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Enhanced Particle Losses

due to Electron Cyclotron Wave Instability

in Magnetic Mirrors

H. Momota and Y. Terashima

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The electron cyclotron wave instability of non-uniform plasmas in magnetic mirrors is investigated. First, the most unstable mode of the instability is studied by the WKB method under an appropriate boundary condition. The non-linear development of the instability is then discussed in terms of the diffusion equation in velocity space for the resonant particles. The particle loss along the magnetic lines of force is found to be appreciable. In aid of the discussions, some numerical examples are presented.

I. INTRODUCTION

Instabilities due to anisotropic velocity distributions have been studied extensively in the literature.[1] One of the subjects is the electron cyclotron wave instability, that is, the electromagnetic instability near the electron gyration frequency. Most of the work done so far treats uniform plasmas with bi-maxwellian distributions by linear theory.[2] On the other hand, we see today many experiments in which the energetic electrons are produced in magnetic mirror configurations. A study of the loss-cone distribution is therefore necessary. In general, in mirror devices cyclotron instabilities of electrostatic nature or loss-cone instabilities[3] can possibly take place. However, the electron cyclotron wave instability is believed to be of primary importance in some experiments[4~6] with which we are concerned.

The purpose of this paper is to study this kind of instability for a non-uniform plasma with loss-cone distribution. Spatial variation of the plasma density as well as of external magnetic field are taken into account. A linear theory of a similar problem has been treated by Scharer[7] in his treatment, however, plasma density and external magnetic field are assumed to be uniform.

In Sec.II, preliminary remarks on a spatially uniform plasma with loss-cone distribution are presented. The diffusion equation in velocity space for the resonant electrons is derived from the quasi-linear theory. Numerical examples are given to compare the growth rate of the instability with the diffusion rate of the resonant particles. In Sec.III, we treat an inhomogeneous system in a magnetic mirror configuration. The mode with maximum growth rate is found by the WKB approximation. Enhanced particle losses along the magnetic lines of force are estimated from the asymptotic form of the velocity distribution.

II. PHENOMENA IN HOMOGENEOUS PLASMAS

In this section we study the electron cyclotron wave instability and the associated particle diffusion in velocity space in a spatially uniform plasma. For the present investigation, ions are assumed to constitute only a fixed and uniform background for charge neutralization. We neglect both collisional and relativistic effects.[8] The distribution function of the electrons $f(\vec{r},\vec{v},t)$ then obeys the collisionless Vlasov equation.

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} - \frac{e}{m}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c} \right)\cdot\frac{\partial f}{\partial \vec{v}} = 0, \tag{1}$$

where m and $-e$ are the mass and the charge of an electron, respectively. The electric field $\vec{E}(\vec{r},t)$ and the magnetic field $\vec{B}(\vec{r},t)$ are determined by Maxwell's equations

$$\mathrm{rot}\ \vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0, \tag{2}$$

$$\mathrm{rot}\ \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} - \frac{4\pi e}{c}\int \vec{v} f\, d^3v. \tag{3}$$

Let the external uniform magnetic field $\vec{B}_0$ be along the z-axis and consider perturbation propagating along $\vec{B}_0$. We apply Fourier analysis in space and time according to

$$f(\vec{r},\vec{v},t,) = f_0(\vec{v},t) - {\sum_{k,\omega}}' f_{k,\omega}(\vec{v})\, e^{i(kx - \omega t)}, \tag{4}$$

where $f_o(\vec{v},t)$ is the unperturbed distribution. The symbol $\sum_m'$ denotes the summation over non-zero m. Similar expressions result for the electric and magnetic fields. Eq.(1) can then be written as

$$\frac{\partial}{\partial t}f_o(\vec{v},t) = \frac{e}{m}\sum_{k,\omega}' [\vec{E}_{k,\omega} + \frac{\vec{v}}{\omega} \times (\vec{k} \times \vec{E}_{k,\omega})] \cdot \frac{\partial}{\partial \vec{v}} f_{-k,\omega}(\vec{v}) \qquad (5)$$

and

$$\vec{\omega}_c \cdot [\vec{v} \times \frac{\partial}{\partial \vec{v}} f_{k,\omega}(\vec{v})] - i(\omega - \vec{k}\cdot\vec{v}) f_{k,\omega}(\vec{v}) - \frac{e}{m}[(1 - \frac{\vec{k}\cdot\vec{v}}{\omega})\vec{E}_{k,\omega}$$
$$+ \frac{\vec{v}\cdot\vec{E}_{k,\omega}}{\omega} k] \frac{\partial}{\partial \vec{v}} f_o(\vec{v},t) = 0, \qquad (6)$$

where $\vec{\omega}_c$ is defined as $e\vec{B}_o/Mc$. In Eq.(6), we have neglected the terms corresponding to the mode-mode coupling (quasilinear treatment).

Together with the following equation derived from Eqs.(2) and (3)

$$\left(\frac{c^2k^2}{\omega^2} - 1 \right) \vec{E}_{k,\omega} = \frac{4\pi e}{i\,\omega} \int \vec{v} f_{k,\omega}(v)\, d^3v, \qquad (7)$$

we have the set of equations (Eqs.(5) ~ (7)) describing the phenomena we are interested in.

II - A. LINEAR INSTABILITY

We summarize briefly the results of the linear theory. Eqs.(6) and (7) are a set of linear equations for $f_{k,\omega}(\vec{v})$ and $\vec{E}_{k,\omega}$. If the external magnetic field is strong enough, the mean distribution function $f_o(\vec{v},t)$ may be expressed as a function of $v_\parallel$ and $v_\perp$, which are defined by $\vec{v} = (v_\perp \cos\phi,\ v_\perp \sin\phi,\ v_\parallel)$ in Cartesian coordinates.

The temporal variation of f_o if assumed to be very small compared to the frequencies of the perturbation and is neglected.

For a growing transverse wave ($\vec{E}_{k,\omega} \perp \vec{k}$ and $Im(\omega) > 0$), Eq.(6) leads to

$$f_{k,\omega}(\vec{v}) = i \sum_{\pm} \frac{e}{m} \frac{E^{\pm}_{k,\omega} e^{\pm i\phi}}{\omega \pm \omega_c - kv_{\parallel}} \left[\left(1 - \frac{kv_{\parallel}}{\omega}\right)\frac{\partial}{\partial v_{\perp}} + \frac{kv_{\parallel}}{\omega} \frac{\partial}{\partial v_{\parallel}} \right] f_o(\vec{v}), \tag{8}$$

where $E^{\pm}_{k,\omega}$ is defined as $\frac{1}{2}(E_{x\ k,\omega} \pm i\ E_{y\ k,\omega})$. A wave of righthand polarization corresponds to $E^{-}_{k,\omega}$ with a positive value of k and $E^{+}_{k,\omega}$ with negative k. Without loss in generality, we shall henceforth restrict ourselves to the case $k > 0$.

Substituting $f_{k,\omega}(\vec{v})$ into Eq.(7), one obtains the following dispersion relation

$$\frac{c^2k^2}{\omega^2} = 1 + \frac{4\pi^2e^2}{\omega m} \int_o^{\infty} dv_{\perp} \int_{-\infty}^{\infty} dv_{\parallel} \frac{v^2}{\omega - \omega_c - kv_{\parallel}} \left[\left(1 - \frac{kv}{\omega}\right) \frac{\partial}{\partial v_{\perp}} + \frac{kv}{\omega} \frac{\partial}{\partial v_{\parallel}} \right] f_o(\vec{v}). \tag{9}$$

After integration with respect to v we have

$$\frac{c^2k^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega^2} \int_{-\infty}^{\infty} \frac{dv_{\parallel}}{\omega - \omega_c - kv_{\parallel}} \left\{ \frac{1}{2} \frac{d}{dv_{\parallel}} [w(v_{\parallel})F(v_{\parallel})] - (\omega - kv_{\parallel})F(v_{\parallel}) \right\}, \tag{10}$$

where $\omega_p = (4\pi ne^2/m)^{\frac{1}{2}}$ is the electron plasma frequency, and $F(v_{\parallel})$ and $W(v_{\parallel})$ are defined as

$$F(v_{\parallel}) = \frac{2\pi}{n} \int_o^{\infty} f_o(v_{\parallel}, v_{\perp}) v_{\perp} dv_{\perp} \tag{11}$$

and

$$W(v_{\|}) = \int_o^\infty v^2 f_o(v_{\|}, v_\perp) v_\perp dv_\perp \Big/ \int_o^\infty f_o(v_{\|}, v_\perp) v_\perp dv_\perp \ . \tag{12}$$

In order to express Eq.(10) in a more convenient form, let us replace the complex ω by $\omega_r + i\gamma$ with real ω_r and γ and assume that $|\omega_r| >> \gamma$. One then obtains for a real value of k

$$\frac{c^2 k^2}{\omega_r^2} = \frac{1 + \dfrac{\omega_p^2}{\omega_r(\omega_c - \omega_r)}}{1 + \dfrac{\omega_p^2}{(\omega_c - \omega_r)^2}\left[\dfrac{1}{2}\dfrac{\langle v^2 \rangle}{c^2} - \dfrac{\omega_c}{\omega_c - \omega_r}\dfrac{\langle v_{\|}^2 \rangle}{c^2}\right]} \tag{13}$$

and

$$\gamma = \frac{\pi}{2}\frac{\omega_p^2}{\omega_r + \omega_c \omega_p^2 / 2(\omega_c - \omega_r)^2}\left[\frac{1}{2}\frac{d}{dv} W(v_{\|})F(v_{\|}) - \frac{\omega_c}{k} F(v_{\|})\right]_{v_{\|} = v_r} \tag{14}$$

where the resonant velocity $v_r(k, \omega_r)$ is defined as $v_r = (\omega_r - \omega_c)/k$. It should be noted that the wave is supported by the electrons as a whole the growth rate, however, is governed only by the resonant electrons.[9] Our sign convention is that ω_r as well as k are taken to be positive. Eq.(13) gives a solution $\omega_r(k) < \omega_c$, then v_r is negative so that the resonant electrons counterstream the wave of $\omega_r(k)/k > 0$.

II - B. DIFFUSION IN VELOCITY SPACE

We shall consider a nonlinear property of this instability by extending the usual 1-dimensional quasilinear method to 2-dimensional one. The mean distribution function f_o must be a function of t, and consequently the growth rate γ depends on time. The basic equation governing the evolution of the instability is Eq.(5) and an instantaneous property of a wave can be described by Eq.(8) ~ (14).

The substitution of Eq.(8) into Eq.(5) gives immediately the following diffusion equation in the $v_\perp - v_\parallel$ space:

$$\frac{\partial}{\partial t} f_o(v_\perp, v_\parallel, t) = \left(\frac{\omega_c}{k} \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} + \frac{\partial}{\partial v_\parallel} \right) D(k, \omega_r, \vec{v}, t) \times \left(\frac{\omega_c}{k} \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} + \frac{\partial}{\partial v_\parallel} \right) f_o(v_\perp, v_\parallel, t) \Bigg|_{k=k(v_\parallel)} , \tag{15}$$

where $k(v_\parallel)$ is the root of the equation $\omega_r - \omega_c - kv_\parallel = 0$, together with the dispersion equation (13). $D(k,\omega_r,\vec{v},t)$ is a non-negative quantity defined by

$$D(k,\omega_r,\vec{v},t) = 4\pi L\left(\frac{e}{m}\right)^2 \frac{v^2}{|v_\parallel|} \left| \frac{k}{\omega_r} \right|^2 |E^-_{k,\omega_r}(t)|^2 . \tag{16}$$

Introducing new variables ξ and η defined by

$$\xi = \frac{1}{2} v_\perp^2 - \int \frac{\omega_c}{k(v_\parallel)} dv_\parallel$$

and

$$\eta = v_\parallel , \tag{17}$$

one can rewrite Eq.(15) simply as

$$\frac{\partial}{\partial t}f_o(\xi,\eta,t) = \frac{\partial}{\partial\eta}\, D(\xi,\eta,t)\, \frac{\partial}{\partial\eta}\, f_o(\xi,\eta,t)\,. \tag{18}$$

The fact that D is non negative leads to the conclusion that f_o diffuses in the region $D > 0$ (hereafter named "resonant region") in the $\xi - \eta$ plane along the lines ξ = constant (hereafter named "Diffusion lines").

It is natural to assume that f_o tends to a stationary value as $t \to \infty$. We then have from Eq.(18)

$$\int_{\eta_1}^{\eta_2} D(\xi,\eta,t)\; \left|\frac{\partial}{\partial\eta}f_o(\xi,\eta,t)\right|^2 d\eta$$

$$= -\frac{\partial}{\partial t}\int_{\eta_1}^{\eta_2} \frac{1}{2}f_o(\xi,\eta,t)\,d\eta \to 0 \quad \text{as} \quad t\to\infty\,, \tag{19}$$

where η_1 and η_2 are supposed to satisfy the conditions

$$D(\xi,\eta,t \to \infty) = 0 \qquad \text{for} \quad \eta \le \eta_1 \text{ and } \eta_2 \le \eta\,,$$

$$D(\xi,\eta,t \to \infty) = 0 \qquad \text{for} \quad \eta_1 < \eta < \eta_2\,.$$

This implies that as a result of the diffusion, f_o should be constant on a diffusion line within the resonant region.[10] The diffusion in velocity space is illustrated schematically in Fig.1, where the variables are restored to $v_\perp$ and $v_\parallel$. It must be noted that since $\omega_c/k(v_\parallel)$ is larger than $|v_\parallel|$, the kinetic energy of a particle generally decreases as $|\eta|$ increases along a diffusion line.

A temporal evolution of the mean distribution function will be traced, in principle, on the basis of Eqs.(13), (14), (16) and (18). Instead we here prefer to give a crude but simple estimation of the diffusion rate. Integrating Eq.(15) multiplied by $2\pi v/n$ with respect to $v_\perp$, one obtains

$$\frac{\partial}{\partial t}F(v_\parallel,t) = 4\pi L\left(\frac{e}{m}\right)^2\frac{\partial}{\partial v}\left|\frac{1}{v_\parallel}\right|\left|\frac{k(v_\parallel)}{\omega_r}\right|^2\left|E^-_{k(v_\parallel),\omega_r}(t)\right|^2$$

$$\times\left\{\frac{\partial}{\partial v_\parallel}\left[W(v_\parallel,t)F(v_\parallel,t)\right] - \frac{2\omega_c}{k(v_\parallel)}F(v_\parallel,t)\right\}$$

$$= 4\pi L\left(\frac{e}{m}\right)^2\frac{\partial}{\partial v_\parallel}\left|\frac{1}{v_\parallel}\right|\left|\frac{k(v_\parallel)}{\omega_r}\right|^2\left[\frac{2\omega_r}{\omega_p^2} + \frac{\omega_c}{(\omega_c-\omega_r)^2}\right]$$

$$\times\frac{d}{dt}\left|E^-_{k(v_\parallel),\omega_r}(t)\right|^2 . \tag{20}$$

If the initial electric field $E(t=0)$ is negligible small, this equation yields

$$\left|E^-_{k,\omega_r}(t)\right|^2 = \frac{1}{4L}\left(\frac{e}{m}\right)^2\omega_r\left\{\frac{2(\omega_c-\omega_r)^2}{\omega_p^2} + \frac{\omega_c}{\omega_r}\right\}^{-1}$$

$$\times X^3\int_0^X\left\{F(v_\parallel,0) - F(v_\parallel,t)\right\}dv_\parallel , \tag{21}$$

where $X = (\omega_c - \omega_r)/k$.

From Eq.(16) we thus obtain

$$D(v_\perp, v_\parallel t) = \frac{\pi}{\omega_c} k^2(v_\parallel)\; v_\perp^2\; v_\parallel^2 \;\left\{ 1 + \frac{2k(v_\parallel)v_\parallel^2}{\omega_p^2 \omega_c} [\omega_c - k(v_\parallel)|v_\parallel|] \right\}^{-1}$$

$$\times \int_0^{|v_\parallel|} \{ F(v_\parallel, 0) - F(v_\parallel, t) \} \; dv_\parallel \qquad (22)$$

In Eqs.(21) and (22), the function $F(v_\parallel, t)$ is still unknown. The asymptotic form of the distribution, $F(v_\parallel, t\to\infty)$ can be found by the consideration that f_o may be constant in the resonant region along the ξ = constant lines as $t\to\infty$. Such estimation will be done for some typical examples in the next section.

We may add the remark that the above discussions, except for Eqs. (21) and (22), hold even for the development of externally or initially applied electron cyclotron waves in a stable plasma.

II - C. EXAMPLES

Let us consider the case in which the electrons initially have the following loss-cone distribution:

$$f_o(v_\perp, v_\parallel) = \frac{N_o}{\pi\sqrt{\pi}} \frac{1}{\alpha^2} \sqrt{\frac{3}{2}}\, h(2v_\perp^2 - v_\parallel^2) \; \exp\left(-\frac{v_\parallel^2 + v_\perp^2}{\alpha^2}\right). \qquad (23)$$

This means that a maxwellian plasma with temperature $T_e = (m/2)\alpha^2$ is initially immersed in a magnetic mirror field of infinite length and the mirror ratio of 3. The function $h(x)$ equals unity if x is non-negative and $h(x) = 0$ if $x < 0$, then Eq.(14) yields

$$\gamma = \sqrt{\pi}\,\omega_c\left(1 - \frac{\omega_r}{\omega_c}\right)^2 \left[\, Y\left(\frac{Y^2}{2} + 1\right) - \frac{\omega_c}{k\alpha} \right] \exp(-Y^2) \ , \qquad (24)$$

where $Y = (\omega_c - \omega_r)/k\alpha$, and k is the solution of Eq.(13).

The numerical values of γ and k are demonstrated schematically in Fig.2, where the parameters are: B = 2 kG, the number density $N_o = 10^{12}$ cm^{-3} and T_e = 20 keV. The characteristic growing time of the instability is faster than 1 nsec. Lower temperature will give smaller growth rate.

Next, we elucidate the diffusion phenomenon in this case. From Fig.2, one finds the spread of the frequency spectrum to extend from 0 to 0.68 ω_c, where the growth rate is positive.[11] Then the resonant region in the $v_\perp - v_\parallel$ space is

$$|\, v_\parallel \,| \ > \ 4\text{x}10^9 \text{ cm/sec.} \qquad (25)$$

The diffusion lines ($\frac{1}{2}v_\perp{}^2 - \int (\omega_c/k(v_\parallel))dv_\parallel$ = const.) are approximately

$$\xi \ \simeq \ \frac{1}{2}\, v_\perp^2 + v_\parallel^2 = \text{const.} \qquad (26)$$

In order to obtain $F(v_\parallel,\ t\to\infty)$, we follow the discussions given previously. Integration of Eq.(18) gives

$$\frac{\partial}{\partial t}\int_{\eta_1}^{\eta_2} f_o(\xi,\eta,t)\ \Big|_{\xi=\text{const.}}\ d\eta = 0. \qquad (27)$$

where $\eta_1 = -v_c$ and $\eta_2 \to -\infty$. Recalling Eq.(19), one obtains

$$f_o(\xi,\eta,t\to\infty) = \int_{\eta_1}^{\eta_2} f_o(\xi,\eta, =0)d\eta \bigg/ \int_{\eta_1}^{\eta_2} d\eta. \tag{28}$$

Then

$$f_o(v_\perp,v_\parallel,t\to\infty) = \frac{N_o}{\pi\sqrt{\pi}\alpha^3}\sqrt{\frac{3}{2}}\;\frac{\displaystyle\int_{v_c}^{\sqrt{4\xi/5}} \left(\frac{\xi+s^2}{\xi-s^2}\right)^{\frac{1}{2}} \exp\left(\frac{-2\xi+s^2}{\alpha^2}\right) ds}{\displaystyle\int_{v_c}^{\sqrt{\xi}} \left(\frac{\xi+s^2}{\xi-s^2}\right)^{\frac{1}{2}} ds}, \tag{29}$$

and therefore one can obtain $F(v_\parallel,t\to\infty) = ({}^1/_{N_o}) \int_0^\infty f(v_\perp,v_\parallel,t\to\infty)\, 2\pi v_\perp dv_\perp$.

The asymptotic function $F(v_\parallel,t\to\infty)$ calculated numerically is shown in Fig.3 together with the function $F(v_\parallel,t = 0)$. Particles which are resonant with the waves amount to about one half of the total particles, and the value $\int_0^\infty [F(v_\parallel, 0) - F(v_\parallel,t\to\infty)]\, dv_\parallel$ is estimated to be of the order of 0.01. Putting $v_\perp \sim v_\parallel \sim \alpha$ into Eq.(22), one can finally find the numerical value of the coefficient D to be about 10^8, and the characteristic time of diffusions to be about 10^{-7} sec (about 10^2 times the characteristic growth time).

From these values, we expect that as the electron cyclotron wave instability develops, the particles suffer rapid diffusion in velocity space, which results in an enhancement of mirror losses in actual finite mirror systems.

III. PHENOMENA IN MIRROR TRAPPED PLASMAS

We now extend our discussions to non-uniform plasmas contained in a mirror machine. Since we are interested mainly in enhanced particle losses along the lines of force due to electron cyclotron wave instability, we consistently ignor the variations across the lines of force. The externally applied magnetic field is assumed to be

$$B(z) = \left(\frac{B_m+B_o}{2}\right) - \left(\frac{B_m-B_o}{2}\right)\cos\left(\frac{\pi z}{L}\right), \qquad (-L \leq z \leq L) \tag{30}$$

The unperturbed distribution function is taken to be

$$f_o(v_\perp, v_\parallel, z, t=0) = \frac{N_o}{\pi\sqrt{\pi}} \frac{1}{\alpha^2\beta} \left(1 + \frac{B_o}{B_m-B_o} \frac{\beta^1}{\alpha^2}\right) \quad h\left(\frac{B_m-B(z)}{B(z)} v_\perp^2 - v_\parallel^2\right)$$

$$\times \exp\left[-\frac{1}{\beta^2}\left(v_\parallel^2 + \frac{B(z)-B_o}{B(z)} v_\perp^2\right) - \frac{1}{\alpha^2} \frac{B_o}{B(z)} v_\perp^2\right], \tag{31}$$

where N_o means the number density of the electrons at $z = 0$ (the center of the mirror) and $h(x)$ is the step function introduced previously. A distribution function really depends on the method of plasma production. This distribution function is chosen so that the bi-maxwellian electrons with the temperatures $T_\parallel = (m/2)\ \beta^2$ and $T_\perp = (m/2)\ \alpha^2$ are produced at the center of the mirror and behave adiabatically inside the mirror. Particle losses inside the loss-cone, however, are taken into account.

Further more we assume that the external magnetic field and the unperturbed distribution change very slowly in space and that the local behavior of the perturbation can be described in terms of that in a homogeneous system (WKB approximation).

III - A. LINEAR INSTABILITY

We aim to predict correctly the frequency for maximum growth rate of the instability. Let us consider a perturbation of the form

$$E(z, t) \propto \exp\ [i\int^{z} k(z)dz - i\omega t\ ]\ ,$$

where $k(z)$ is the locally defined complex wave number and ω the complex angular frequency. Then the real part of $k(z)$ is related to the real part of ω by

$$k_r(z) = \frac{\omega_r}{c}\ \{\ 1 + \frac{\omega_p^2(z)}{\omega_r[\omega_c(z)-\omega_r]}\ \}^{\frac{1}{2}}\ , \tag{32}$$

which is obtained immediately from Eq.(13) as a local solution. We set $\omega = \omega_r + i\omega_i$ and also $k(z) = k_r(z) + ik_i(z)$.

Since the equation governing the perturbation is a partial differential equation, the growth rate of the perturbation must be obtained under a suitable boundary condition. By considerations of symmetry we postulate as such a boundary condition that the amplitudes of the perturbed electric field at both the ends of the mirror are equal. Then, by making use of the relation (see Appendix)

$$k_i(z) = [\ \omega_i - \gamma(z)\ ]\ /\ (\partial\omega_r/\partial k_r)\ , \tag{33}$$

we obtain[12]

$$\omega_i = \frac{\int_{-L}^{L} dz\gamma(z)\left(\frac{\partial\omega_r}{\partial k_r}\right)^{-1}}{\int_{-L}^{L} dz\ \left(\frac{\partial\omega_r}{\partial k_r}\right)^{-1}}\ . \tag{34}$$

Here $\gamma(z)$ is the local growth rate which is obtained by solving the local dispersion equation under the condition $k_i = 0$. Namely from Eq.(14),

$$\gamma(z) = \frac{\sqrt{\pi}}{2}\,\omega_p^2(z)\ \{\ \omega_r + \frac{\omega_c(z)}{2}\ [\ \frac{\omega_p(z)}{\omega_c(z)-\omega_r}\ ]^2\ \}^{-1}\ \exp(-X^2)$$

$$\times\ \{\ X\ [\ \frac{B(z)}{B_m-B(z)}X + \frac{B(z)}{B(z)-B_o(\alpha^2-\beta^2)/\alpha^2}\ ] - \frac{\omega_c(z)}{K(z)}\ \}\ , \tag{35}$$

where $X = (\omega_c(z) - \omega_r)/K(z)$ and $K(z) = k_r(z)\ \beta\{[B_m - B(z)]/2[B(z) - B_o(1 - \beta^2/\alpha^2)]\}^{\frac{1}{2}}$. The local group velocity $\partial\omega_r/\partial k_r$ is obtained from Eq.(32) as

$$\frac{\partial\omega_r}{\partial k_r} = c\,\frac{\{1 - \dfrac{\omega_p^2(z)}{\omega_r[\omega_c(z)-\omega_r]}\}^{\frac{1}{2}}}{1 + \dfrac{\omega_c(z)}{2\omega_r}\ [\ \dfrac{\omega_p(z)}{\omega_c(z)-\omega_r}\ ]} \ . \tag{36}$$

A typical example is illustrated schematically in Fig.4. The plasma parameters are chosen as $T_\perp = T_\parallel = 20$ keV, $N_o = 10^{12}$ cm^{-3} and $B_m/B_o = 3$ with $B_o = 2$ kG. The growing mode has the maximum growth rate at the frequency $\omega_r = 0.62\omega_c$ $(z = 0)$.

III - B. DIFFUSION IN VELOCITY SPACE AND MIRROR LOSS

As is shown in II - B, the cyclotron wave instability gives rise to particle diffusions in velocity space. In a mirror this diffusion causes particle losses along the lines of force. Let us take as variables the kinetic energy $\varepsilon = (m/2)(v_\perp^2 + v_\parallel^2)$ and the magnetic moment $\mu = mv_\perp^2/2B(z)$, instead of $v_\perp$ and $v_\parallel$. Then the unperturbed distribution function (31) becomes in the μ - ε space

$$f_o(\varepsilon,\mu,z,t=0) = \frac{N_o}{\sqrt{\pi}} \frac{B(z)}{T_\perp \sqrt{T_\parallel}} \left(1 + \frac{B_o}{B_m - B_o} \right)^{\frac{1}{2}} \frac{h(B_m\mu-\varepsilon)}{[\varepsilon-\mu B(z)]^{\frac{1}{2}}}$$

$$\times \exp\{-\frac{1}{T_\parallel} [\varepsilon + (\frac{T_\parallel}{T_\perp} - 1)\mu B_o]\} . \qquad (37)$$

Particles whose kinetic energy ε is greater than μB_m disapear because they are lost through the mirror ends, and particles whose kinetic energy ε is less than $\mu B(z)$ have turning points inside the interval $(-z, z)$, therefore these particles do not contribute to Eq.(37).

As was stated before, the unperturbed state changes so slowly that for diffusions in velocity space the scheme described previously for the uniform plasma still holds locally in the nonuniform plasma. The frequency spectrum of the perturbation is thought to be extremely sharp around, say, $\omega_r = \omega_o$. The local resonant region in velocity space is then reduced to a narrow band

$$\varepsilon = \mu B(z) + \frac{m}{2} V_r^2(\omega_o,z), \qquad (38)$$

where $V_r(\omega_o,z)$ is the locally defined resonant velocity $[\omega_o - \omega_c(z)]/k(\omega_o,z)$ and is determined by Eq.(32) as

$$V_r(\omega_o,z) = \frac{c}{\omega_o} [\omega_o - \omega_c(z)] \left\{ 1 + \frac{\omega_p^2(z)}{\omega_o[\omega_c(z)-\omega_o]} \right\}^{-\frac{1}{2}} \qquad (39)$$

The local diffusion lines are

$$\varepsilon - \frac{\omega_o}{\omega_c(z=0)} \mu B_o = \text{const.} \tag{40}$$

Let the turning point of the particle with μ and ε be z_t, which is determined by $\varepsilon = \mu B(z = z_t)$. Consider a particle which satisfies

$$\varepsilon > \mu\, B_o + \frac{m}{2} V_r^2(\omega_o,\ z=0). \tag{41}$$

This particle will satisfy the resonance condition (38) at some place $z = z_r$, $- z_t < z_r < z_t$. From the fact that $\omega_c(z)$ increases as $|z|$ while $\omega_p(z)$ decreases, $V_r(\omega_o,\ z)$ is found to be an increasing function of $|z|$. In the resonant region, the particles really suffer from changes in energy and magnetic moment by interaction with waves. In the μ - ε space, the resonant particles diffuse along the diffusion lines and are, on the average, directed to smaller value of μ. Once the particles enter into the resonant region, they continue to keep the resonance condition with their decreasing μ values (Fig.5). Finally they fall into the loss-cone and escape from the confined field.

As a result, the distribution function finally tends to

$$f_o(\varepsilon,\mu,z,t\to\infty) = f_o(\varepsilon,\mu,z,t=0) \times h[\mu B_o + \frac{m}{2}V_r^2(\omega_o,z=0) - \varepsilon\ ]\ . \tag{42}$$

Actually we have a finite spread in the frequency spectrum of the excited waves. For a particle with given μ and ε, the finite spread of the spectrum gives some width to the resonant line. This effect can be incorporated into our considerations that the resonant particles diffuse within the resonant region. The other effect of the finite spectrum is to make a little vague the edge of the resonant region which corresponds to the resonant velocity at $z = 0$ (see Fig.5). These considerations lead to the conclusion that the finite spread in the frequency spectrum has little influence on the asymptotic distribution in Eq.(42).

We estimate the particle loss for the example presented above by Eq.(42). The linear theory (Fig.4) suggests to assume that the spectrum is very sharp around $\omega_o = 0.62\omega_c$. Consequently we have $|V_r(\omega_o, z=0)| = 5.4 \times 10^9$ cm/sec. From Eq.(42) with Eq.(37), we have

$$N(z,t\to\infty) = \int_0^\infty d\varepsilon \int_0^\infty d\mu \; f_o(\varepsilon,\mu,t\to\infty)$$
$$= \frac{3}{2} N_o \left\{ 0.424\left[\frac{B_m - B(z)}{B_o}\right]^{\frac{1}{2}} - 0.608\left[\frac{B(z)-B_o}{B_o}\right]^{\frac{1}{2}} \right.$$
$$\left. \times D\left\{\left[0.206\,\frac{B_m - B(z)}{B(z)-B_o}\right]^{\frac{1}{2}}\right\} \right\} , \tag{43}$$

where

$$D(x) = e^{-x^2} \int_o^x e^{t^2} dt. \tag{44}$$

At the middle point of the mirror ($z = 0$), this is evaluated immediately to be $N(z = 0,\ t \to \infty) = 0.734\ N_o$. The value $N(z,t \to \infty)$ is shown as a function of z, in Fig.6.

The enhanced mirror loss caused by the instability is estimated in this case to be about 30 % of the initially contained particles. The characteristic time of the particle loss can be assumed to be about 10^2 times of the growing time as inferred from the example given in II - C. We conclude that the electron cyclotron wave instability is serious to the plasma containment in mirror fields, though the instability is of electromagnetic nature.

IV. CONCLUSIONS

In the discussions above, we see that by the development of electron cyclotron wave instability, the resonant electrons promptly diffuse along the diffusion lines in velocity space. Energyflow in the uniform plasma is an follows: The perpendicular energies of the resonant electrons are converted partially into the field energy and also partially into their parallel energies. The diffusion equation in velocity space, Eq.(18), will be applicable even to the problem of electron cyclotron heating.

For the non-uniform plasma produced in mirror devices, it is shown that the diffusion in velocity space results in an enhanced particle loss. The conclusion is based upon the knowledge of the mode with the maximum growth rate. Our arguments are, however, valid only so far as the medium which supports the waves does not change appreciable during the diffusion of the resonant electrons. In some experiments we have much contamination of cold electrons. The cold electrons support the waves, while the hot electrons are in resonance. Our analysis will be applicable to this case in a good approximation especially when the diffusion time of the resonant particles is faster than the flight times between mirror ends of the cold electrons.

It should be noted that the velocity distribution of the electrons remaining in the mirrors after some electrons are lost by the instability, has larger anisotropy than that of the initial state. This implies that the regenerated plasma is still unstable for perturbation of the same kind.

The ions are thought to be the uniform background. As long as the ion temperature is low, the ion loss is mainly determined by the loss of the energetic electrons. If the ion temperature becomes high, the ion dynamics comes into problem.

Throughout our discussions, the variations across the magnetic field are ignored because we are interested only in the electromagnetic mode propagating along the magnetic field. For other kinds of instabilities such as electrostatic cyclotron instability or drift instability, the radial variations must be taken into account.

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APPENDIX

We here show that the relation (33) holds under certain conditions. Let a local dispersion relation be

$$k(z) = Q(z,\omega), \tag{A1}$$

where we take the perturbation of the form $\exp[i\int^z k(z)dz - i\omega t]$ with complex $k(z)$ and ω, and characteristic lengths of unperturbed quantities are assumed to be much larger than k^{-1}. The function Q is also assumed to be analytic function of ω. Let ω, $k(z)$ and $Q(z,\omega)$ decompose into the real and imaginary parts, $\omega = \omega_r + i\omega_i$.etc. We also assume that $|\omega_r| >> |\omega_i|$ $|k_r| >> |k_i|$ and $|Q_r| >> |Q_i|$. Then Eq.(A1) leads

$$k_r(z) = Q_r(z,\omega_r) \tag{A2}$$

and

$$k_i(z) = Q_i(z,\omega_r) + \omega_i \frac{\partial Q_r}{\partial \omega_r}(z,\omega_r) \ . \tag{A3}$$

Note that $Q(z,\omega)$ does not depend on boundary conditions but only on the local properties of plasma of interest. Let us consider a local perturbation with a boundary condition of $k_i(z) = 0$. The temporal growth rate of this perturbation, $\gamma(z)$, is found to be

$$\gamma(z,\omega_r) = -Q_i(z,\omega_r) \Big/ \frac{\partial Q_r}{\partial \omega_r}(z,\omega_r) \ . \tag{A4}$$

From Eq.(A2) we have

$$\frac{\partial Q_r}{\partial \omega_r}(z,\omega_r) = \left[\frac{\partial \omega_r}{\partial k_r}(z,\omega_r) \right]^{-1} . \tag{A5}$$

Then Eq.(A3) can be written as

$$k_i(z,\omega_r) = [\omega_i(\omega_r) - \gamma(z,\omega_r)] \Big/ \frac{\partial \omega_r}{\partial k_r}(z,\omega_r) . \tag{A6}$$

In order to determine the values $k_i(z)$ and ω_i, we must ask for the boundary condition for the perturbation. A periodic boundary condition, for example, yields

$$\omega_i(\omega_r) = \frac{\int dz\ \gamma(z,\omega_r) \Big/ \frac{\partial \omega_r}{\partial k_r}(z,\omega_r)}{\int dz \Big/ \frac{\partial \omega_r}{\partial k_r}(z,\omega_r)} . \tag{A7}$$

This relation is Eq.(34), in III - A.

Another example of fixed boundary condition ($\omega_i = 0$) gives

$$k_i(z,\omega) = -\gamma(z,\omega) \Big/ \frac{\partial \omega}{\partial k_r}(z,\omega) . \tag{A8}$$

The equation used in III - A (Eq.(34)) is just the same as the above equation (A7).

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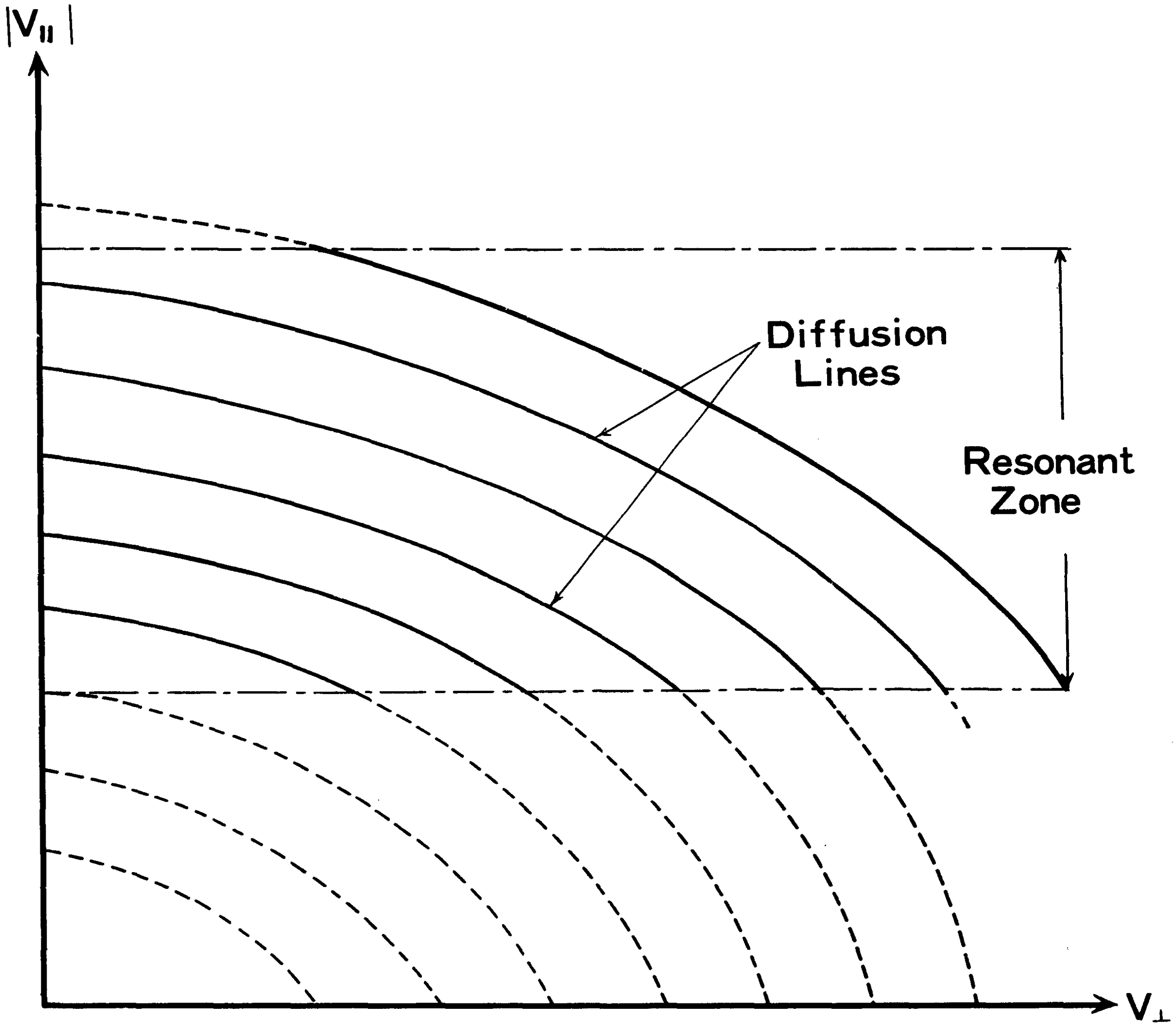

Fig.1 Diffusions in velocity space. The resonant region is the interval of $|v_{\parallel}| = |\omega_c - \omega_r(k)\ |/k$ where k and $\omega_r(k)$ are the quantities of the non-vanishing $E^{-}_{k,\ \omega_r}$. The diffusion lines are the ξ = const. lines where ξ is defined by Eq.(17).

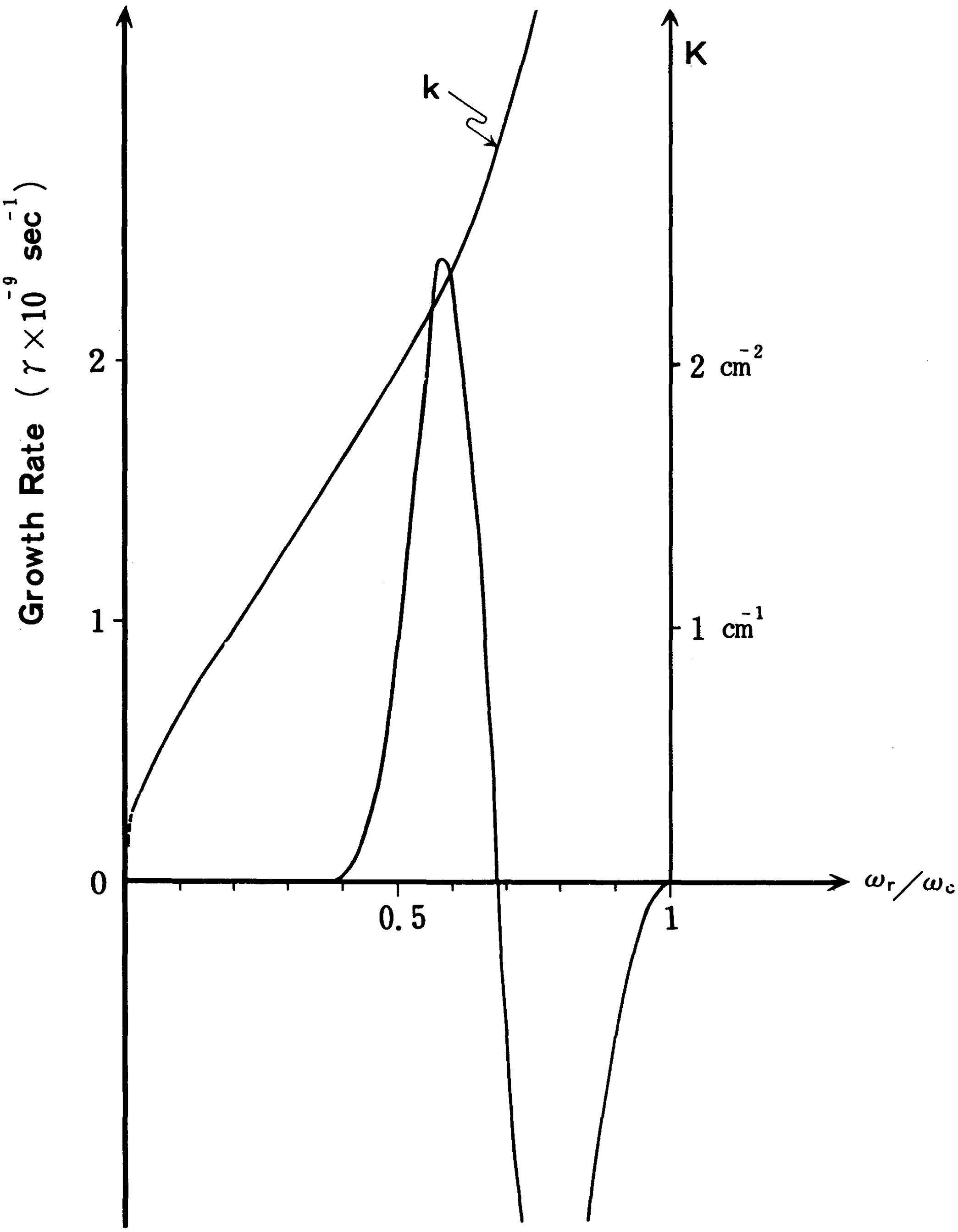

Fig.2 The growth rate γ and the wave number k versus the frequency ω_r for the case of Eq.(23) with $N_o = 10^{12}cm^{-3}$, T_e = 20 keV and B = 2 kG.

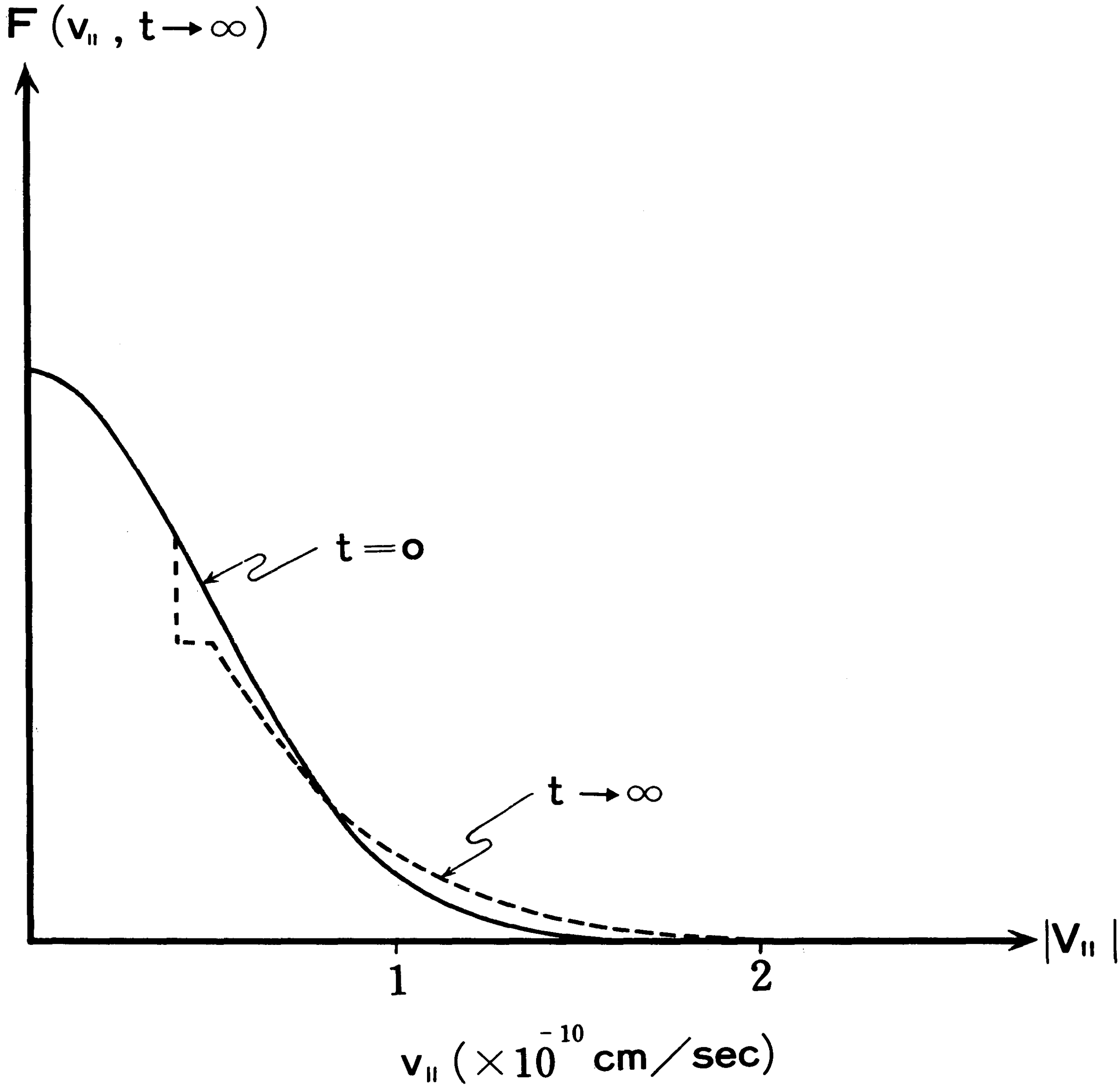

Fig.3 The asymptotic distribution $F(v_\parallel , t \to \infty)$ and the initial distribution $F(v_\parallel , t = 0)$ which is obtained by integrating Eq.(23) times $2\pi v_\parallel / N_o$ over $v_\perp$.

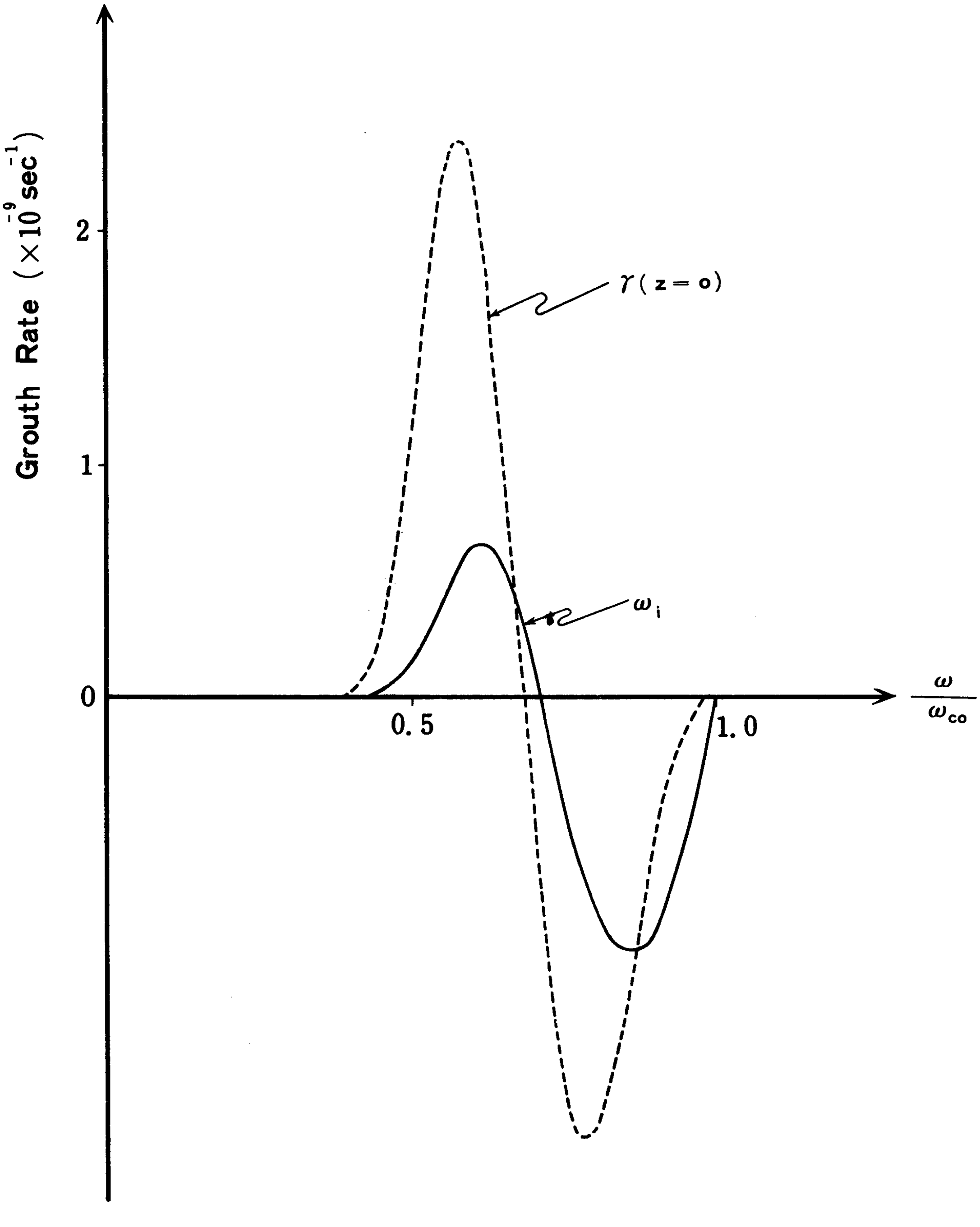

Fig.4 The solid line denotes the true growth rate ω_i defined by Eq.(34) for the case of Eqs.(30) and (31) with $T_\perp = T_\parallel$ = 20 keV, $N_o = 10^{12}\mathrm{cm}^{-3}$, B_o = 2 kG and B_m/B_o = 3. ω_{co} is the electron gyration frequency at $B = B_o$. The corresponding growth rate of the local solution at z = 0 is shown by the dotted line for comparison.

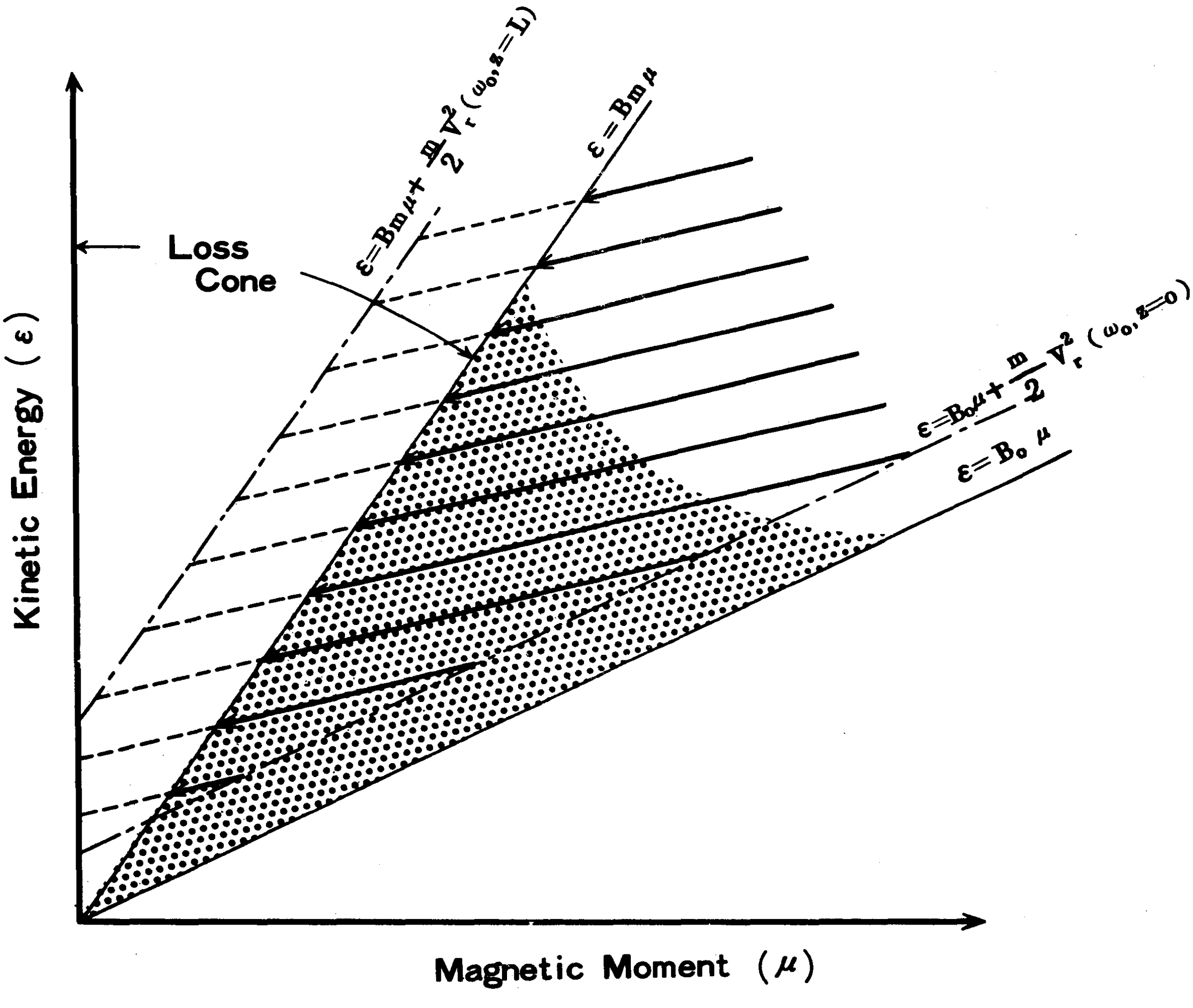

Fig.5 Diffusion in the μ - ε space where ε is the kinetic energy and μ the magnetic moment. The frequency spectrum of the unstable waves is assumed to be very sharp around $\omega_r = \omega_o$

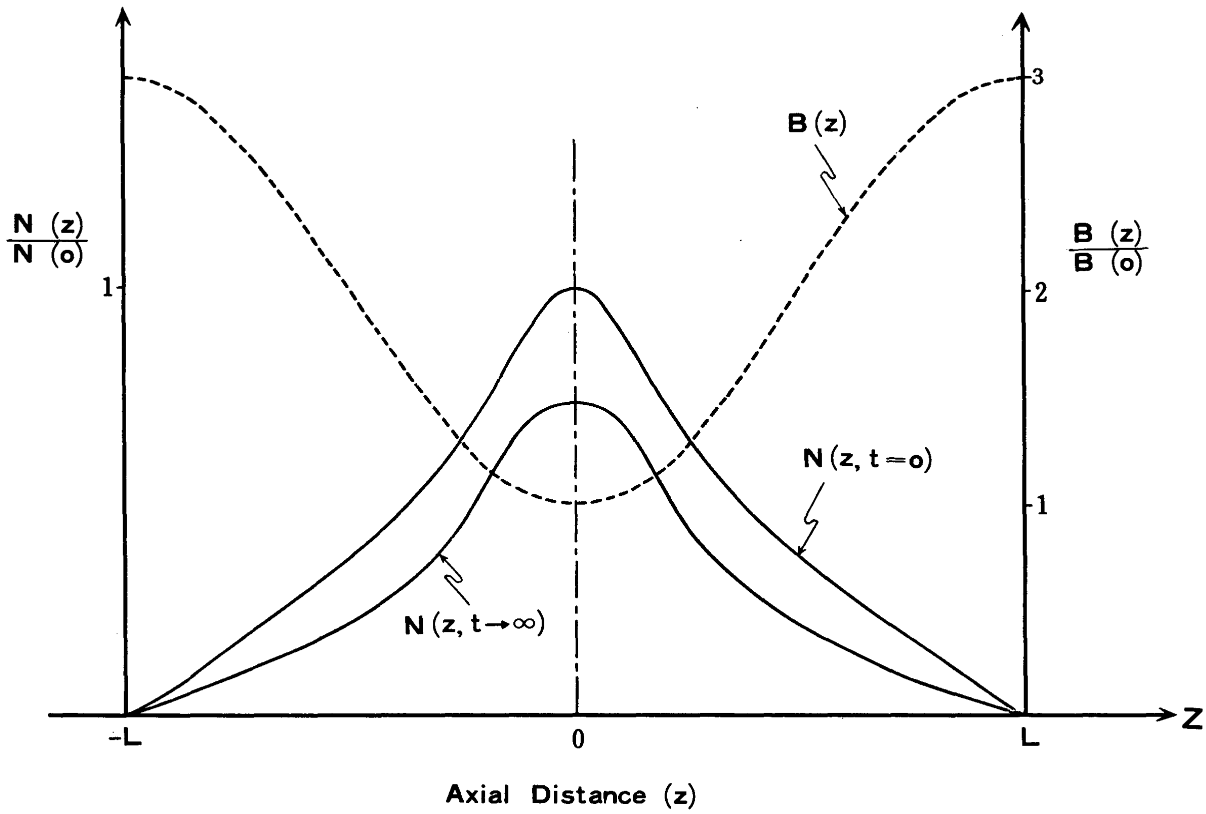

Fig.6 The change of the electron density distribution and the profile of the magnetic field.