INSTITUTE OF PLASMA PHYSICS

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RESEARCH REPORT

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Unstable BGK Solution due to Excitation of an Ion Acoustic Wave

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, JAPAN.

Abstract

Some comments on the stability of the BGK solutions are given using idealized models, and a possibility is shown of excitation of an ion acoustic wave by trapped electrons in a large amplitude potential trough.

Text

The object of this report is to investigate a response of trapped electrons in a periodic potential trough produced in a plasma to a low frequency perturbation and to discuss the stability of BGK solutions.[1]

It is well-known that a stationary large amplitude wave is allowed to be an exact solution of the Vlasov equation, a BGK solution. We suppose that this wave is a plasma oscillation with a frequency ω_0 and a wave number k_0. The distribution function can be represented with one variable, the total energy E in the wave frame which moves with the phase velocity v_p (= ω_0/k_0). When the total energy of an electron is less than the maximum potential energy, the electron is called to be trapped. Otherwise it is called to be untrapped. Since we are interested in a response of the trapped particles to a perturbation, it is important what shape of the distribution of the trapped particles is chosen. The distribution function for the trapped region may be either concave, convex, or flat. For convenience of calculation, we choose the following two extreme distributions displayed in Figs. 1 and 2 for the trapped region.

The regions I, II and III in both Figs.1 and 2 represent the trapped regions corresponding to a deformed, a flat part of the distribution function, and the untrapped region, respectively. In Fig.1, we suppose that N_1 electrons are distributed in the elliptic-cylindrical portion, which is hereafter called A, above the level of the distribution of II and III. In Fig.2, we have, on the other hand, a cavity of the same volume as A.

It is not certain whether any periodic potential be possible or not in such a distribution as is shown in Fig.1. However, a case in which the particle density in the trapped region is larger than that in the untrapped region has been obtained in the limit of small amplitude by Kito and Kaji.[2]

It is shown, on the other hand, after a computer experiment by Berk and Roberts[3)] that a distribution of Fig.2 causes a periodic potential.

Let us first consider the case of Fig.1. From the knowledge of polarizations of dielectrics, it is easily understood that the contribution to the polarisation arises from the electrons only of the portion A. Suppose the region I is sufficiently narrower than the region II, then all the trapped electrons in I oscillate at a roughly equal frequency in the potential trough, though a frequency of an electron is slightly different from the others according to its total energy. These electrons, also, move with the large amplitude wave, with the phase velocity v_p. Thus these electrons should be able to act like a beam composed of a sequence of lumps of electrons where the period is $2\pi/k_0$; each lump includes N_1 electrons oscillating with an eigen frequency ω_B defined later i.e. a bouncing frequency in a potential trough. We might expect something similar to a two-stream instability.[4)]

We consider an application of a low frequency perturbation like $E(k, \omega)\exp(ikx-i\omega t)$ to our beam-plasma system. We choose the frequency ω and the wave number k to be of the order of those of an ion acoustic wave. In the wave frame, when the Doppler-shifted frequency, $-(\omega - kv_p)$, tends to the eigen frequency ω_B of the trapped electrons, we expect the oscillation with the frequency ω to be excited and the the energy of the beam to be transferred to the plasma. Supposing that $\omega/k \simeq c_s$ (= sound velocity = $\sqrt{T_e/M}$, T_e: electron temperature, M: ion mass) and v_p >> electron thermal velocity >> c_s , we may anticipate that an instability takes place when $k \simeq \omega_B/v_p$ ($\ll k_0$) and $\omega \simeq \omega_B c_s/v_p$.

We shall express the above argument in the form of equations. The perturbation is supposed to cause a deviation δx from an unperturbed orbit of a trapped electron $x_N + \tilde{x} + v_p t$. Here x_N denotes the N-th bottom of the potential trough and $\tilde{x}$ corresponds to the intrinsic oscillation without the perturbation. Since the region I has been supposed to occupy a very small domain centered on a point of potential minimum, the potential $\phi(\tilde{x})$ may be approximated to $\phi(0)k_0^2\tilde{x}^2/2$. The equation for δx, then, is given by

$$\frac{d^2}{dt^2}\,\delta x = -\,\omega_B^2\delta x - (e/m)E(k,\ \omega)\exp\{ik(x_N+\tilde{x}+\delta x)-i(\omega-kv_p)t\}, \tag{1}$$

and $\omega_B^2 = e\phi(0)k_0^2/m$. Note $k|\tilde{x}| < \pi k/k_0 \ll 1$ and we neglect the $k(\tilde{x} + \delta x)$ in the exponential on the right hand side and also use a continuous variable x instead of the discrete one x_N. It should be noted that this simplified model of continuous beam may be legitimated since we are considering a large scale perturbation i.e. $k \ll k_0$. Now we have a solution

$$\delta x = (e/m)E(k,\ \omega)\exp\{ikx-i(\omega-kv_p)t\}/\{(\omega-kv_p)^2-\omega_B^2\}, \tag{2}$$

or in terms of Fourier component

$$\delta x(k,\ \omega - kv_p) = (e/m)E(k,\ \omega)/\{(\omega - kv_p)^2 - \omega_B^2\}.$$

Therefore the polarization due to the trapped particles, $P^t(k,\ \omega)$, is represented in terms of the polarization $P_b^t(k,\ \omega)$ in the wave frame as

$$P^t(k,\ \omega) = P_b^t(k,\ \omega - kv_p) = -\,en_T\delta x(k,\ \omega - kv_p)$$

$$= -\,\omega_T^2\,E(k,\ \omega)/4\pi\{(\omega - kv_p)^2 - \omega_B^2\},$$

where $\omega_T^2 = 4\pi n_T e^2/m$ and $n_T = N_1 k_0/2\pi$.

We next examine polarizations due to the untrapped electrons $P^u(k, \omega)$ and due to ions $P^i(k, \omega)$. Their dependence on the large amplitude wave is not so drastic as the dependence of $P^t(k, \omega)$, then we assume the polarizations can be given in the same expressions that yield the dispersion relation of an ion acoustic wave, namely

$$P^u(k, \omega) = (k_e^2/4\pi k^2)E(k, \omega), \quad P^i(k, \omega) = -(\omega_i^2/4\pi\omega^2)E(k, \omega),$$

where k_e and ω_i are the electron Debye constant and the ion plasma frequency, respectively.

Now we have the dispersion relation to our system as follows:

$$1 - \omega_i^2/\omega^2 + k_e^2/k^2 = \omega_T^2/\{(\omega - kv_p)^2 - \omega_B^2\}. \tag{3}$$

This form of equation can be obtained after some simplifications of the treatment by Kruer et al[4] and also of the more precise formulation by Goldman.[5] It should be noted that the definition of the trapped electron density n_T is different from that of the Ref.4. The dispersion relation (3) is schematically displayed in Fig.3. The width of the region of k in which ω has complex values is denoted by δk and the center of this region is named with k_M. We obtain in the form

$$\left.\begin{aligned} &\delta k = (2\omega_T\omega_B/k_e v_p^2)\sqrt{c_s/v_p}, \qquad k_M = (\omega_B/v_p)(1 + \omega_i/k_e v_p), \\ &\mathrm{Max}(Im(\omega)/k_M c_s) = (\omega_T/2\omega_i)\sqrt{c_s/v_p}. \end{aligned}\right\} \tag{4}$$

The $\mathrm{Im}(\omega)$ takes a maximum value at $k = k_M$. Let us estimate the above quantities by using experimental data by Wharton et al.[6] We have $c_s \delta k \simeq 2.6 \times 10^{-2}\mathrm{Hz}$, $c_s k_M \simeq 5.7 \times 10^5\mathrm{Hz}$, and $\mathrm{Max}(\mathrm{Im}(\omega)/c_s k_M) \simeq 2.6 \times 10^{-2}$, where we have used the proton mass as the ion one.

Next, we consider the case of Fig.2 where a hole due to a lack of electrons is present. Recall that the volume of the hole is the same as that of A of Fig.1. It is easily seen that the situation is the same as in the case of Fig.1 except for the inverse direction of the polarization. We then have the dispersion relation only by replacing ω_T^2 by $-\omega_T^2$ in Eq.(3) and we can conclude after a repetition of the previous procedure that the system becomes unstable. It is worth remarking that the instability in the case of Fig.1 is convective and the one in the case of Fig.2 is absolute, and then the latter is more dangerous though every quantity shown in (4) is almost same as those in Fig.1.

Now we may conjecture the followings. When any unevenness of the distribution function is present in the region of trapped particles whether it is smooth or not, it casuses a polarization which is resonant with low frequency oscillations, for example, ion acoustic waves, and then the system becomes unstable. Therefore, for a BGK solution to be stable, the distribution of the trapped region must be at least flat and be smoothly connected with that of the untrapped region.

Experimentally, the above conclusion suggests that when a relatively strong field is applied to a plasma and an excess or a lack of trapped electrons is provided, ion acoustic waves may be excited if the parameters (4) are fit to the experiment.

References

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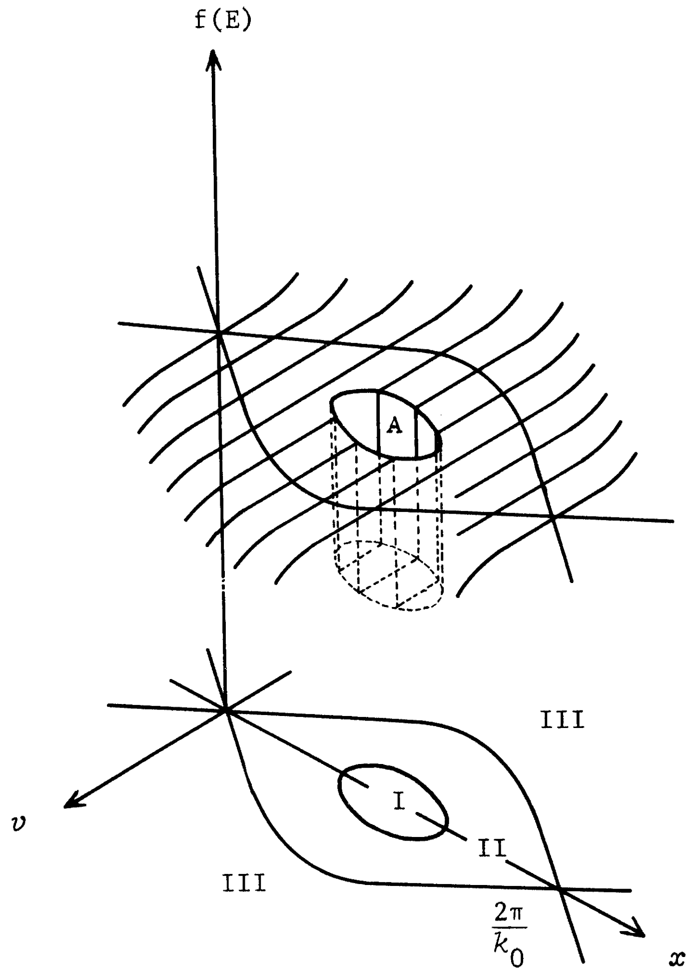

Fig.1 A distribution function $f(E)$ which has an elliptic-cylindrical projection, named by A, corresponding to the region I in the phase space represented in the wave frame. In the region II, the distribution function is flat and smoothly connected with that of the region III. Here is shown only a unit domain of the whole system with a spacial periodicity of $2\pi/k_0$.

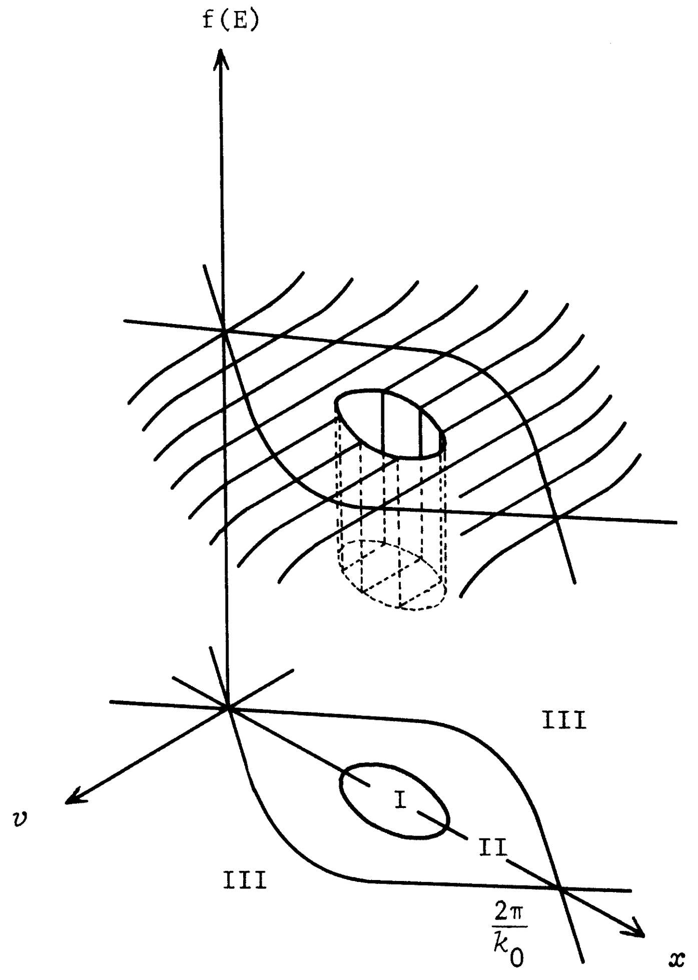

Fig.2 A distribution function which has an elliptic-cylindrical depression, whose volume is identical with that of A, corresponding to the region I in the phase space.

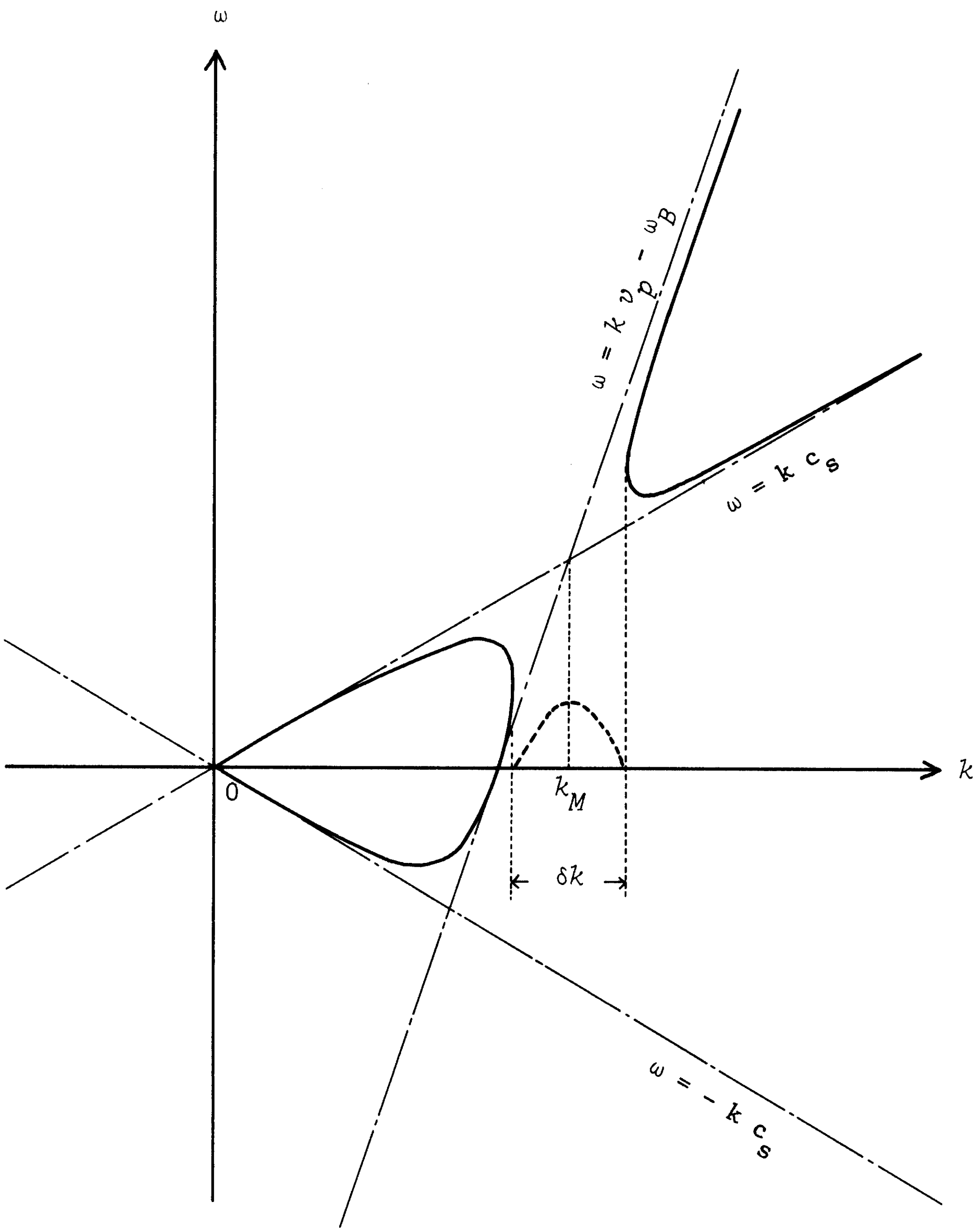

Fig.3 The dispersion curves due to Eq.(3) dealing with the case of Fig.1. Only the relevant branches are drawn for the positive k - space. The bold dotted line means the Im(ω) in an arbitrary unit.

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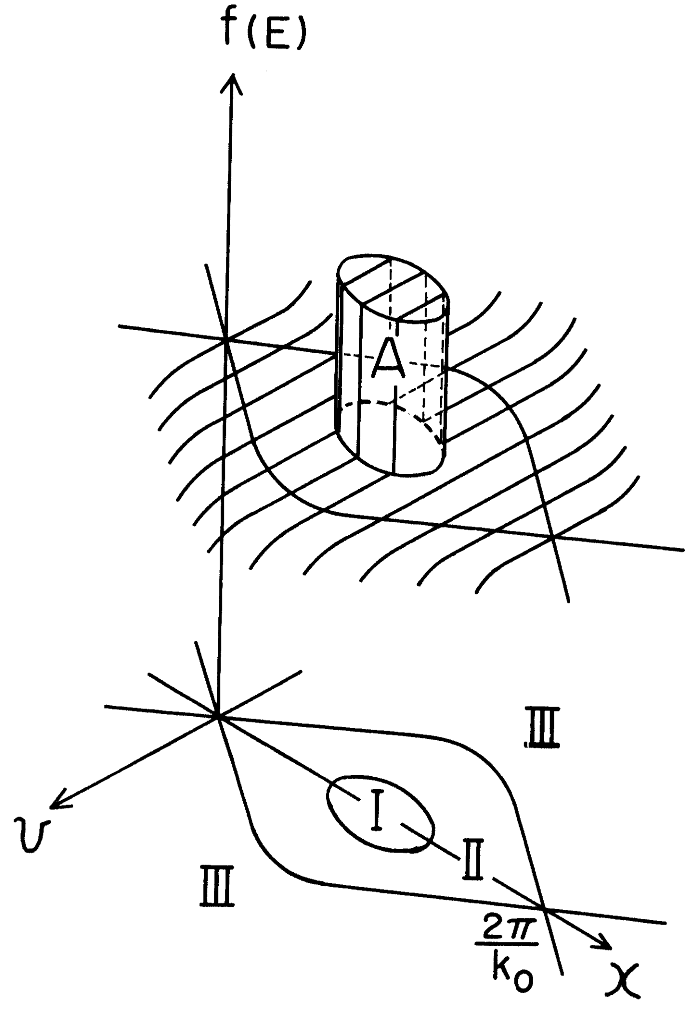
The Fig. 1 on the page 8 should be replaced with the following figure.



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