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Correction Term of Landau Damping Due to Trapped Particles

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Abstract

An approximate expression for the damping coefficient of an electron plasma wave with a finite amplitude is obtained. It is given in the form,

$$\gamma_{NL}(\tau) = \gamma_L[1 - \frac{\overline{E}_0{}^2}{64} (\frac{2}{3}\tau^3 - \tau^2 + \tau - \frac{1}{2} + \frac{1}{2}e^{-2\tau})]$$

where γ_L and γ_{NL} stand for "the linear and the non-linear Landau damping coefficients". Here τ and $\overline{E}(\tau)$ are adequately stretched time and electric field, and $\overline{E}_0 = \overline{E}(0)$. The above expression is valid for $\tau \ll \omega_B^{-1}$ where ω_B is the bouncing frequency of electrons trapped in a potential trough of the wave.

In a previous paper[1)], a set of simultaneous equations which represent a non-linear Landau damping are presented. These equations, in contrast to Ref. (2) and Ref. (3), are obtained in the form that every quantity included can be determined self-consistently: The reaction of the distribution function to the field as well as the inverse reaction is included correctly.

Though it is desirable to solve the equations analytically for a general case, it is difficult so that only a limiting case is solved explicitly, that is, the paper I shows that a sufficiently strong initial field does not damp and that its amplitude oscillates over several periods of the oscillation.

In this paper we use dimensionless quantities: namely the time t, the velocity v and the position x are normalized in terms of the inverse of the plasma frequency, ω_p^{-1}, the thermal velocity v_{th}, and the Debye length L_D, respectively; then the normalization of the electric field E is given by $4\pi e n_0 L_D$ (charge $-e$, density n_0). We show that when the time considered is small compared with the oscillation period of trapped electrons. We are able to solve the set of equations to find how the damping deviates from the usual Landau value.

The equations which describe the amplitude oscillation are given, following the previous paper I, by

$$\frac{dv^{(1)}}{d\tau} + E_1^{(1)} \exp ikx^{(0)} + E_1^{(1)*} \exp(-ikx^{(0)}) = 0 \qquad (1)$$

$$\frac{dE_1^{(1)}}{d\tau} + \frac{k^2}{2\pi^2} \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} dx^{(0)} \exp(-ikx^{(0)}) \int_{-R}^{R} v^{(1)}(0)\, dv^{(1)} = 0. \qquad (2)$$

Here $(x^{(0)}, \varepsilon v^{(1)})$ are the phase space coordinates of a resonant particle as measured from the wave frame moving with the phase velocity λ and ε is the expansion parameter defined by γ_L/ω_{pe}, Landau damping coefficient divided by the plasma frequency. The quantity $E_1^{(1)}(\tau)$ is the lowest order term in the expansion of the electric field $E(x, t)$,

$$E(x,t) = \varepsilon \sum_{l=-\infty}^{\infty} \sum_{\alpha=1} \varepsilon^{\alpha} E_l^{(\alpha)}(\tau) e^{ilk(x-\lambda t)} \tag{3}$$

and the perturbation is applied like exp ikx(0) at $\tau = 0$ where τ is the stretched time defined by $\tau = \varepsilon t$. The quantity $v^{(1)}(0)$ appeared in (2) is the initial value of $v^{(1)}(\tau)$ and must be represented in terms of $v^{(1)}(\tau)$ and $x^{(0)}(\tau)$ at time τ; i.e., $v^{(1)}(0) = V(x^{(0)}(\tau), v^{(1)}(\tau), \tau)$. Notice that the upper and the lower limits R and $-R$ of the integral in (2) are sufficiently large and the final result does not depend on them, so that we may take the range of integration as $[-\infty, \infty]$.

The integral variables $x^{(0)}$ and $v^{(1)}$ in eq.(2) can be transformed into $x^{(0)}(0)$ and $v^{(1)}(0)$ and this transformation makes the integration simpler. The Jacobian of this transformation is easily seen to be unity and the range of integration over $x^{(0)}$ can be taken as same as before due to the fact that the $kx^{(0)}$ has the period of 2π owing to eq.(1)(Appendix). According to this transformation, $x^{(0)}(\tau)$ and $v^{(1)}(\tau)$ are expressed in terms of $x^{(0)}(0)$ and $v^{(1)}(0)$.

As a variation of the phase ϕ of $E_1^{(1)} = |E_1^{(1)}|e^{i\phi}$ in a unit time is much smaller than that of $|E_1^{(1)}|$ itself, we regard the phase as a constant or $-\pi/2$ for convenience in the time scale under consideration. The smallness of the variation of ϕ can be proved in the same manner as described below. Now we rewrite the eqs.(1) and (2) in the form

$$\frac{du}{d\tau} = \frac{d^2 y}{d\tau^2} = - \overline{E} \sin y \tag{4}$$

$$\frac{d\overline{E}}{d\tau} = - \frac{1}{\pi^2} \int u_0 du_0 \int_{-\pi}^{\pi} dy_0 \sin y \tag{5}$$

Here we have employed the following transformations:
$E_1^{(1)} = (1/2ik)\overline{E}$, $\overline{E}_0 = \overline{E}(0)$, $y = kx^{(0)}$, $u = kv^{(1)}$, $y_0 = y(0)$, $u_0 = u(0)$.

We try to solve eqs. (4) and (5) approximately. The equation of motion (4) is formally solved and is given by

$$y = y_0 + u_0\tau + \Delta y$$

$$\Delta y = \int_0^{\tau} d\tau'(\tau' - \tau)\overline{E}(\tau')\sin(y_0 + u_0\tau' + \Delta y(\tau')) \tag{6}$$

Noting that

$$|\Delta y| \leqq -\overline{E}_0 \int_0^{\tau} d\tau'(\tau' - \tau) = \frac{\overline{E}_0\tau^2}{2} \tag{6'}$$

we can expand the right hand side of eq.(7) with respect to $\Delta y(\tau')$ when $\overline{E}_0\tau^2/2 << 1$ (which is sufficient for the validity of the expansion). After iteration we get

$$\Delta y = \int_0^\tau S(\tau,\tau')d\tau' + \int_0^\tau C(\tau,\tau')d\tau' \int_0^{\tau'} S(\tau',\tau'')d\tau''$$
$$+ \int_0^\tau C(\tau,\tau')d\tau' \int_0^{\tau'} C(\tau',\tau'')d\tau'' \int_0^{\tau''} S(\tau'',\tau''')d\tau'''$$
$$- \frac{1}{2}\int_0^\tau S(\tau,\tau')d\tau' \int_0^{\tau'} S(\tau',\tau'')d\tau'' \int_0^{\tau'} S(\tau',\tau''')d\tau''' + \ldots.$$

where

$$\left.\begin{matrix} S(\alpha,\beta) \\ \\ C(\alpha,\beta) \end{matrix}\right\} = -(\alpha - \beta)\overline{E}(\beta) \times \left\{\begin{matrix} \sin(y_0 + u_0\beta) \\ \\ \cos(y_0 + u_0\beta) \end{matrix}\right. .$$

Likewise $\sin y$ on the right hand side of (5) is expanded with respect to Δy after replacing y by $y_0 + u_0\tau + \Delta y$. Eq.(5) then becomes

$$\frac{d\overline{E}}{d\tau} = -\frac{1}{\pi^2}\int_{-\pi}^{\pi} dy_0 \int_{-\infty}^{\infty} du_0 u_0 \cos(y_0+u_0\tau) \int_0^\tau S(\tau,\tau')d\tau'$$
$$- \frac{1}{\pi^2}\int_{-\pi}^{\pi} dy_0 \int_{-\infty}^{\infty} du_0 u_0 \cos(y_0+u_0\tau) \int_0^\tau C(\tau,\tau')d\tau' \int_0^{\tau'} C(\tau'.\tau'')d\tau'' \int_0^{\tau'} S(\tau',\tau''')d\tau'''$$
$$+ \frac{1}{2\pi^2}\int_{-\pi}^{\pi} dy_0 \int_{-\infty}^{\infty} du_0 u_0 \cos(y_0+u_0\tau) \int_0^\tau S(\tau,\tau')d\tau' \int_0^{\tau'} S(\tau',\tau'')d\tau'' \int_0^{\tau'} S(\tau',\tau''')d\tau'''$$
$$+ \frac{1}{\pi^2}\int_{-\pi}^{\pi} dy_0 \int_{-\infty}^{\infty} du_0 u_0 \sin(y_0+u_0\tau) \int_0^\tau S(\tau,\tau')d\tau' \int_0^\tau C(\tau,\tau'')d\tau'' \int_0^{\pi''} S(\tau'',\tau''')d\tau'''$$
$$+ \frac{1}{6\pi^2}\int_{-\pi}^{\pi} dy_0 \int_{-\infty}^{\infty} du_0 u_0 \cos(y_0+u_0\tau) \int_0^\tau S(\tau,\tau')d\tau' \int_0^\tau S(\tau,\tau'')d\tau'' \int_0^\tau S(\tau,\tau''')d\tau'''$$
$$+ \ldots.. \qquad (7)$$

where we have used the orthogonality of trigonometric functions. Integrations in the first term on the right hand side of eq.(7) can be carried out and we get $-\overline{E}$ which gives the Landau damping. The other terms are integrated over y_0 and u_0. The integrations over τ', τ'' and τ''', however, are almost impossible unless an explicit form of $\overline{E}(\tau)$ is assumed. We solve eq.(7) by the procedure of iteration; Noting that the first term yields $-E$, we have $\overline{E} = \overline{E}_0 e^{-\tau}$ as the first order approximation and we put this into $S(\alpha,\beta)$ and $C(\alpha,\beta)$ in the higher order terms. we get finally

$$\frac{d\overline{E}}{d\tau} = - \overline{E}\ [1 - \frac{\overline{E}_0{}^2}{64} (\frac{2}{3}\tau^3 - \tau^2 + \tau - \frac{1}{2} + \frac{1}{2} e^{-2\tau})] \tag{8}$$

or in the usual way

$$\gamma_{NL}(\tau) = \gamma_L [1 - \frac{\overline{E}_0{}^2}{64} (\frac{2}{3}\tau^3 - \tau^2 + \tau - \frac{1}{2} + \frac{1}{2} e^{-2\tau})] \tag{8'}$$

where γ_{NL} stands for "non-linear Landau damping coefficient".

Let us examine the above equation. When $\overline{E}_0$ is getting larger, Eq.(8) holds only for a small τ as we see from eqs.(6) and (6'). In the limit of $\tau \to 0$ Eq.(8) becomes

$$\frac{d\overline{E}}{d\tau} = - \overline{E}\ \{1 - \frac{\overline{E}_0{}^2}{12} (\frac{\tau}{2})^4 \} \tag{9}$$

or in the scale of the bouncing time $\omega_B{}^{-1} = (\sqrt{\overline{E}}\ )^{-1}$

$$\frac{d\overline{E}}{d\eta} = - \frac{\overline{E}}{\sqrt{\overline{E}_\eta}} \{1 - \frac{1}{12} (\frac{\eta}{2})^4 \} , \qquad \eta = \omega_B \tau . \tag{9'}$$

These show the correction term of Landau damping is proportional to τ^4 or η^4 in an early stage of time τ. Eq.(9) says that the deviation from the usual Landau damped waves becomes outstanding at a time, which is independent of E_0 in the scale of η, because E_0 does not appear explicitly in the curely bracket. Also the field hardly damps due to the factor $(\overline{E}_0)^{-1/2}$ as E_0 becomes large. This tendency coincides with the experiment[4].

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Appendix

The original form of eq.(5) is

$$\frac{d\bar{E}}{d\tau} = -\frac{1}{\pi^2}\int_{-\infty}^{\infty} du(\tau)\int_{-\pi}^{\pi} dy(\tau)u_0[u(\tau),y(\tau),\tau]\cos(\tau) \qquad \text{(A-1)}$$

Suppose that we choose $u(\tau')$ and $y(\tau')$ as the integral variables, where $0 \le \tau' \le \tau$. Noting that the Jacobian of this transformation is unity and that the periodicity with respect to y gives

$$y[u(\tau'),\ y(\tau')-2\pi,\ \tau',\ \tau] = y[u(\tau'),\ y(\tau'),\ \tau',\ \tau]-2\pi$$

$$u[u(\tau'),\ y(\tau')-2\pi,\ \tau',\ \tau] = u[u(\tau'),\ y(\tau'),\ \tau',\ \tau]$$

we have

$$\frac{d\bar{E}}{d\tau} = -\frac{1}{\pi^2}\int_{-\infty}^{\infty} du(\tau')\int_{-\pi}^{\pi} dy(\tau')\cos y[u(\tau'),\ y(\tau'),\ \tau',\ \tau]$$

$$\times\ u_0[u(\tau'),\ y(\tau'),\ \tau',\ \tau]$$

Suppose $\tau' = 0$, then we get eq.(5).