac{da}{dt} = | \mu |^{1/2} \sqrt{\frac{2}{3} - \frac{2}{3} a^2 + \frac{1}{6} a^4} \qquad (4.15)$$

(It should be noted that we deals with a wave with an amplitude $|\mu|^{1/2} a$). This is a typical example of the wave which is stable in the linear theory but is unstable against finite amplitude perturbations[5].

§ 5. Conclusions

In this paper it has been shown that we should distinguish the exceptional case from the normal case in the discussion of the effect of nonlinearity on the behaviors of a wave, which is monochromatic and stationary in the first approximation. Since the normal cases have been discussed already by many authers, we have given no concrete example. In this paper the emphasis is laid upon the characteristic behaviors of nonlinear waves in the exceptional cases.

In a dissipationless plasma, s_i and α_i vanish. Accordingly we see that $A_2 = 0$ in the normal case. It can be also shown that all $A_n = 0$, so that in a normal case a stationary wave in the linear approximation remains stationary in nonlinear theory. In an exceptional case this is not always true. As we have seen, the nonlinearity can change the growth rate of the wave. But consulting to eq.(2.23) and considering that $\nu = 0$ in a dissipationless system, we may conclude that either A_1 or ω_1 must be zero; i.e. we have either non-zero growth rate or non-zero frequency shift, but not both, in an exceptional case in a dissipationless plasma.

Two-stream instabilities are typical examples of the exceptional case. We can also show that in a fluid description the instability can be either suppressed or enhanced depending on the choice of parameters characterizing the two-stream system. When we discuss the instability in the kinetic theory, we will have to take account of the Landau damping and collisions, too. Particle trapping may also become important. But even in that case, the two-stream instability near its threshold should be discussed as an exceptional case in our sense.

Detailed discussions of the two-stream instability will be given in papers to follow.

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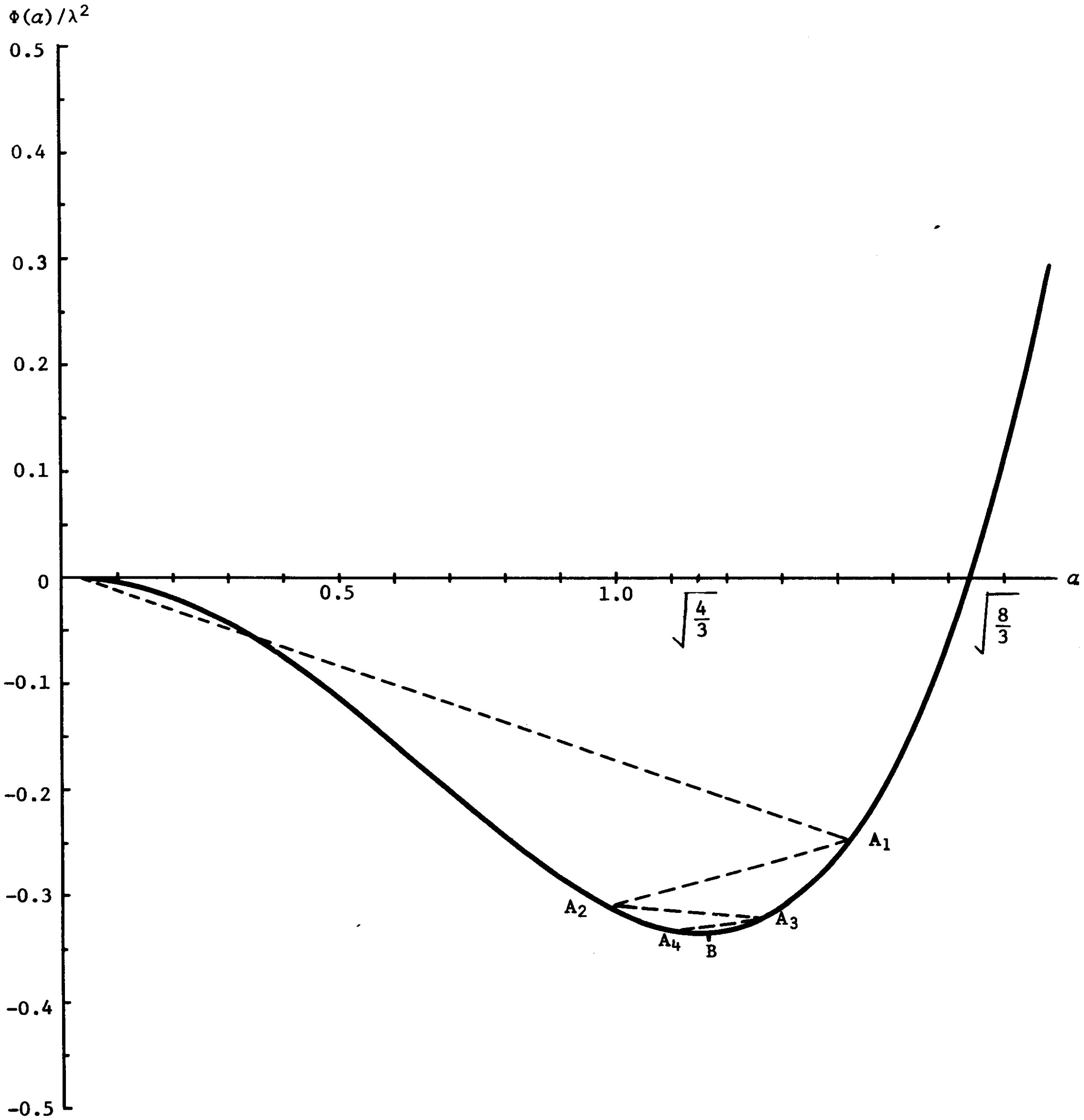

Fig. 1: The Shape of the "potential" $\Phi(a)$ given by eq.(3.14). The points A_1, A_2, ... indicate the "reflection" points and B is the "bottom" of the potential.

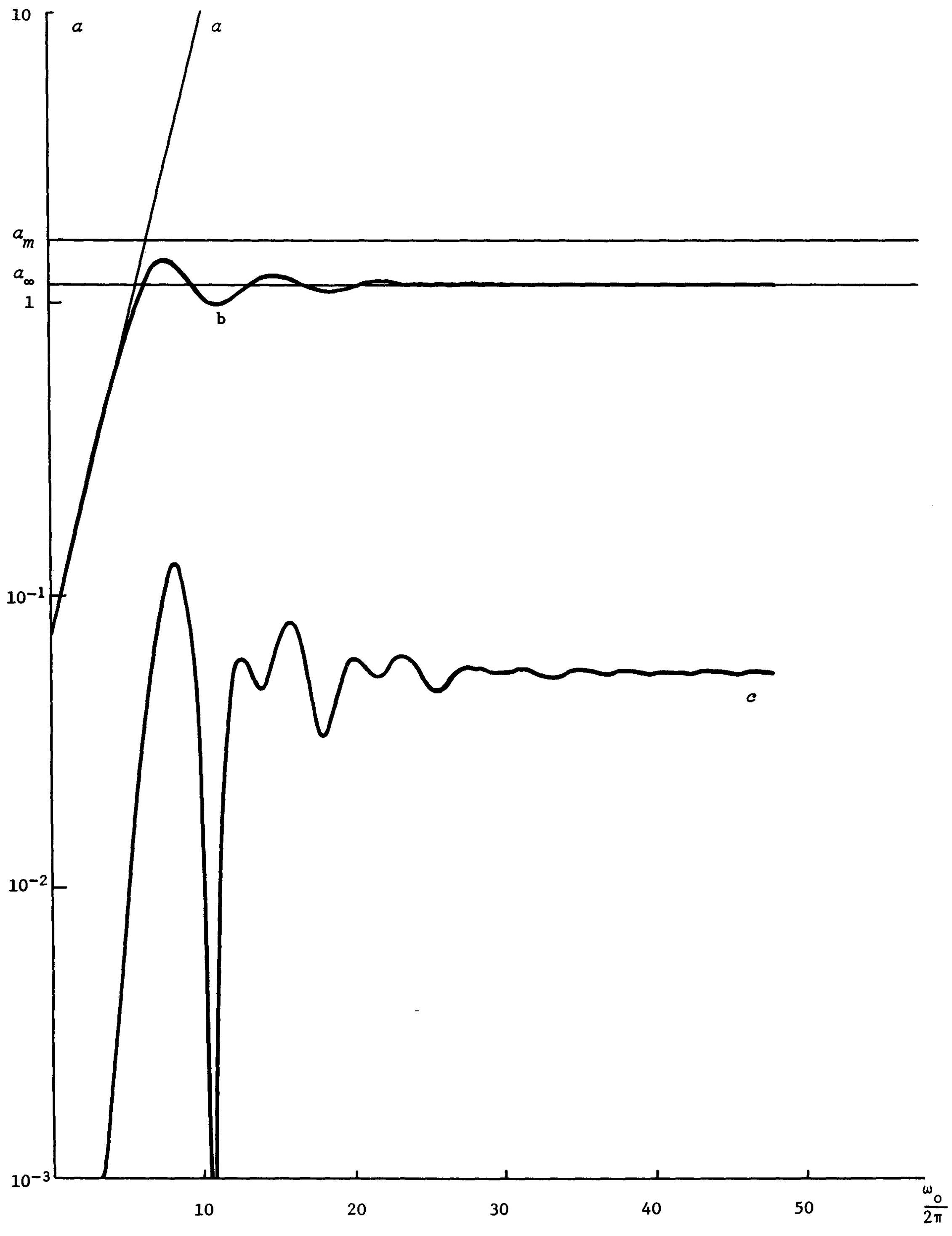

Fig. 2.: An example of the amplitude of the waves obtained through the numerical solution of eq.(3.15). The values of parameters are $V = 1$, $\ell = 1$, $\lambda = 0.1$, $\nu = 0.5$, $k = 1.0$.

a : The solution due to the linear theory.

b : The amplitude of the fundamental mode obtained through the numerical solution of eq.(3.15).

c : The amplitude of the second harmonic.

$a_\infty = 2/\sqrt{3}$: The asymptotic value of a predicted by the theory.

$a_m = \sqrt{8/3}$: The maximum value which would be attained if ν were zero.