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On the Nonlinear Schrödinger Equation for Langmuir Waves

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Abstract

A direct derivation of the nonlinear Schrödinger equation for Langmuir waves is presented, based upon the nonlinear wave packet ansatz of Karpman and Krushkal. Both fluid and Vlasov equation formulations are used. The results obtained are essentially equivalent to those found earlier by Taniuti, et al. using reductive perturbation theory, including the importance of wave particle resonances at the group velocity for the long time behavior of the amplitude of modulated waves. Separating the wave packet considerations from the calculation of the nonlinear frequency shift makes it possible to attack the latter with whatever method facilitates the analysis of that part of the problem. In addition, certain ambiguities concerning singularities in velocity integrations are resolved, and the connection with a well-posed initial value problem is made somewhat clearer. This method can be used equally well for other waves, and may be of help particularly in situations where it is not clear, a priori, what scaling to adopt in applying reductive perturbation theory.

I. Introduction

Weakly nonlinear plasma waves, in regimes where the linear dispersion relation gives a phase velocity nearly independent of the wave number, k, can be described by the Kortweg-deVries (KdV) equation,[1] whose mathematical and physical properties have been rather thoroughly investigated. For dispersive regimes —— e.g. ion acoustic waves, where k is an appreciable fraction of the Debye wavenumber, k_D, or waves, such as electrostatic electron plasma (Langmuir) waves, which have non-zero frequency, ω, for $k \to 0$ —— the amplitude can be described by a nonlinear Schrödinger (NLS) equation,[2] provided the amplitude variation in x and t is slow compared to k and ω.

The general problem of slow amplitude variations due to nonlinear effects has been approached from several, apparently disparate, points of view. Whitham and Lighthill[3] start with the exact, nonlinear periodic solution (often referred to as "uniform wave trains") which can readily be obtained for typical nonlinear wave equations and, assuming the amplitude, wave number, etc. to be slowly varying functions of x and t, derive equations for these quantities, using rather elegant Lagragian techniques. A quite different point of view was adopted by Taniuti[4] and coworkers, who introduced a consistent scaling of the amplitude, wave number spread and space-time variables which is known as "reductive perturbation theory" and used it to derive both the KdV and NLS equations for the appropriate situations. Finally, Karpman and Krushkal[5] used a simple, albeit heuristic,nonlinear generalization of the familiar wave-packet formalism, again obtaining the NLS equation.

Each of these points of view has its advantages and drawbacks. In the examples treated by Whitham and Lighthill, the amplitude is found to obey not the NLS equation but, rather, a hyperbolic equation, with the usual consequences as regards steepening (of the envelope, rather than the wave itself) at large times. This is somewhat disturbing, since in all of these theories it is precisely the long time behavior which is allegedly described, and the physical significance of this aspect is still not clear. Reductive perturbation theory is elegant and, for the most part, fairly rigorous, but involves a good deal of formal manipulation in the course of which the physics is not always easy to follow. In addition, the use of multiple time variables introduces some questions regarding the relation to a well-posed initial value problem, a difficulty common to most theories of the behavior of systems at large times. The Karpman and Krushkal approach is short, sweet and general, but does not lead to specific expressions for the coefficients in the NLS equation, as does reductive perturbation theory, for example.

We present here a simple derivation of the NLS equation for electron plasma (Langmuir) waves which, in essence, follows through on the Karpman-Krushkal paper by actually calculating the coefficients for a particular case of interest. The results obtained are in agreement with those obtained earlier by Asano, Taniuti and Yajima[6] for the fluid equations and by Ichikawa and Taniuti[7] for the kinetic formulation (Vlasov equation), using reductive perturbation theory. Although less elegant, our approach has the virtue of exhibiting more clearly the "inner workings", so to speak, and of being more closely related, in the kinetic case, to the conventional perturbation analysis[8]. In addition, the point of view adopted here breaks the theory into two, nearly disjoint parts: one is concerned with wave packet ideas and leads, in a simple and general (albeit non-rigorous) way to the NLS equation; the other is concerned only with the calculation, for any particular case of interest, of the lowest order nonlinear corrections to the dispersion relation.

In section II, we derive the NLS equation a la Karpman and Krushkal[5], generalizing their discussion and that of Brinca[9] to allow for a non-local structure in the nonlinear term. In section III we present what we believe to be the least laborious way of calculating the nonlinear dielectric function ε_{NL} for the case of Langmuir waves treated in the fluid approximation, and in section IV we carry out a calculation of ε_{NL} for Langmuir waves using the Vlasov equation. Conclusions and discussion of the results are given in section V.

II. General Derivation of the Nonlinear Schrödinger Equation

As already noted, we divide the calculation into two parts. In the first, treated in this section, we assume the existence of a suitable nonlinear dispersion equation,

$$\varepsilon(k,\omega,A) = 0 \, , \tag{1}$$

where A is the wave amplitude, and derive the NLS equation; the explicit calculation of ε is carried out in the next two sections. In deriving the NLS equation, we follow here the slightly tidier version of the original Karpman-Krushkal formulation given recently by Brinca.[9] If initially ($t = 0$) the wave amplitude (in our example, the electrostatic potential) is

$$\phi(x,0) = \int dk\phi_k \exp[ikx] + c.c. \, ,$$

with ϕ_k peaked around $k = k_o$, then in linear theory, the long time behavior of the system (after the various transient effects have died away) is given by

$$\phi(x,t) = \int dk\phi_k \exp[i(k_x - \omega t)] + c.c. \tag{2}$$

where $\omega=\omega(k)$ is the least damped solution of the linear dispersion relation. (In the case of Langmuir waves, for example, the exact temporal behavior is given by a superposition of decaying exponentials, corresponding to the Landau poles.

All but one of these will become negligible after a few electron plasma periods, while, for a suitable zero order distribution function (with small slope at the phase velocity), the least damped term can have a time constant of many plasma periods. This is the case of interest, of course, and is the one we shall assume here.) Our "derivation" of the NLS rests on the assumption that we can continue to use (2), simply substituting for ω the corresponding solution of the nonlinear dispersion relation (1). Of course, this superposition is <u>not</u> valid for a nonlinear problem, and could be justified only by a careful ordering procedure in which it is shown that the errors involved are of higher order. (We eschew this step, since rigor is not our aim, but note that the agreement of our results with those of reductive perturbation theory, where a systematic ordering procedure is in fact carried through, suggests that such a proof could be constructed.)

We consider the case where both the amplitude and the spread of k values, defined by (1), are small, so that an expansion of the dispersion relation, first in A^2 and then in $\kappa = k - k_o$ is appropriate. Thus, solving

$$\varepsilon(k,\omega,A) = \varepsilon(k,\omega,0) + A^2(\partial\varepsilon/\partial A^2) + \cdots = 0 \qquad (3)$$

for ω we have

$$\omega = \Omega(k) + \mu A^2 = \omega_o + \nu$$

where $$\nu = v_g\kappa + v_g'\kappa^2/2 + \mu A^2 + \cdots \tag{4}$$

and $\Omega(k)$ is the solution of the linear dispersion relation

$$\varepsilon[k,\Omega(k),0] = 0$$

with $\omega_o = \Omega(k_o)$; $v_g = \Omega'(k_o)$; $v_g' = \Omega''(k_o)$; and $\mu = \partial\omega/\partial A^2 \equiv -(\partial\varepsilon/\partial A^2)(\partial\varepsilon/\partial\omega)^{-1}$ evaluated at $k = k_o$, $\omega = \omega_o$, $A = 0$. (A trivial generalization of the formalism is required if we wish to expand about a non-zero amplitude, $A = A_o$, rather than $A = 0$. For simplicity, we treat only the case of $A_o = 0$.) Substituting (4) into (2) we have, as a consequence of perfectly straightforward algebra,

$$\phi(x,t) = \exp[i(k_o x - \omega_o t)]\int d\kappa\,\phi_{k+\kappa}\exp[i(\kappa x - \nu t)] + \text{c.c.}$$

$$= \psi(x,t)\exp[i(k_o x - \omega_o t)] + \text{c.c.} \tag{5}$$

with

$$i(\partial\psi/\partial t + v_g\partial\psi/\partial x) + (v_g'/2)\partial^2\psi/\partial x^2 - \mu|\psi|^2\psi = 0 \tag{6}$$

identifying $|\psi|$ as the amplitude A of (1). This is conventionally written in the frame moving with the group velocity, v_g:

$$i\psi_t + p\psi_{xx} + q|\psi|^2\psi = 0 \tag{7}$$

where

$$p = v_g'/2 = (\partial^2\omega_o/\partial k^2)/2$$

$$q = -\mu = -\partial\omega/\partial A^2 \tag{8}$$

The calculation of p is, of course, quite simple, requiring only the linear dispersion relation. Finding q is more difficult —— and also more interesting, since its sign, relative to that of p, determines whether or not plane wave solutions of (7) will be stable or unstable.[2] Before considering the actual calculation of the coefficients, however, we note that assuming ε to be linear in A^2 for small A does not guarantee that the dependance on A will be as given in (3). A more general expression for such a linear dependance would involve a convolution,

$$\int dx' s(x - x') A^2(x')$$

where s(x) is a kernel to be determined by nonlinear analysis, just as we must determine the coefficient $\partial\varepsilon/\partial A^2$ in the special case where (3) is appropriate. More explicitly, whereas the linear analysis gives as an equation for ϕ

$$\varepsilon_o(k,\omega)\phi(k,\omega) = \Phi_o(k,\omega); \qquad \varepsilon_o(k,\omega) \equiv \varepsilon(k,\omega,0) , \qquad (9)$$

the inhomogeneous term, Φ_o, arising from the initial value conditions, the nonlinear analysis may be expected to give

$$\varepsilon_o(k,\omega)\phi(k,\omega) - \Phi_o$$

$$= \int dk'dk''\tilde{\phi}(k - k')\tilde{\phi}(k' - k'')\tilde{\phi}(k'')G(k,k',k'') \qquad (10)$$

$$\tilde{\phi}(k) \equiv \phi[k,\omega_o(k)]$$

where the right hand side has the most general form consistent with the circumstance that the nonlinear interaction originates as a product in configuration space, and hence a convolution in k space. Given that ϕ is peaked around k_o, $\phi(k_o + q) = \psi(q) = 0$ unless $q \approx 0$, we might expect that we could replace $G(k,k',k")$ in (10) by

$$G_o = G(k_o, 0, k_o)$$

[plus similar terms to account for the necessary symmetry under $k_o \rightarrow -k_o$] obtaining expressions of the form

$$G_o \int dq' dq" \psi(q - q') \psi(q' - q") \psi^*(-q") \qquad (11)$$

This is just the Fourier transform of

$$G_o \psi(x) |\psi(x)|^2 \qquad (12)$$

in which case (10) reduces essentially to the form (3), with $\varepsilon_{A^2} = G_o$. However, as we shall see in the example treated in section III, $G(k,k',k")$ may be singular for $k' \rightarrow 0$, in which case we can only reduce the right side of (10) to the form

$$\int dq' dq" \psi(q - q') \psi(q' - q") \psi^*(-q") S(q') \qquad (13)$$

where

$$S(q') = G(k_o, q', k_o) + G(k_o, q', -k_o)$$

(plus terms with appropriate perturbations of the arguments, as explained in detail in Section IV.) Since (13) is the Fourier transform of

$$\psi(x)\int dx' s(x - x')|\psi(x')|^2$$

where s(x) is the Fourier transform of S(k), we replace our former ansatz (3) by the more general form

$$\varepsilon(k,\omega,A) = \varepsilon(k,\omega,0) + \int dx' s(x - x')A^2(x') + \cdots \qquad (14)$$

with $A(x) \equiv |\psi(x)|$. Then (4) is replaced by

$$\nu = v_g\kappa + v_g'\kappa^2/2 - \int dx' Q(x - x')A^2(x') \qquad (15)$$

$$Q(x) = s(x)/\varepsilon_\omega$$

and the nonlinear term in the NLS equation becomes nonlocal, giving an equation of the form

$$i\psi_t + p\psi_{xx} + \psi\int dx' Q(x - x')|\psi(x')|^2 = 0 \qquad (16)$$

Of course, Q(x) may have a part proportional to $\delta(x)$, corresponding to a non-singular part of $G(k_o,0,k_o)$ in (10), which then gives a local term like that in (7).

This completes the first portion of the calculation, the "derivation" of the nonlinear and nonlocal Schrödinger equation (16). For any particular kind of wave, it remains only to calculate the coefficients. Singularities in G, which can give rise to the nonlocal character seen in (16), can arise from wave-particle resonances. In a fluid theory these cannot occur, so we need only calculate the scalar coefficients, p and q, of (17). This is illustrated in section III, using a fluid model (isothermal electrons in fluid approximation, moving in a background of cold, massive ions). The more general case of a kinetic treatment, requiring the calculation of the kernel, Q(x), is illustrated, again for Langmuir waves, in section IV.

III. Nonlinear Langmuir Waves ; in the Fluid Theory

Choosing $k_D^{-1} = (T/4\pi ne^2)^{1/2}$, $\omega_p^{-1} = (k_D^2 T/m)^{-1/2}$, and T/e as units of space, time and potential, respectively, we have as our basic equations the continuity equation, the momentum equation and Poisson's equation:

$$\partial u/\partial x + \partial \nu/\partial t + u\partial \nu/\partial x = 0$$

$$\partial u/\partial t + \partial \nu/\partial x + \partial \phi/\partial x + \partial/\partial x\ (u^2/2) = 0 \qquad (17)$$

$$\partial^2 \phi/\partial x^2 + e^{\nu} - 1 = 0$$

where

$$\nu = \log n/n_o\ , \qquad (18)$$

n_o being the density of the infinitely massive background ions. Linearizing these equations gives the well-known dispersion relation

$$\omega_o^2 = 1 + k^2 \qquad (19)$$

To obtain the nonlinear corrections to this, and hence the coefficient $\mu = \partial\omega/\partial A^2$, we could use either of the following approaches: 1) Carry out a perturbation analysis of the original equations (17); 2) Work in the wave frame, moving with velocity V relative to the lab frame, in which case there is no time dependance and the first two equations become conservation laws which can be used to eliminate u and express ν as a function of ϕ. Substituting this into

Poisson's equation gives a nonlinear ordinary equation

$$\phi'' = F(\phi, V)$$

from which we can find the dependance of $\omega = kV$ on the amplitude of ϕ using either simple perturbation theory or the Bogolyubov-Krylov technique.[10]

As might be expected, the wave-frame approach is the simpler one and has the additional virtue of providing a link with the Whitham approach,which is based on such uniform wave train solutions. In the wave frame ($\partial/\partial t = 0$) the first two equations of (17) give

$$ue^{\nu} = V \tag{20}$$

and

$$u^2/2 + \phi + \nu = V^2/2 \tag{21}$$

if we assume $u = V$ where $n = n_o$. Since ϕ is a potential, we can always choose it to vanish at the point where $u = V$, $n = n_o$. From (20) and (21) we have

$$\nu = \log V/u \tag{22}$$

and so

$$u^2/2 - \log u = V^2/2 - \log V - \phi \tag{23}$$

which (implicitly) defines u as a function of V and ψ, and hence also

$$F(\phi,V) = 1 - e^{\nu} = 1 - V/u \tag{24}$$

Thus, everything is reduced to the single nonlinear (Poisson) equation

$$\phi'' = F(\phi,V) = F_1\phi + F_2\phi^2 + F_3\phi^3 + \cdots \tag{25}$$

where the coefficients F_n are known explicit functions of V.

In the linear (small amplitude) approximation, we have simply

$$\phi'' = F_1\phi \tag{26}$$

giving sinusoidal oscillations with wave number

$$k_o = (-F_1)^{1/2} , \tag{27}$$

independent of amplitude, and hence a frequency

$$\omega_o = k_o V . \tag{28}$$

Including higher order terms in (25) we find that, of course, higher harmonics give non-sinusoidal, albeit still purely harmonic, solutions. (When $V < 1$, soliton solutions are also possible, but these are not of interest here, since we just want the change in the dispersion relation and that involves only periodic waves.) More significantly, they also give a shift in the wave number,

$$k \to k_o + \Delta k$$

where Δk is proportional to the square of the wave amplitude,

$$\Delta k = g(V) A^2 \qquad (24)$$

Thus, for given V, which is the phase velocity of the wave in the lab frame, there is a shift in k. Conversely, if we keep k fixed, then V must change,

$$V \to V + \Delta V = V + h(k) A^2 \qquad (30)$$

Since this implies

$$\omega = kV \to kV + kh(k) A^2 , \qquad (31)$$

we have the desired coefficient needed for the NLS equation (7),

$$q = -\mu = - \partial\omega/\partial A^2 = -kh(k) \ . \qquad (32)$$

There remain only the formal calculations of F_1, F_2, F_3 and of h(k). For the former, we introduce

$$N(\phi) = 1/u$$

which satisfies

$$\log N + 1/2N^2 + \phi = V^2/2 - \log V \ . \tag{33}$$

Then

$$dN/d\phi = N' = N^3(1 - N^2)^{-1}$$

$$N'' = N^5(N^2 - 3)(N^2 - 1)^{-3} \tag{34}$$

$$N''' = N^7(N^4 - 4N^2 + 15)(1 - N^2)^{-5} \ .$$

Since

$$F_1 = -VN' \ , \qquad F_2 = -VN''/2 \ , \qquad F_3 = -VN'''/3 \ , \tag{35}$$

with all of the derivatives evaluated at $\phi = 0$, where $N = V^{-1}$, we have

$$F_1 = (1 - V^2)^{-1}$$

$$F_2 = (3V^2 - 1)(1 - V^2)^{-3}/2 \tag{36}$$

$$F_3 = (15V^4 - 4V^2 + 1)(1 - V^2)^{-5}/6 \ .$$

From (27) we have

$$k_o = (V^2 - 1)^{-1/2} \tag{37}$$

which gives

$$V^2 = (k_o^2 + 1)/k_o^2 = \omega_o^2/k_o^2 \tag{38}$$

and hence the usual dispersion relation (19). To find Δk, it would seem easiest to use the method of Bogolyubov and

Krylov.[10] In fact, however, to lowest order this gives a contribution to Δk proportional to F_3A^2. To get the full A^2 contribution to Δk, we must go to second order in the Bogolyubou-Krylov expansion, thereby picking up an additional term proportional to $(F_2A)^2$. However, this is a lengthy calculation, and while it gives the correct answer,[11] we shall follow here the easier path of simply solving (25) by straightforward harmonic analysis and perturbation theory.

We look for a periodic solution of (25)

$$\phi = \sum_{-\infty}^{\infty} \phi_n \exp(inkx) \tag{39}$$

with

$$\phi_n = \phi^*_{-n} \ .$$

To third order, we then have from (25)

$$-n^2k^2\phi_n = F_1\phi_n + F_2\Sigma\phi_{n'}\phi_{n''} + F_3\Sigma\phi_{n'}\phi_{n''}\phi_{n'''} \tag{40}$$

with n' + n" = n in the first sum and n' + n" + n'" = n in the second. We next make a perturbation expansion in the amplitude of ϕ_n,

$$\phi_n = \sum_{1}^{\infty} \phi_n^{(p)} \varepsilon^p \tag{41}$$

and also in the wave number,

$$\lambda \equiv - k^2 = \sum_{0}^{\infty} \varepsilon^p \lambda_p \tag{42}$$

with

$$\phi_{\pm 1}^{(1)} = 1 \qquad \phi_n^{(1)} = 0 \;, \qquad n \neq 0 \tag{43}$$

Substituting (41) and (42) into (40), we find to first order

$$\lambda_o = F_1 = (1 - V^2)^{-1} \tag{44}$$

as expected. In second order, we find that

$$\begin{aligned} \phi_o^{(2)} &= -2\, F_2/F_1 \qquad \phi_{\pm 2}^{(2)} = F_2/3F_1 \\ \phi_n^{(2)} &= 0 \qquad |n| > 2 \end{aligned} \tag{45}$$

while $\phi_{\pm 1}^{(2)}$ is undetermined. In third order, we find

$$\begin{aligned} \lambda_2 &= 2F_2[\phi_2^{(2)} + \phi_o^{(2)}] + 3F_3 \\ &= -10F_2^2/3F_1 + 3F_3 \\ &= (1 - 9V^2)/3(V^2 - 1)^5 \;. \end{aligned} \tag{46}$$

Since

$$k^2 = -\lambda = (V^2 - 1)^{-1} + \varepsilon^2(9V^2 - 1)/3(V^2 - 1)^5 \tag{47}$$

We have, to order ε^2, for fixed k

$$\omega^2 = k^2V^2 = (k^2 + 1) + \varepsilon^2(9V^2 - 1)/3(V^2 - 1)^4$$

$$= \omega_o^2 + \varepsilon^2 k^6 (8\omega^2 + 1)/3 \qquad (48)$$

Thus,

$$\mu = \partial\omega/\partial\varepsilon^2 = k^6(8\omega^2 + 1)/6\omega$$

so

$$q = -\mu = -k^6(8\omega^2 + 1)/6\omega \qquad (49)$$

This agrees with the Asano-Taniuti-Yajima result[4] obtained via reductive perturbation theory. They find (in our units)

$$q_{ATY} = -k^2(8\omega^2 + 1)/6\omega \ , \qquad (50)$$

but this is the coefficient for a NLS equation in

$$\tilde{n} = n - 1 = k^2\phi + \cdots$$

If q_ϕ if the coefficient for a NLS equation in ϕ and q_n the corresponding quantity for $\tilde{n}$, we should have $q_\phi = k^4 q_n$ which just accounts for the difference of k^4 between (49) and (50).

IV. Nonlinear Langmuir Waves; Kinetic Theory

We continue to assume a uniform background of massive ions, but use the Vlasov equations

$$[\partial/\partial t + v\partial/\partial x + (\partial\phi/\partial x)\partial/\partial v]f = 0 \tag{51}$$

$$\partial^2\phi/\partial x^2 = \int dvf(x,v,t) - 1 \tag{52}$$

where the units of x,t,ϕ are the same as in section III. Following the standard technique,[8,12] we Fourier transform the space dependance and Laplace transform the time dependence:

$$\begin{aligned} f(x,v,t) &= F(v) + \int dp \exp[i(kx - \omega t)]f(p,v) \\ \phi(x,t) &= \int dp \exp[i(kx - \omega t)]\phi(p) \end{aligned} \tag{53}$$

where $p = (k,\omega)$ and $dp \equiv dkd\omega/(2\pi)^2$. Then (51) gives

$$\begin{aligned} f(p,v) = (\omega - kv)^{-1}\{&if(k,v,t{=}0) + k\phi(p)F'(v) \\ &+ \int dp'\phi(p')k'Df(p - p',v)\} \end{aligned} \tag{54}$$

with

$$D = \partial/\partial v$$

(In the Laplace convolution integral in (54) we must choose the ω' contour to lie above all singularities of $\phi(k,\omega')$ while $\omega - \omega'$ lies above all singularities of $f(k,v,\omega-\omega')$. As usual, ω must lie above all singularities of $f(k,v,\omega)$.

From these initial locations, of course, we can deform the ω' contour in the usual ways. Similar remarks apply to all of the convolution integrals which arise in the subsequent analysis.) Iterating (54) through the third order in ϕ gives

$$f(p,v) = I(p,v) + (\omega - kv)^{-1}\{k\phi(p) + \int dp'k'\phi(p')D(k - k')\phi(p - p')[\omega - \omega' - (k - k')v]^{-1} + \int dp'dp''k'\phi(p')Dk''\phi(p'')[\omega - \omega' - (k - k')v]^{-1}D(k - k' - k'')\phi(p - p' - p'')[\omega - \omega' - \omega'' - (k - k' - k'')v]^{-1}\}F' \tag{55}$$

where I consists of all the inhomogeneous terms, i.e. those linear in the initial perturbation $f(k,v,t=0)$ and each D operates on all functions of v to its right. Substituting (55) into the transform of (52) gives the basic integral equation

$$-k^2\varepsilon(p)\phi(p) = \Phi_o(p) + \int dp'M_2(p,p')\phi(p')\phi(p - p') + \int dp'dp''M_3(p,p',p'')\phi(p')\phi(p'')\phi(p - p' - p'') \tag{56}$$

with

$$\Phi_o(p) = \int dv\ I(p,v)$$
$$\varepsilon(p) = 1 - \int dvF'/k^2(v - \omega/k) \tag{57}$$

$$M_2(p,p') = -kk'(k - k')\int dv(\omega - kv)^{-2}F'/[\omega - \omega' - (k - k')v] \tag{58}$$

$$M_3(p,p',p'',) = -kk'k''(k - k' - k'')\int dv(\omega - kv)^{-2} \cdot [\omega - \omega' - (k - k')v]^{-1} DF'/[\omega - \omega' - \omega'' - (k - k' - k'')v] \tag{59}$$

(We have eliminated one D from (58) and from (59) by partial integration.)

We use (56) to find the nonlinear frequency shift. Since we cannot solve (56) directly, we again resort to a perturbation theory, iterative approach. From the resulting solution we will later, in Eq.(94), find the second order frequency shift. The analysis between this point and Eq. (91) involves simply the iterative solution of (56) three terms of third order in ϕ,

$$\phi = \phi^{(1)} + \phi^{(2)} + \phi^{(3)} \tag{60}$$

We find immediately

$$\phi^{(2)}(p) = -\int dp' M_2(p,p')\phi^{(1)}(p')\phi^{(1)}(p - p')/k^2\varepsilon(p) \tag{61}$$

$$\phi^{(3)}(p) = -\int dp'dp''\{M_3(p,p',p'') - [M_2(p,p') + M_2(p,p-p')]M_2(p - p', p'')/[(k - k')^2 \cdot \varepsilon(p - p')]\}\phi^{(1)}(p')\phi^{(1)}(p'')\phi^{(1)}(p - p' - p'')/k^2\varepsilon(p) \quad (62)$$

For the case of interest here, namely $\phi(k,\omega)$ peaked around $k = \pm k_o$, the second order term (61) makes no direct contribution, since we cannot have, simultaneously, $|p|$, $|p'|$ and $|p - p'|$ all equal to k_o. However, it does contribute to (62) via the M_2 M_2 terms. The ω' and ω'' integrations in (62) can be easily carried out if we assume that for each k there is a single (least damped) root, $\Omega(k)$, i.e., that on the right side of (61) we can set

$$\phi(p) = i\hat{\phi}(k)[\omega - \Omega(k)]^{-1} \qquad \mathrm{Im}[\omega - \Omega(k)] > 0 \quad (63)$$

If we close the ω' and ω'' contours in (62) on an infinite semi-circle in the lower half plane, then we enclose poles at $\omega' = \Omega(k') \equiv \Omega'$ and $\omega'' = \Omega(k'') \equiv \Omega''$ from the ϕ plus one at $\omega' = k'v$ in $M_2(p, p - p')$. Define

$$\varepsilon(k) = \varepsilon(k,\Omega(k))$$

$$R_2(k,k') = M_2(k,\Omega;\ k',\Omega') \quad (64)$$

$$R_3(k,k',k'') = M_3(k,\Omega;\ k',\Omega';\ k'',\Omega'')$$

and similarly for R_3. Then (62) becomes

$$-k^2\varepsilon(p)\phi^{(3)}(p) = -(2\pi)^{-2}\, i\int dk'dk''R(k,k',k'')$$

$$\hat{\phi}(k')\hat{\phi}(k'')\hat{\phi}(k - k' - k'') \qquad (65)$$

where

$$\{R(k,k',k'') = R_3(k,k',k'') - [R_2(k,k') + R_2(k, k - k')]$$

$$R_2(k - k', k'')/(k - k')^2\varepsilon(k - k')\}[\omega - \Omega' - \Omega'' -$$

$$\Omega(k - k' - k'')]^{-1} + T(k,k',k'') \qquad (66)$$

The term

$$T(k,k',k'') = kk'(k - k')\int dvM_2(k - k', \omega - k'v; k'',$$

$$\Omega'']F'/(\omega - kv)^2[\omega - k'v - \Omega'' - \Omega(k - k'$$

$$- k'')][(k'v - \Omega' + i\varepsilon)(k - k')^2 \qquad (67)$$

$$\varepsilon(k - k', \omega - k'v)]$$

arises from the aforementioned $\omega' = kv$ pole in M_2, but makes no contributions to the final result since, unlike the other terms in (65), it does not have a factor of $(\omega - \Omega)^{-1}$. However, we carry it thru the intermediate analysis to ensure that it gives rise to no singular terms. [To avoid notational ambiguities, we should emphasize that R coefficients in (66) with arguments like k - k' are to be interpreted as

$$R_2(k - k', k'') \equiv M(k - k', \Omega - \Omega'; k'', \Omega''),$$

etc., and similarly for $\varepsilon(k - k')$. Somewhat later, we shall find it convenient to express k, k', k" in terms of small deviations q, q', q" from k_o. In that case, we will write

$$R_2(k,k') = R_2(k, k_o+q') \equiv M[k,\Omega; k_o+q', \Omega(k_o+q')]$$

etc.]

As already indicated, we assume the initial perturbation, $f(k,v,t = 0)$ to be different from zero only for $k \approx \pm k_o$. Then $\phi(k,\Omega)$, the inhomogeneous term in (56), will have a similar character, and so will $\Phi(k)$. We make this explicit by writing

$$\Phi(k) = [\psi(k + k_o) + \psi^*(-k - k_o)]$$

where $\psi(q)$ is non-zero only for $q \approx 0$. Since (63) and (68) imply a contribution to (53) having a wave-packet-like form,

$$\phi(x,t) = [\psi(x,t)\exp[i(k_o x - \omega_o t)] + c.c.] \qquad (68)$$

with

$$\psi(x,t) = (2\pi)^{-1}\int dq\ \psi(q)\exp[i(qx - \nu t)]$$

$$\nu = \Omega(k_o + q) - \Omega(k_o)$$

there is a formal similarity to the considerations of section I. However, we emphasize again that here we are simply concerned with finding the nonlinear corrections to the dispersion relation, not primarily with wave packet considerations, and are obliged to keep $\psi(q)$ as a peaked function of non-zero width only in order to treat properly the singularities which arise in the delta function limit for $\psi(g)$.

The integrand in (65) is non-zero only if the magnitudes of k', k", and (k - k' - k") are all near k_o. This happens when two of them are near k_o and the third is near $-k_o$, which allows three possibilities. Thus (65) becomes

$$-k^2\varepsilon(p)\phi^{(3)}(p) = (2\pi)^{-2}i\int dq'dg''\psi(q')\psi^*(-q'')\psi(q''')$$

$$[R(k_o + q, k_o + q', -k_o + q'') + R(k_o + q, k_o + q', k_o +$$

$$q''')+R(k_o + q, -k_o + q'', k_o + q')] \equiv (2\pi)^{-2}i\int dq'dq''$$

$$S(q,q',q'')\psi(q')\psi^*(-q'')\psi(q - q' - q'')[\omega - \Omega(k)]^{-1} \qquad (69)$$

with $q''' = q - q' - q''$.

Insofar as the matrix elements R are non-singular at $q' = q'' = 0$, we evaluate them there. Inspection of (58), (59), (64), (66) and (67) shows that singularities occur only in $R_3(k_o + q, k_o + q', k'')$, in $R_2(k_o + q, k_o + q')$, and in $R_2(q - q', k'')$. These all have terms proportional to $(q - q')^{-1}$. We must also deal with the $(k - k')^2\varepsilon(k - k')$ in the denominator of the second term of (66). Everywhere else, we can set $q = q' = q'' = 0$ in the matrix elements without encountering any difficulty. Of course, there are many denominators of the $(\omega - kv)$ type involving resonance of particles at the phase velocity, but there are handled by the usual Landau prescription, since in this part of the calculation we are simply solving for the (nonlinear) dispersion relation which is, by definition, an initial value problem, with all resonant integrals defined via analytic continuation from the upper half ω plane. It is just the fact that the $(q - q')^{-1}$ type singularities can not be dealt with in this canonical fashion which necessitates a more careful treatment of them.

Consider first

$$R_2(k,k') = -kk'(k-k')\int dvF'/[\omega - \Omega')v](\omega - kv)^{-2} \qquad (70)$$

(Actually, we want this for $\omega = \Omega(k)$, but the use of ω reminds us that, by the usual Laplace transform rules, we must have Im ω above any singularities of the integrand, which in the present case simply requires $\mathrm{Im}\omega > 0$. Thus, we can replace ω by $\Omega + i\eta$, where $\eta \to 0_+$.) The matrix element (70) is finite at $k' = k$, since the factor $(k' - k)$ in the numerator compensates the singular denominator. In the velocity integration, we will have resonances at the phase velocity

$$v = u(k) \equiv \Omega(k)/k$$

and in the limit $k' \to k$, also at the group velocity

$$v = v_g(k) \equiv \partial\Omega/\partial k$$

The former resonance, at the phase velocity, is of a familiar sort and hence not particularly interesting, being just responsible for linear Landau damping. If F and its derivatives are much smaller at $v = u$ then at $v = v_g$, as will be the case for a Maxwellian with $k << k_D$, then the effects associated with the phase velocity resonance will be negligible. We assume this to be the case here, in order to focus attention on the less familiar effects of the resonance at v_g. If it is not, then linear Landau damping effects are likely to dominate all of the modulational terms considered here, which are third order in ϕ. In any case, the correct evaluation of (70) and the other coefficients taking into account the $v = u$ resonance is an entirely straightforward extension of the analysis presented here, provided there is some dispersion, i.e. that $u \neq v_g$.

Neglect of the phase velocity resonance implies treating $(\omega - kv)^{-1}$ as a principal value integral or, in a more approximate but simpler rein, replacing $(\omega - kv)^{-2}$ by just ω^{-2} in integrals like (70). For the resonance at v_g, we have, for any non-singular function, $h(v)$

$$\lim_{k \to k'} (k - k') \int dv\ h(v)/[(k - k')v - \omega - \Omega']$$

$$= \lim_{k' \to k} \int dv\ h(v)/[v - v_g - i\eta(k - k')^{-1}]$$

$$= P\int dv\ h(v)(v - v_g)^{-1} + \mathrm{sgn}(k - k')\pi i\ h(v_g) \qquad (71)$$

with

$$\text{sgn } x \equiv x/|x| \tag{72}$$

Consequently, we can let $k,k' \to k_o$ in R_2, but the result is still a function of $(k' - k)$. With

$$k = k_o + q, \qquad k' = k_o + q', \qquad (q,q',q-q',q") \to 0 \tag{73}$$

we find

$$R_2(k,k') \to W(k_o, q - q') \tag{74}$$

where

$$W(k,q) \equiv \int dv\ F'/(v - u)^2 (v - v_g - i\varepsilon/q)$$

$$= W_1(k) + i \text{ sgn } q\ W_2(k) \tag{75}$$

$$W_1 = P\int dv\ F'(v)/(v - u)^2 (v - v_g)$$

$$W_2 = \pi F'(v_g)/(v_g - u)^2 \tag{76}$$

We consider next the R_3 terms for which $k - k' \to 0$. These diverge, as $(k - k')^{-1}$, but the combination which occurs in (69) is convergent, irrespective of the order in which we allow $q-q'$ and $q"$ to approach zero. That is, in the limit (73),

$$R_3(k,k_o+q',k_o+q''') + R_3(k,k_o+q',-k_o+q") \to C(k_o,q-q') \tag{77}$$

with

$$C(kq) \equiv -\int dv(v-u)^{-2}(v-v_g-i\eta/q)^{-1}DF'(v-v_g)(v-u)^{-2}=C_1+iC_2\text{sgn}(q), \tag{78}$$

and

$$C_1 = -P\int dv\,(v-u)^{-1}(v-v_g)^{-1}DF'/(v-v_g)(v-u)^2$$

$$C_2 = -\pi F'(v_g)/(v_g-u)^4 \tag{79}$$

Similarly, we find in the limit (73)

$$R_2(k-k',k_o+q''') + R_2(k-k',-k_o+q'') = W(k_o,q-q') \tag{80}$$

where W is defined in (75).

The last of the (k - k') singularities arises from the $(k - k')^2\varepsilon(k - k')$ denominator of (66). We have, in the limit (73),

$$(k - k')^2\varepsilon[k-k', \Omega-\Omega'] \rightarrow -\Delta(k, q-k') \tag{81}$$

with

$$\Delta(k,q) = \int dv\; F'(v)/(v - v_g - i\eta/q)$$

$$= \Delta_1(k) + i\;\mathrm{sgn}(q)\Delta_2(k) \tag{82}$$

$$\Delta_1 = P\int dv\; F'(v)/(v - v_g) \tag{83}$$

$$\Delta_2 = \pi F'(v_g) \tag{84}$$

Since it is the reciprocal of (82) which occurs in (66) we note that

$$\Delta^{-1}(k,q) = (\Delta_1 - i\;\mathrm{sgn}\;(q)\Delta_2)(\Delta_1^2 + \Delta_2^2)^{-1} \tag{85}$$

We now list the remaining, non-singular coefficients.

$$R_3(k,-k,k) = (1/2)\int dv(v - u)^{-3} DF'(v - u)^{-1} \equiv -B/2$$

$$R_2(k,-k) = -A \equiv -\int dv\ F'(v - u)^{-3}$$

$$R_2(k,2k) = 2A \tag{86}$$

$$R_2(2k,k) = A/2$$

$$R_2(k,k-k') = 0$$

$$T(k,k,\pm k) = 0 \tag{87}$$

Substituting these and the singular terms (74), (77), (80) and (81) into S, eq.(69) we have

$$S(q,q',q'') = S_1(k) - S_2(k)i\ \mathrm{sgn}(q - q') \tag{88}$$

with

$$S_1 = -(A^2/6k^2+B/2) + C_1 + [(W_1^2-W_2^2)\Delta_1 + 2W_1W_2\Delta_2]/[\Delta_1^2+\Delta_2^2] \tag{89}$$

$$S_2 = -C_2 - [2\Delta_1W_1W_2-\Delta_2(W_1^2-W_2^2)]/[\Delta_1^2+\Delta_2^2] \tag{90}$$

We are now ready to return to our basic integral equation (56) for ϕ and find the frequency shift. In the approximation that ϕ is peaked around k_o, we have shown that (56) becomes

$$-k^2\varepsilon(p)\phi(p) = \phi(p) + i(2\pi)^{-2}\int dq'dq''S(q')$$

$$\psi(q-q')\psi(q'-q'')\psi^*(-q'')[\omega-\Omega(k)]^{-1} \qquad (91)$$

where we have changed the integration variable from q' to q - q' so that

$$S(q') \equiv S(0, q-q', 0) = S_1(k) - iS_2(k)\,\mathrm{sgn}(q') \qquad (92)$$

If we assume for $\phi(p)$ on the left side of (91) the same form,

$$\phi(p) = i[\psi(k-k_o) + \psi^*(-k-k_o)][\omega-\Omega(k)]^{-1} \qquad (93)$$

used to reduce the right hand side, then for k near k_o, i.e., $k = k_o + q$, (91) gives

$$\varepsilon(p)\psi(q) + (2\pi k)^{-2}\int dq'dq''S(q')\psi(q-q')\psi(q'-q'')\psi^*(-q'')$$

$$= -i(\omega-\Omega)\Phi_o/k^2 \qquad (94)$$

If the second term on the left side of (94) were proportional to $\psi(q)$, then the coefficient of $\psi(q)$ would just give us the nonlinear shift in ε from which we obtain the frequency shift needed for (15) and (16). Although this term is not proportional to $\psi(q)$, its Fourier transform is proportional to $\psi(x)$. That is, if we Fourier transform (94) with respect to q, treating $p = [k,\Omega(k)]$ as constant, then we have for the left side of (94)

$$[\varepsilon(p) + \int dx' S(x-x')|\psi(x')|^2/k^2]\psi(x) \tag{95}$$

This has the form (14) if we identify $|\psi|$ with A and $S(x)/k^2$ with $s(x)$, where $S(x)$ is the Fourier transform of the $S(q')$ given by (92),

$$S(x) = S_1\delta(x) + (S_2/\pi)P(1/x) \tag{96}$$

The kernel, $Q(x)$ which appears in the NLS equation (16) is therefore

$$Q(x) = [S_1\delta(x) + (S_2/\pi)P(1/x)]k^2\varepsilon_\omega \tag{97}$$

To summarize, we find that the nonlinear dielectric function for Langmuir waves, treated with the Vlasov equation, has the form postulated in section I, Eq.(14), and hence that the NLS equation for this case has the form (16), that is,

$$i\psi_t + p\psi_{xx} + Q_1\psi|\psi|^2 + (Q_2/\pi)\psi P\int dx'|\psi(x')|^2/(x-x') = 0 \tag{98}$$

with

$$Q_1 = S_1/k^2\varepsilon_\omega = -(v_g-u)S_1/2k$$

$$Q_2 = -(v_g-u)S_2/2k \tag{99}$$

where S_1 and S_2 are functions of k given by (89) and (90). As before, p is given by (8). Explicit evaluation of these for the case of $k << k_D$, when our neglect of phase velocity resonance effects is certainly justified, has been carried out by Ichikawa and Taniuti[7], who obtained equivalent results using reductive perturbation theory.

V. Conclusions and Discussion of Results

As compared with reductive perturbation theory, the method of derivation presented here is rather pedestrian, lacking the elegance of the former and its property of being rigorous and completely consistent, once a set of assumptions about the scaling have been made. As in any formulation involving multiple time or space scales, the reductive perturbation theory eliminates such philosophically troubling points as the appearance, in the same equation, of complementary variables like k and x. At the same time, the scaling makes it difficult to treat the problem as a straightforward initial value problem, which is, of course, the surest way to remove ambiguities concerned with resonant singularities. From a purely pragmatic point of view, the present method deals with these questions quite easily, since computation of the dielectric function involves, by definition, the solution of an initial value probelm, so that all velocity singularities are unambiguously defined.

Taking a broader point of view, however, we note that there is generally lacking a derivation relating an equation like the NLS equation, which describes the long time asymptotic behavior, to a well-posed initial value problem. Presumably, if at the initial time, $t = 0$, a weakly modulated wave is present in the system, then the long time behavior of the amplitude is correctly described by the NLS equation, but no clear demonstration of this has been given. If one approaches it using standard perturbation theory (with no scaling), then it seems likely that summation of an infinite subset of terms to remove secularities would be necessary, after which the amplitude could be shown to satisfy the NLS equation at large times. The present approach provides some guidance towards the development of such a demonstration, since we can at least see fairly clearly just how the connection of the initial and asymptotic behaviors comes about. One crucial step consists in the initial Karpman-Krushkal ansatz, (2). As we have already noted, this use of superposition for a nonlinear solution requires justification. In addition, we see that assuming only one, least-damped, pole in the ω plane for each k, eliminates the transient behavior, something which could presumably be shown in a formal way by the summation of terms referred to earlier. In addition, in evaluating the ω', ω'' integrals in (62), we again neglected the other Landau poles, this time with less justification than in the case of (2), where at least the $\exp(-i\omega t)$ makes this a plausible step.

Aside from these more theoretical questions, we feel that the present approach has the virtue of disentangling the sometimes onerous algebra required to find the nonlinear frequency shift from the wave-packet considerations, making it possible to use for the former whatever method is most convenient, e.g., in the example treated in section III, the use of uniform wave train solutions. In this connection, it might be supposed that the analogous exact large amplitude solutions of the Vlasov equation[13] should provide an easier way to obtain the results of section IV, but so far we have not succeeded in demonstrating this.

Finally, by exposing the admittendly messy details of the calculation, which are mercifully somewhat hidden in the elegant formalism of the reductive perturbation theory, it becomes somewhat easier to assess the difficulties, e.g., the connection between initial and asymptotic properties, discussed above, and to extend the calculation to cases which do not fit the rather stringent scaling requirements of that theory. For example, if linear Landau damping becomes comparable in magnitude with the effects treated here, then it appears very naturally in the formalism, since there is no restriction that $\omega_o = \Omega(k_o)$ be real. Of course, one must then consider also the phase velocity resonance contributions in evaluating the velocity integrals involved in the kernel, $Q(x)$.

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