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Correction to 'Generalization of Hamiltonian mechanics to a three-dimensional phase space'

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In a recent paper [N. Sato, Prog. Theor. Exp. Phys. 2021, 6, 063A01 (2021)] we introduced a generalization of Hamiltonian mechanics to three-dimensional phase spaces in terms of closed 3-forms. This correction addresses an error in the proof of theorem 3, which concerns the existence of a coordinate change transforming a closed 3-form into a constant form. Indeed, invertibility of a 3-form is not sufficient to ensure the existence of a solution X_t to eq. (77) when $n > 3$. The theorem can be corrected by restricting the class of 3-forms to those that are relevant to generalized Hamiltonian mechanics. Although the new theorem requires a stronger hypothesis, the formulation of dynamical systems with 2 invariants in terms of closed 3-forms, which is the key contribution of the paper, holds.

1. Correction

The formulation of theorem 3 at p. 15, Sect. 5 of [1] is not correct when the dimension of the manifold Ω is $n > 3$. This is because the invertibility of the 3-form w_t, i.e. the existence of a 3-tensor $Q_t^{jk\ell}$ such that $w_{tijk}Q_t^{jk\ell} = \delta_i^\ell$ is not sufficient to infer the existence of a solution X_t to eq. (77) therein, which reads

$$\sigma_{tjk} = -X_t^i w_{tijk}. \tag{1}$$

Indeed, the space of 2-forms σ_t, denoted by $\bigwedge^2 T^*\Omega$ has dimension $n(n-1)/2$. Hence, $\dim\left(\bigwedge^2 T^*\Omega\right) > \dim(\Omega) = n$ whenever $n > 3$. This means that the map $\hat{w}_t(X_t) : T\Omega \to \bigwedge^2 T^*\Omega$ defined by eq. (1), which sends vectors into 2-forms, can never be surjective for $n > 3$, i.e. there exist 2-forms σ_t with no generating vector field X_t. Only when $\sigma_t \in \mathrm{Im}(\hat{w}_t)$ we have a solution $X_t^i = -Q_t^{ijk}\sigma_{tjk}$ of (1).

In Sect. 2 of this correction we provide an amended version of theorem 3. Relevant text amendments are listed in Sect. 3. An additional theorem, which applies to closed 3-forms of the type $w = \omega\wedge dG$, with ω a 2-form and dG an exact 1-form, is proven in Sect. 4. A further result (proposition 1 of Sect. 4) is also proven explaining the relevance of this class of 3-forms for generalized Hamiltonian mechanics, intended as the ideal dynamics of systems with 2 invariants.

Below, we consider a smooth manifold Ω of dimension n and assume smoothness of the involved quantities.

2. Theorem 3

***Theorem*3.** *Let $w \in \bigwedge^3 T^*\Omega$ be a closed 3-form. Let w_{ijk}, $i, j, k = 1,\ldots, n$ denote the components of w with respect to a coordinate system $(x^1,\ldots, x^n)$ in Ω,*

$$w = \sum_{i<j<k} w_{ijk} dx^i \wedge dx^j \wedge dx^k. \tag{2}$$

Suppose that the $n \times n^2$ matrix $w_{i(jk)}$ has rank n. Take a sufficiently small neighborhood U of any $\boldsymbol{x}_0 \in \Omega$. Let $w_0 = w_{0ijk} dy^i \wedge dy^j \wedge dy^k$ denote the constant (flat) 3-form with components $w_{0ijk} = w_{ijk}(\boldsymbol{x}_0)$ in a coordinate system $(y^1,\ldots, y^n)$. Further assume that Moser's 2-form σ_t, $t \in [0, 1]$, such that $d\sigma_t = dw_t/dt$ in U, belongs to the image of the map $\hat{w}_t : T\Omega \to \bigwedge^2 T^*\Omega$ defined by $\hat{w}_t(X_t) = -i_{X_t} w_t$, i.e. *$\sigma_t \in \mathrm{Im}(\hat{w}_t)$ for some $X_t \in T\Omega$. Then, w_t has a right inverse $\mathfrak{J}_t$ in U. Furthermore, there exists a coordinate change $(x^1,\ldots, x^n) \to (y^1,\ldots, y^n)$ generated by the vector field $X_t = -\mathfrak{J}_t^{jk\ell} \sigma_{tjk} \partial_\ell$ such that*

$$w = w_0 \quad \text{in} \quad U. \tag{3}$$

Proof. We follow the steps of the classical proof of the Lie-Darboux theorem based on Moser's method [6,25]. Let w_0 denote the constant form on $\mathbb{R}^n$,

$$w_0 = \sum_{i<j<k} A_{ijk} dy^i \wedge dy^j \wedge dy^k, \tag{4}$$

with A_{ijk}, $i, j, k = 1,\ldots, n$, real constants. Consider a family of vector fields $X_t \in T\Omega$, $0 \leq t \leq 1$, defined in a neighborhood U of a point $\boldsymbol{x}_0 \in \Omega$ that generates a one-parameter group of diffeomorphisms g_t as follows,

$$\frac{d}{dt} g_t(\boldsymbol{x}_0) = X_t(g_t(\boldsymbol{x}_0)), \quad g_0(\boldsymbol{x}_0) = \boldsymbol{x}_0. \tag{5}$$

Next, define the family of 3-forms

$$w_t = w_0 + t(w - w_0). \tag{6}$$

We wish to obtain X_t, and thus g_t, so that the following property is satisfied

$$g_t^* w_t = w_0. \tag{7}$$

Here $g_t^* w_t$ denotes the pullback of w_t by g_t. Equation (7) implies that

$$\frac{d}{dt} g_t^* w_t = g_t^* \left(\frac{dw_t}{dt} + d i_{X_t} w_t \right) = 0, \tag{8}$$

where we used the fact that w_t is a closed differential form. By the Poincaré lemma, in a sufficiently small neighborhood W of $\boldsymbol{x}_0$, the closed differential form dw_t/dt is exact, i.e. there exists a 2-form $\sigma_t = \sum_{j<k} \sigma_{tjk} dx^j \wedge dx^k$ such that

$$\frac{dw_t}{dt} = d\sigma_t \quad \text{in} \quad W. \tag{9}$$

Hence, equation (8) can be solved in W by finding a vector field X_t satisfying

$$\sigma_t = -i_{X_t} w_t. \tag{10}$$

In components, Eq. (10) is equivalent to

$$\sigma_{tjk} = -X_t^i w_{tijk}, \quad j, k = 1, \ldots, n. \tag{11}$$

Next, observe that by hypothesis the $n \times n^2$ matrix $w_{i(jk)}$ has rank n. Similarly, setting the components of w_0 in the variables $(y^1,\ldots, y^n)$ to be given by the constant tensor $A_{ijk} = w_{ijk}(\boldsymbol{x}_0)$, the

$n \times n^2$ matrix $A_{i(jk)}$ has rank n. Furthermore, at the point $\boldsymbol{x}_0$ we may assume $w(\boldsymbol{x}_0) = w_0(\boldsymbol{x}_0)$ because the matrices w_{ijk} and A_{ijk} coincide there. Then, for $0 \leq t \leq 1$,

$$w_t(\boldsymbol{x}_0) = w_0(\boldsymbol{x}_0). \tag{12}$$

This implies that the $n \times n^2$ matrix $w_{ti(jk)}(\boldsymbol{x}_0)$ has rank n at $\boldsymbol{x}_0$. By continuity of the tensor w_{tijk} it follows that there exists a neighborhood V of $\boldsymbol{x}_0$ where the rank of the $n \times n^2$ matrix $w_{ti(jk)}$ is n. Define $U = W \cap V$. Then, the matrix $w_{ti(jk)}$ has a right-inverse inverse $\mathfrak{J}_t^{(jk)\ell}$. Since by hypothesis $\sigma_t \in \mathrm{Im}(\hat{w}_t)$, in U the solution X_t of equation (11) can be written in terms of the right inverse as

$$X_t^{\ell} = -\mathfrak{J}_t^{jk\ell}\sigma_{t\,jk}, \quad \ell = 1, \ldots, n. \tag{13}$$

The vector field (13) gives the desired local change of coordinates. □

3. List of text amendments

Statements pertaining to the notion of invertibility of 3-forms should be amended as follows:

(1) At p. 8, after Eq. (31), remove *'of rank n (the definition of rank will be given later)'*

(2) At p. 13, 1st line. After '*Then, we say that $\mathfrak{J}$ is the inverse of w.*' add the sentence: '*More generally, we say that $\mathfrak{J} \in \bigwedge^3 T\Omega$ is a weak inverse of w whenever the solution X of the equation $i_X w = -dH \wedge dG$ can be cast in the form $X^i = \mathfrak{J}^{ijk} G_j H_k$ (the notion of weak invertibility will be discussed in detail in a subsequent publication).*'

(3) After Eq. (60), correct as '*Let us derive necessary conditions …*'

(4) After Eq. (63), correct as '*Therefore, the notion of invertibility in Eq. (59) for the tensor w_{ijk} is related to …*'

(5) After Eq. (67), remove the sentence 'Indeed, the only invertibility condition… right inverse given by Eq. (59).'

(6) Replace the first paragraph of section 5 with the following: '*This section is dedicated to the proof of Lie-Darboux type theorems (theorems 3 and 4) in the generalized Hamiltonian framework with a three-dimensional phase space $N = 3$, $n \geq 3$. A direct consequence of these theorems is the local existence of an invariant (Liouville) measure. In particular, we prove a Lie-Darboux theorem (theorem 4) for closed 3-forms of the type $w = \omega \wedge dG$, with ω a 2-form and dG an exact 1-form. A further result (proposition 1) is also proven explaining the relevance of this class of 3-forms for generalized Hamiltonian mechanics, intended as the ideal dynamics of systems with 2 invariants. Below, we consider a smooth manifold Ω of dimension n and assume smoothness of the involved quantities. We have the following:* '

(7) Replace the old version of theorem 3 with theorem 3 of Sect. 2 of this correction.

(8) In Sect. 5 of the manuscript, add theorem 4 and proposition 1 of this correction.

(9) Two lines after Eq. (79): replace '*Theorem 3 has a number of consequences. First, any generalized Hamiltonian system possesses an invariant (Liouville) measure*' with '*We conclude this section with some observation concerning invertible 3-forms w that admit a constant (flat) expression $w_0 = \sum_{\mathrm{i<j<k}} A_{\mathrm{ijk}} dy^i \wedge dy^j \wedge dy^k$, $A_{ijk} \in \mathbb{R}$, by a suitable change of coordinates. First, any such form induces an invariant (Liouville) measure*'

(10) After Eq. (81), correct as 'Multiplying by the inverse B^{ijk} of A_{ijk}…'

(11) After Eq. (85), correct '*Nevertheless… introduced in Sect. 3*' with '*Furthermore, even if $n = 3m$ with m an integer, for canonical triplets of variables ($p^1, \ldots, p^m$, $q^1, \ldots, q^m$,*

$r^1, \dots, r^m$) to locally exist in the neighborhood of all points $\boldsymbol{x}_0 \in \Omega$, it is not sufficient that $w_{ijk}(\boldsymbol{x}_0)$ can be transformed by a linear change of basis into the generalized Levi-Civita symbol E_{ijk} (the covariant version of the tensor (44) introduced in Sect. 3), because the applicability of theorem 3 also requires that the relevant Moser 2-form $\tilde{\sigma}_t$ belongs to the image of the map $\hat{\tilde{w}}_t$.'

(12) In the concluding remarks section, correct as '*When the components of w define an $n \times n^2$ matrix of rank n, the form w has a right inverse. If the right-inverse corresponds to an antisymmetric 3-tensor, it defines a generalized Poisson operator $\mathfrak{J}$.*'

(13) In Sect. 6, last paragraph, replace '*the sufficient conditions*' with '*necessary conditions*'

4. Addendum

***Theorem* 4**. *Let $\omega \in \bigwedge^2 T^*\Omega$ denote a (not necessarily closed) 2-form of constant rank $2m = n - s$ and $dG \in T^*\Omega$ an exact 1-form such that $(x^1, \dots, x^n)$ defines a coordinate system in Ω with $x^n = G$. Define the 3-form $w \in \bigwedge^3 T^*\Omega$ as $w = \omega \wedge dG$ and suppose that $dw = 0$. Then, for every $\boldsymbol{x}_0 \in \Omega$ there exist a neighborhood U of $\boldsymbol{x}_0$ and a coordinate system $(p^1, \dots, p^\ell, q^1, \dots, q^\ell, G^1, \dots, G^\tau)$ with $n = 2\ell + \tau$ such that*

$$w = \omega_0 \wedge dG, \quad \omega_0 = \sum_{i=1}^{\ell} dp^i \wedge dq^i \quad \text{in} \quad U, \tag{14}$$

with $\ell = m$ if $\partial_n \in ker(\omega)$ and $2\ell \leq n - 1$ if $\partial_n \notin ker(\omega)$. Furthermore, given a 1-form $dH \in T^*\Omega$, linearly independent from dG, the phase space measure $d\Pi = dp^1 \wedge \dots \wedge dp^\ell \wedge dq^1 \wedge \dots \wedge dq^\ell \wedge dG^1 \wedge \dots \wedge dG^\tau$ is an invariant measure in U for the generalized Hamiltonian system $X \in TU$ such that

$$i_X w = -dH \wedge dG, \tag{15}$$

provided that such X exists. In addition,

$$i_X \omega_0 = -\tilde{d}H \quad \text{in} \ \ \Sigma_G, \tag{16}$$

where $\Sigma_G = \{\boldsymbol{x} \in U : G(\boldsymbol{x}) = c \in \mathbb{R}\}$ and $\tilde{d}$ denotes the differential operator on Σ_G.

Proof. Since $dw = d\omega \wedge dG = 0$, it follows that $\tilde{d}\omega = 0$ in any level set Σ_G. On the other hand,

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j = \sum_{i=1}^{n-1} \omega_{in} dx^i \wedge dG + \sum_{i<j}^{n-1} \omega_{ij} dx^i \wedge dx^j. \tag{17}$$

Define $\tilde{\omega} = \sum_{i<j}^{n-1} \omega_{ij} dx^i \wedge dx^j$. Evidently $w = \tilde{\omega} \wedge dG$. Since w is closed, this implies $\tilde{d}\tilde{\omega} = 0$. If $\partial_n \in \ker(\omega)$, from (17) it follows that $\omega = \tilde{\omega}$ and $\mathrm{rank}(\tilde{\omega}) = 2\ell = 2m = n - s$. Conversely, if $\partial_n \notin \ker(\omega)$ the forms ω and $\tilde{\omega}$ are different, with $\mathrm{rank}(\tilde{\omega}) = 2\ell = n - 1 - u \leq n - 1$. In either case, by the Lie-Darboux theorem for all $\boldsymbol{x}_0 \in \Omega$ there exist a neighborhood U of $\boldsymbol{x}_0$ and $n - 1$ local coordinates $(p^1, \dots, p^\ell, q^1, \dots, q^\ell, G^1, \dots, G^{s-1})$ or $(p^1, \dots, p^\ell, q^1, \dots, q^\ell, G^1, \dots, G^u)$ such that

$$\tilde{\omega} = \omega_0 = \sum_{i=1}^{\ell} \tilde{d}p^i \wedge \tilde{d}q^i \quad \text{in} \ \ \Sigma_G. \tag{18}$$

By smoothness, the coordinates $p^i, q^i : C^\infty(\Sigma_G) \to \mathbb{R}$ also define smooth functions $p^i, q^i : C^\infty(U) \to \mathbb{R}$. Then,

$$w = \sum_{i=1}^{\ell} \tilde{d}p^i \wedge \tilde{d}q^i \wedge dG = \sum_{i=1}^{\ell} dp^i \wedge dq^i \wedge dG = \omega_0 \wedge dG. \tag{19}$$

Now consider a solution $X \in TU$ of system (15). Recalling that, by hypothesis, dH and dG are linearly independent and noting that $0 = i_X i_X w = -(i_X dH)dG + (i_X dG)dH$, it follows that $i_X dH = i_X dG = 0$. On the other hand, $i_X w = i_X \omega_0 \wedge dG = -\tilde{d}H \wedge dG$, which implies

$$i_X \omega_0 = -\tilde{d}H \quad \text{in} \;\; \Sigma_G. \tag{20}$$

Since $\tilde{d}\tilde{\omega} = 0$, equation (20) defines a Hamiltonian system with invariant measure $\left(\bigwedge_{i=1}^{\ell} \tilde{d}p^i \wedge \tilde{d}q^i\right) \wedge \tilde{d}G^1 \wedge \ldots \wedge \tilde{d}G^{s-1}$ if $\partial_n \in \ker(\omega)$ or $\left(\bigwedge_{i=1}^{\ell} \tilde{d}p^i \wedge \tilde{d}q^i\right) \wedge \tilde{d}G^1 \wedge \ldots \wedge \tilde{d}G^u$ if $\partial_n \notin \ker(\omega)$ on Σ_G. Set $(G^1,\ldots, G^\tau) = (G^1,\ldots, G^{s-1}, G)$ if $\partial_n \in \ker(\omega)$ and $(G^1,\ldots, G^\tau) = (G^1,\ldots, G^u, G)$ if $\partial_n \notin \ker(\omega)$. It follows that

$$d\Pi = \left(\bigwedge_{i=1}^{\ell} dp^i \wedge dq^i\right) \wedge dG^1 \wedge \ldots \wedge dG^\tau, \tag{21}$$

defines an invariant measure for X in U. □

Proposition 1. *Let $w \in \bigwedge^3 T^*\Omega$ denote a closed 3-form and $dG \in T^*\Omega$ an exact 1-form such that $(x^1,\ldots, x^n)$ defines a coordinate system in Ω with $x^n = G$. Suppose that for any exact 1-form $dH \in T^*\Omega$ such that dH and dG are linearly independent there exists a vector field $X \in T\Omega$ solving*

$$i_X w = -dH \wedge dG. \tag{22}$$

Further assume that the 2-tensor $\omega_{ij} = w_{ijn}$ is invertible on the level sets $\Sigma_G = \{\boldsymbol{x} \in \Omega : G(\boldsymbol{x}) = c \in \mathbb{R}\}$ with inverse $\mathcal{J} \in \bigwedge^2 T\Sigma_G$ such that

$$\sum_{j=1}^{n-1} \omega_{ij}\mathcal{J}^{jk} = \delta_i^k, \quad i,k = 1,\ldots,n-1. \tag{23}$$

Then, on each level set Σ_G there exists a closed 2-form $\tilde{\omega} \in \bigwedge^2 T^*\Sigma_G$ such that

$$i_X \tilde{\omega} = -\tilde{d}H, \tag{24}$$

where $\tilde{d}$ denotes the differential operator on Σ_G. Furthermore,

$$w = \tilde{\omega} \wedge dG, \tag{25}$$

and

$$X = \mathfrak{J}(dH, dG), \tag{26}$$

with $\mathfrak{J} = \mathcal{J} \wedge \partial_n$.

Proof. Eq. (22) implies that

$$X^i w_{ijk} = H_k G_j - H_j G_k. \tag{27}$$

Since $x^n = G$, we have $X^i w_{ijn} = -H_j$ for $j = 1,\ldots, n-1$. Hence,

$$i_X \omega = -\tilde{d}H, \tag{28}$$

where $\omega \in \bigwedge^2 T^*\Omega$ is the 2-form $\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$ and $\tilde{d}$ is the differential operator on the level sets Σ_G. Since $i_X dG = 0$, the equations of motion (22) and (28) give

$$i_X (w - \omega \wedge dG) = 0. \tag{29}$$

Let $\xi \in \bigwedge^3 T^*\Omega$ denote a 3-form such that

$$i_X \xi = \sum_{j<k} X^i \xi_{ijk} dx^j \wedge dx^k = \sum_{j<k} \mathcal{J}^{i\ell} H_\ell \xi_{ijk} dx^j \wedge dx^k = 0. \tag{30}$$

It follows that

$$w - \omega \wedge dG = \xi. \tag{31}$$

On the other hand w, and thus ξ, cannot depend on H by construction. Therefore, we must have

$$\mathcal{J}^{i\ell}\xi_{ijk} = 0 \quad \forall \ \ell = 1, \ldots, n-1, \quad j, k = 1, \ldots, n. \tag{32}$$

However, the tensor $\mathcal{J}$ is invertible on Σ_G by hypothesis (equation (23)). Hence, $\xi = 0$ must be the zero 3-form. Then, Eq. (29) can be expressed in the form

$$w = \omega \wedge dG. \tag{33}$$

Using the closure of w, we therefore arrive at the equation

$$0 = d\omega \wedge dG = \tilde{d}\omega \wedge dG. \tag{34}$$

However, the 3-form $\tilde{d}\omega$ can be expanded on the basis elements $dx^i \wedge dx^j \wedge dx^k$ with $i < j < k$ and $i, j, k = 1,\ldots, n-1$, which satisfy $dx^i \wedge dx^j \wedge dx^k \wedge dG \neq 0$. It follows that

$$\tilde{d}\omega = 0, \tag{35}$$

i.e. the 2-form $\omega \in \bigwedge^2 T^*\Sigma_G$ is closed. The theorem is proven by noting that $X = \mathfrak{J}(dH, dG)$ with $\mathfrak{J} = \mathcal{J} \wedge \partial_n$ and by setting $\tilde{\omega} = \omega$. □

We remark that proposition 1 applies to the case in which n is odd, because the invertibility of ω implies that $n = 2m + 1$ for some $m \in \mathbb{N}$. The case in which n is even can be handled by a further integrability assumption on the kernel of ω.

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Statements and declarations

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Competing interests

The author has no competing interests to declare that are relevant to the content of this article.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] N. Sato, Prog. Theor Exp. Phys. **2021**, 063A01 (2021).