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Note on the Gauge Invariance and the Conservation Laws for a Class of Nonlinear Partial Differential Equations

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SYNOPSIS: Relations between gauge invariance and the conservation laws are discussed in relevant to Korteweg-deVries' and its concerning equations. The technics employed here is conventional in the realm of field theory; the canonical formalism related to the invariant variation problem. The function of constants of motion, when represented in terms of canonical variables is to generate infinitesimal partial gauge transformations of any order, under which the variation problem is invariant. On this point of view, one more example is presented together with the general theory.

1. Introduction

The equation derived from the invariant variation problem

$$\delta \int dt\ \bar{L}[\phi] = 0 \tag{1}$$

$$\bar{L}[\phi] = \int dx \left\{ \frac{d\phi}{dx}\frac{d\phi}{dt} + \frac{1}{3}\left(\frac{d\phi}{dx}\right)^3 - \left(\frac{d^2\phi}{dx^2}\right)^2 \right\} \tag{2}$$ *

is known as the Korteweg-deVries equation, which yields a number of conservation laws. Whitham investigated this kind of variation problem,(1) and Gardner succeedingly discussed on this problem along the Hamiltonian form related to this Lagrangian for a limited case.(2)

Main advantages of the theory of invariant variation, however, exist in the facilitation in demonstrating the relation between the invariance and the conservation laws; (Noether; 1918).(3) For example, the Lagrangian (2) is Galilean-invariant. The invariance of this kind was discussed by Kruskal and his group.(4) Another invariance to be pointed out here is a kind of gauge invariance. The quantity $\phi(x,t)$ in (2) is unphysical but a "potential". That is, only its gradient, namely "strength",

$$u = d\phi/dx \tag{3}$$

is "physical". This circumstance implies the gauge invariance of the variation problem; just as in the Maxwell electromagnetic theory.(5)

The purpose of the work is to show the relation of the gauge invariance to such conservation laws just as contained in the KdV equation in the canonical frame.[6] The potential ϕ and the strength $d\phi/dx$ are treated as field quantities, here. Except for ϕ the quantities are assumed to vanish on the boundary of the space region, so also their independent small variations $\delta\phi$, $\delta(\frac{d\phi}{dx})$, $\cdots$etc.

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* Throughout this article, the differential operators with respect to space-time are denoted by $\frac{d}{dx}, \frac{d}{dt}$ instead of $\frac{\partial}{\partial x}, \frac{\partial}{\partial t}$, while ones with respect to the "quantities" and their derivatives are denoted by $\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi_x} \cdots\cdots$ etc. The functional defined by the "density" $G(\phi, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2} \cdots)$ is

$$\bar{G}[\phi] = \int dx\, G(\phi, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \cdots\cdots) \quad .$$

In particular, a point function can be regarded as a functional when it is represented in terms of Dirac's delta function as in

$$\phi(x) = \int dx'\phi(x')\delta(x'-x) \quad ,$$

$$\frac{d\phi}{dx} = -\int dx'\phi(x')\delta'(x'-x) \quad ,$$

where $\delta'(x)$ stands for $d\delta/dx$.

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2. Lagrangian and Hamiltonian;

The independent variation problem

$$\delta \int \bar{L}[\phi]dt = 0 \quad , \tag{4}$$

$$\bar{L}[\phi] = \int dx\{\frac{d\phi}{dx}\frac{d\phi}{dt} + \frac{1}{3}(\frac{d\phi}{dx})^3 + \frac{\alpha^2}{36}(\frac{d\phi}{dx})^4 - (\frac{d^2\phi}{dx^2})^2\} \tag{5}$$

with respect to $\phi(x,t)$ under fixed boundary values, leads to

$$\frac{d}{dt}\left(\frac{d\phi}{dx}\right) + \left\{\frac{d\phi}{dx} + \frac{1}{6}\alpha^2\left(\frac{d\phi}{dx}\right)^2\right\}\frac{d^2\phi}{dx^2} + \frac{d^4\phi}{dx^4} = 0 \quad , \tag{6}$$

which is referred from the work of Miura and his group.[7] (This equation tends to KdV equation as $\alpha \to 0$.) The variation problem (4) and the equation are both invariant under the gauge transformation

$$\phi \to \phi(x,t) + \Lambda(t) \quad , \tag{7}$$

where $\Lambda(t)$ stands for an arbitrary function of t independent of x. In fact, (7) adds on the right-hand side of (5) a term dependent only on the boundary values of ϕ, alien to the variation problem.

Because of the linear dependence of $\bar{L}[\phi]$ on $\frac{d\phi}{dt}$, the Legendre transformation $\frac{d\phi}{dt}$ to p (the canonical momentum conjugate to ϕ) is not unique. The situation yields many possibilities of Hamiltonian. They are formally different from each other, and have different functions in the frame work of canonical theory, although they give equal values of "energy" under a certain compatible condition as is seen shortly. A choice for the Hamiltonian

$$\bar{H}[p,\phi] = \int dx\ H \quad , \tag{8}$$

$$H = -\left(\frac{1}{2}p^2 + \frac{\alpha^2}{18}p^3 + \frac{d^2p}{dx^2}\right)\frac{d\phi}{dx} + \frac{1}{6}p^3 + \frac{\alpha^2}{36}p^4 \quad , \tag{9}$$

(See Appendix I), leads to the proper forms of canonical field equations,

$$\frac{dp}{dt} = - \frac{\delta \bar{H}}{\delta \phi} = - \frac{d}{dx}(\frac{1}{2}p^2 + \frac{\alpha^2}{18}p^3 + \frac{d^2 p}{dx^2}) \quad , \qquad (10)$$

$$\frac{d\phi}{dt} = \frac{\delta \bar{H}}{\delta p} = - (p + \frac{\alpha^2}{6} p^2)\frac{d\phi}{dx} - \frac{d^3\phi}{dx^3} + \frac{1}{2}p^2 + \frac{\alpha^2}{9} p^3 \quad . \qquad (11)^*$$

Eq. (10) is essentially identified to the KdV equation at $\alpha = 0$, while Eq. (11), after x-differentiation, gives

$$\frac{d}{dt} (p - \frac{d\phi}{dx}) + \frac{d}{dx}\{ (p + \frac{\alpha^2}{6} p^2 + \frac{d^2}{dx^2}) (p - \frac{d\phi}{dx}) \} = 0 \qquad (12)$$

with the aid of (10). Thus, the quantity $(p - \frac{d\phi}{dx})$ itself is a conserved density of eqs. (15), (16), but, if we put at a time, (e,g, t=0)

$$p - d\phi/dx = 0 \quad , \qquad (13)$$

then (13) holds forever. At a sight, the choice of the Hamiltonian is different from the kind of Gardner's. The functions of these Hamiltonians are different, while these two have equal values of the "energy" under the condition (13).

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* The functional derivatives are defined by

$$\frac{\delta \bar{G}}{\delta \phi} = \frac{\partial G}{\partial \phi} - \frac{d}{dx} \frac{\partial G}{\partial \phi_x} + \frac{d^2}{dx^2} \frac{\partial G}{\partial \phi_{xx}} - \cdots\cdots \quad .$$

(See Ref. 5) .

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3. Poisson Bracket:

The Poisson Bracket, defined by

$$[\bar{F},\bar{G}] = \int dx \left\{ \frac{\delta \bar{F}}{\delta p(x)} \cdot \frac{\delta \bar{G}}{\delta \phi(x)} - \frac{\delta \bar{G}}{\delta p(x)} \cdot \frac{\delta \bar{F}}{\delta \phi(x)} \right\} \tag{14}$$

leads to

$$[p(x,t),\ \phi(x',t)] = \delta(x-x') \quad ,$$
$$[p(x,t),\ p(x',t)] = [\phi(x,t),\ \phi(x',t)] = 0 \quad . \tag{15}$$

Also, we have

$$\frac{dp}{dt} = -[p(x,t),\ \bar{H}] \quad ,$$
$$\frac{d\phi}{dt} = -[\phi(x,t),\ \bar{H}] \quad , \tag{16}$$

the canonical equations in the P.B. form just as in the usual analytical dynamics. For any well-defined functional $\bar{G}[p,\phi]$,

$$\frac{d\bar{G}}{dt} = -[\bar{G},\bar{H}] \tag{17}$$

gives the time-development of the dynamical variable $\bar{G}$, and implies that $\bar{G}$ is a constant of motion if $\bar{G}$ "commutes" with $\bar{H}$. Of course, other properties of the Poisson bracket such as Jacobi's identity hold also in our case.

A dynamical variable $\bar{F}[p,\phi]$ is displaced by an appropriate functional $\bar{G}[p,\phi]$ according to

$$\delta\bar{F} = \varepsilon[\bar{F},\bar{G}] \quad , \tag{18}$$

where ε stands for an infinitesimal parameter of a continuous transformation, (See Appendix II). Eq. (7) implies $\bar{F}[p,\phi]$ to be invariant under the $\bar{G}$-transformation, if $\bar{F}$ commutes with $\bar{G}$, vice versa. The functional containing many linearly independent parameters α_n,

$$\bar{G}[p,\phi] = \sum \alpha_n \bar{G}_n [p,\phi] \tag{19}$$

does similar rôle for a certain displacement on the variables. In particular, any $\bar{G}_n[p,\phi]$ makes a "partial" displacement on the variables, if all $\bar{G}_n$'s are compatible:

$$[\bar{G}_m, \bar{G}_n] = 0 \quad . \qquad \text{for all n, m} \tag{20}$$

and then $[\bar{G}_n, \bar{H}] = 0$, if $[\bar{G},\bar{H}] = 0$. In other words, $\bar{H}$ is invariant also under the partial displacement.

4. Gauge Transformation:

The functional

$$\varepsilon\lambda(t)\bar{p} = \varepsilon\lambda(t) \int dx \; p(x,t) \tag{21}$$

generates the infinitesimal gauge transformation on $\phi(x,t)$

$$\delta\phi = \varepsilon\lambda(t)[\phi(x,t),\bar{p}] = \varepsilon\lambda(t) \quad . \tag{22}$$

The Hamiltonian (8) is evidently gauge-invariant. This invariance implies

$$[\bar{p}, \bar{H}] = 0 \quad , \tag{23}$$

i.e. $\bar{p}$ stands for a constant of motion.

Now, suppose a generator

$$\bar{\phi}[p,\phi'] = \int dx(p + i\alpha\frac{dp}{dx} + \frac{1}{6}\alpha^2p^2)\phi' \quad . \tag{24}$$

The old pair (p,ϕ) turns into the new pair (p',ϕ') by the time-independent, finite canonical transformation generated by $\bar{\phi}$;

$$p'(x,t) = p(x,t) + i\alpha\frac{d}{dx}p(x,t) + \frac{1}{6}\alpha^2p(x,t)^2 \tag{25}$$

$$\phi(x,t) = (1 - i\alpha\frac{d}{dx} + \frac{1}{3}\alpha^2p(x,t))\phi'(x,t)$$

The new momentum p' satisfies the proper KdV-equation,[7] i.e. $\alpha=0$. Thus, we can regard this canonical transformation as the one to make α vanish. In terms of the new variables, consequently, the Hamiltonian is written as

$$\bar{H}'[p',\phi'] = \int dxH' \quad ,$$

$$H' = -(\frac{1}{2}p'^2 + \frac{d^2p'}{dx^2})\frac{d\phi'}{dx} + \frac{1}{6}p'^3 \quad . \tag{26}$$

The new momentum $\bar{p}'$ can be considered as the generator of infinitesimal gauge transformation on the new ϕ', since P.B. relations do not change themselves under canonical transformations. Moreover, time-independent canonical transformations do not change the canonical equations. This fact yields

$$\frac{d\bar{p}}{dt} = 0 = [\bar{p}, \bar{H}' [p',\phi']] \quad . \tag{27}$$

The old momentum p has been formally solved in terms of p' stepwisely

$$p = p' - i\alpha\frac{dp'}{dx} + \sum\alpha^nT_n(p', \frac{dp'}{dx}, \cdots\cdots) \quad . \tag{28}$$

The densities $T_n(p', \frac{dp'}{dx}, \cdots)$ are given as linearly independent polynomials (See Ref. 7). Thus, (27) and (28) lead to

$$\frac{d\bar{T}_n}{dt} = [\bar{T}_n, \bar{H}'] = 0 \quad . \tag{29}$$

It is essential that all of $\bar{T}_n[p']$ are the functionals of p' only, not dependent on ϕ'. It implies the compatibility relation

$$[\bar{T}_n[p'], \bar{T}_m[p']] = 0 \qquad \text{for all } n, m \tag{30}$$

to be a truism. Since $\bar{p}$ generates an infinitesimal gauge transformation on ϕ, $\bar{T}_n[p']$ generates the infinitesimal partial gauge transformation of an order n also on $\phi[\phi',p']$. The Hamiltonian is, therefore, invariant under the partial gauge transformation of any order. This invariance thus yields many constants of motion of KdV equation.

The parameter α, running over all real non-negative values, makes a class of equations, any element of which turns into another by means of a canonical transformation. This class is hereafter called "KdV class" for short, since the class contains the KdV equation as its element α=0.

5. General Remarks;

So much for the KdV class, but now for general schemes. The Lagrangian functional of general form

$$\bar{L}[\phi] = \int dx \frac{d\phi}{dx}\frac{d\phi}{dt} - \bar{F}[d\phi/dx] \quad , \tag{31)*$$

where

$$\bar{F}[d\phi/dx] = \int dx\ F\left(\frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \cdots\right) \quad , \tag{32}$$

contains (2) and (5) as special cases. The variation problem of (31) is certainly invariant under the transformation (7); linear dependence of (31) on $\frac{d\phi}{dt}$ leads to non-uniqueness of Legendre transformation, and consequently to many possibilities of the Hamiltonian formalism. As is seen in Appendix I, however, the Hamiltonian

$$\bar{H}[p,\phi] = \bar{F}[p] - \frac{1}{2}\int dx \, \frac{\delta\bar{F}}{\delta p}(p - \frac{d\phi}{dx}) \qquad (33)$$

is a reasonable choice among them, and (33) leads to canonical equations

$$\frac{dp}{dt} = -\frac{\delta\bar{H}}{\delta\phi} = \frac{1}{2}\frac{d}{dx}\frac{\delta\bar{F}}{\delta p} \quad , \qquad (34)$$

$$\frac{d\phi}{dt} = \frac{\delta\bar{H}}{\delta p} = -\frac{\delta^2\bar{F}}{\delta p^2}(p - \frac{d\phi}{dx}) + \frac{1}{2}\frac{\delta\bar{F}}{\delta p} \quad . \qquad (35)^{**}$$

Eq. (35) is equivalent to eq. (34) only under the condition $p = d\phi/dx$, which should be given as an initial condition. On the other hand, Eq. (34) is formally similar to Gardner's representation of KdV equation (Cf. Ref. 2, eq.(5) and Eq. (31)). However, the concrete functional form of $\bar{F}[p]$ now remains free. Thus, the gauge invariant variation problem with respect to (31), in general, yields the Gardner-type equation (34), which naturally implies $[\bar{H}, \bar{p}] = 0$.

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* In general, the Lagrangian may be in the form

$$G(\frac{d\phi}{dx}, \frac{d^2\phi}{dx^2}, \ldots\ldots)\frac{d\phi}{dt} - F[\frac{d\phi}{dx}] \quad ,$$

which gives a gauge-invariant variation problem, but this general form can be modified into (33) by means of an appropriate canonical transformation.

** $\frac{\delta^2\bar{F}}{\delta p^2}$ in (35) must be regarded as an operator to the immediate right member. (See App. I)

Next, suppose a one-parametric canonical transformation by the generator

$$\bar{\Phi}[p,\phi'] = \int dx\ \phi'\ S(p, \frac{dp}{dx}, \cdots;\ \alpha) \quad , \tag{36}$$

where α stands for the parameter of the transformation, and

$$p = S(p, \frac{dp}{dx}, \cdots;\ 0) \tag{37}$$

is assumed for simplicity. Evidently this transformation

$$p' = \frac{\delta\bar{\Phi}}{\delta\phi'} = S(p, \frac{dp}{dx}, \cdots;\ \alpha) \tag{38}$$

preserves the gauge properties. In terms of new canonical variables (p',ϕ'), therefore, another Gardner-type canonical equation

$$\frac{dp'}{dt} = \frac{1}{2}\frac{d}{dx}\frac{\delta\bar{F}'}{\delta p'} \tag{39}$$

comes out, where the identity

$$\bar{F}'[p'] = \bar{F}[p] \tag{40}$$

defines new $\bar{F}'[p']$. So, eqs. (34) and (39) imply respectively

$$\frac{d\bar{p}}{dt} = 0 \quad , \qquad \frac{d\bar{p}'}{dt} = 0 \quad . \tag{41}$$

On the other hand, p can be expressed in terms of p' in the formal power series of α

$$p = p' + \sum\alpha^n A_n(p', \frac{dp'}{dx}, \cdots\cdot) \tag{42}$$

by solving (38). It is concluded that

$$\frac{d}{dt}\bar{A}_n[p'] = 0 \tag{43}$$

holds for all n, because (34), (39) and (41) must be correct for all values of α. In other words, every A_n is a conserved density of eq. (39).

The compatibility condition

$$[\bar{A}_n, \bar{A}_m] = 0$$

is also a triviality, and the variables $\bar{p}$ and $\bar{p}'$ give rise to the infinitesimal gauge transformations on ϕ and ϕ', respectively. So, every $\bar{A}_n[p']$ gives rise to partial gauge transformation on ϕ'.

6 Another Example :

The KdV class is an example for the consideration. We now intend to show another. Let us look for the properties of a class of nonlinear equation,

$$\frac{du}{dt} - \exp(2i\alpha\frac{du}{dx})\,[\frac{du}{dx} + 2i\alpha\{(1-u)\frac{d^2u}{dx^2} - (\frac{du}{dx})^2\} - \alpha^2(u\frac{du}{dx}\frac{d^2u}{dx^2} + \frac{d^3u}{dx^3}) + 4i\alpha^3(\frac{d^2u}{dx^2})^2] = 0 \quad , \tag{44}$$

where α runs over all real values. Eq. (44) is of course an artificial one, and appears somewhat complicated, but it is employed as a simple in investigating the similar properties as occurred in the KdV class. Eq. (44) is derived from the invariant variation problem of

$$\bar{L}[\phi] = \int dx\{\frac{d\phi}{dx} \cdot \frac{d\phi}{dt} - (\frac{d\phi}{dx})^2 \exp(2i\alpha\frac{d^2\phi}{dx^2})\} \quad , \tag{45}$$

where $u = d\phi/dx$. Consequently, by putting

$$\bar{F}[p] = \int dx\ p^2 \exp(2i\alpha\frac{dp}{dx}) \tag{46}$$

in the Gardner-type equation (34), we get eq. (44); (there u must be replaced by p).

Next, suppose the canonical transformation

$$p' = \delta\bar{\Phi}/\delta\phi' = p \exp(i\alpha\frac{dp}{dx}) \tag{47}$$

$$\phi = \frac{\delta\bar{\Phi}}{\delta p} \tag{48}$$

generated from

$$\bar{\Phi}[p, \phi'] = \int dx\ p \exp(i\alpha\frac{dp}{dx})\phi' \tag{49}$$

The second expression (48) is no longer referred in detail, because it has less important roles. After the transformation (47) $\bar{F}[p]$ turns into

$$\bar{F}'[p'] = \int dx\ p'^2 \quad , \tag{50}$$

and the equation turns into

$$\frac{dp'}{dt} = \frac{dp'}{dx} \tag{51}$$

which implies

$$\frac{d\bar{p}'}{dt} = 0 \tag{52}$$

for all real values of α. The old momentum p can be expressed in terms of the new momentum p' in the formal power series of α

$$p = p' + \alpha A_1 + \alpha^2 A_2 + \cdots\cdots \tag{53}$$

where

$$\begin{aligned} A_1 &= -i\,\frac{dp'}{dx}\frac{d^2p'}{dx^2} \quad , \\ A_2 &= \frac{1}{2}p'\left(\frac{dp'}{dx}\right)^2 + \frac{1}{2}\,\frac{d^2}{dx^2}\left\{p'\left(\frac{dp'}{dx}\right)^2\right\} \quad , \quad \cdots\cdots\cdot , \end{aligned} \tag{54}$$

The conserving property of A_1 is trivial; i.e. $d\bar{A}_1/dt = 0$ holds without eq. (51), while that of A_2 is essential; $d\bar{A}_2/dt = 0$ holds if and only if p' is a solution of eq. (51). Thus, we have two explicit examples to show the relation between the gauge invariance and the conservation laws of a class of nonlinear equations; one KdV class, the other shown above.

7. Concluding Remarks

The conclusion of this work is that;
the existence of a number of conserved densities originates in the gauge invariance of variation problem.

The gauge transformation (7) contains an arbitrary function (t). According to Noether, if any transformation, under which the variation problem remains invariant, possesses r parameters, then the Euler equations have r conserved density. An arbitrary function, as in our case, behaves as a set of infinite number of parameters, and correspondingly results derivation of many conserved densities.

It is possible to construct arbitrarily more examples by employing appropriate concrete $\bar{F}[p]$. Moreover, similar situations are expected when the field variable ϕ has a number of components. That may motivate wider examinations for classes of nonlinear differential equation.

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Appendix I Many Faces of Hamiltonian;

Two propositions from classical variation calculus are mainly relied in the following arguments:

(i) Results of variation problem remain unaltered after Euler equations are partially brought into the original problem.

(ii) The relations among variables, if any, must be considered as subsidiary conditions, and can be taken into the calculation by means of indefinite multipliers.

Let's rename $\frac{d\phi}{dt}$ to $\dot{\phi}$ in (31), and consider $\frac{d\phi}{dt} = \dot{\phi}$ as a subsidiary condition of the independent variables ϕ and $\dot{\phi}$. By using a multiplier p, the variation problem turns into

$$\delta \int dt \ [\int dx (\frac{d\phi}{dx} \dot{\phi}) - \bar{F}[\frac{d\phi}{dx}] - \int dx \ p \ (\dot{\phi} - \frac{d\phi}{dt})] = 0 \quad ,(A.1)$$

which is performed with respect to ϕ, $\dot{\phi}$ and p independently. One of the Euler equations derived from (A.1)

$$p = \frac{\delta \bar{L}}{\delta \dot{\phi}} = \frac{d\phi}{dx} \quad , \qquad (A.2)$$

usually regarded as the Legendre transformation, $(\phi, \dot{\phi})$ to (ϕ, p), does not contain $\dot{\phi}$ in this case. So, the equation, an imposed relation between p and ϕ, must be considered as a new subsidiary condition.

The lack of relation p to $\dot{\phi}$ leads to multiplicity of the related Hamiltonians. One of tentative choices among them is

$$\bar{H}[p,\phi,v] = \int dx \frac{\delta\bar{L}}{\delta\dot{\phi}} \dot{\phi} - \bar{L}[\frac{d\phi}{dx}] + \int dx \; v \; (p - \frac{d\phi}{dx})$$

$$= \bar{F} \; [\frac{d\phi}{dx}] + \int dx \; v \; (p - \frac{d\phi}{dx}) \quad , \qquad (A.3)$$

where v stand for the indefinite multiplier. This Hamiltonian yields $p = d\phi/dx$ as one of the Euler equations, which may be put into $\bar{F}$. Another tentative Hamiltonian is, then,

$$\bar{H}[p,\phi,v] = \bar{F}[p] + \int dx \; v \; (p - \frac{d\phi}{dx}) \quad , \qquad (A.3)$$

the last term of which is necessary to derive that condition.

In the next step, what we have to do is to eliminate v so as to get the proper form of $\bar{H}$, which must be alien to the additional variable other than the canonical variables (p,ϕ). The Euler equations from (A.3) are

$$\frac{dp}{dt} = - \frac{\delta\bar{H}}{\delta\phi} = - \frac{dv}{dx} \quad ,$$

$$\frac{d\phi}{dt} = \frac{\delta\bar{H}}{\delta p} = \frac{\delta\bar{F}}{\delta p} + v \quad , \qquad (A.4)$$

$$p = d\phi/dx \quad ,$$

which yield

$$v = - \frac{1}{2} \frac{\delta\bar{F}}{\delta p} \quad . \qquad (A.5)$$

By putting (A.5) into (A.3), we obtain

$$\bar{H}[p,\phi] = \bar{F}[p] - \frac{1}{2} \int \frac{\delta \bar{F}}{\delta p} (p - \frac{d\phi}{dx})\, dx \quad , \qquad (A.6)$$

which leads to the canonical equation

$$\frac{dp}{dt} = \frac{1}{2} \frac{d}{dx} \frac{\delta \bar{F}}{\delta p} \quad ,$$
$$\qquad (A.7)$$
$$\frac{d\phi}{dt} = - \frac{1}{2}[\frac{\delta^2 \bar{F}}{\delta p^2} (p - \frac{d\phi}{dx})] + \frac{1}{2} \frac{\delta \bar{F}}{\delta p} \quad .$$

The first equation has obviously the general Gardner-type, while the second together with the first is deformed into

$$\frac{d}{dt} (p - \frac{d\phi}{dx}) - \frac{1}{2} \frac{d}{dx} [\frac{\delta^2 \bar{F}}{\delta p^2} (p - \frac{d\phi}{dx})] = 0 \quad , \qquad (A.8)$$

where by the notation $\delta^2\bar{F}/\delta p^2$ we understand an operator to the immediate right member: e.g. provide $F = -\frac{1}{2}(\frac{dp}{dx})^2$, $\frac{\delta^2 \bar{F}}{\delta p^2}$ functions as the differential operater $\frac{d^2}{dx^2}$. Eq. (A.8) implies $(p - \frac{d\phi}{dx})$ itself being a conserved density; if it is valued to zero at an initial moment, then it vanishes forever. This statement is never triviality, although $p = d\phi/dx$ is partially used in the way of traversing from the Lagrangian to the Hamiltonian formalism, since p and ϕ have to be treated as independent variables in the canonical frame of work.

Thus, by (A.6) we understand the terminal choice of Hamiltonian; which must be alien to v, and yields the canonical equations equivalent to the original equation if $p - \frac{d\phi}{dx} = 0$ is given as an initial condition.

Appendix II Canonical Transformation.

A canonical transformation $(p,\phi) \rightarrow (p',\phi')$ must be obtained from a functional $\bar{W}[p,\phi']$ by

$$p' = \delta\bar{W}/\delta\phi'(x,t) \quad ,$$
$$\phi = \delta\bar{W}/\delta p(x,t) \quad . \qquad \text{(A.9)}$$

Clearly the transformation (A.9) does not alter the Poisson Bracket (14); and also it does not change the canonical equations of motion if $\bar{W}$ does not explicitly contain t.

In particular,

$$\bar{W}[p,\phi'] = \int dx\, p\phi' + \varepsilon\bar{G}[p,\ \phi'] \quad ,\varepsilon\text{: infinitesimal} \qquad \text{(A.10)}$$

gives rise to the infinitesimal displacement on p and ϕ;

$$\delta\phi = \phi' - \phi = \varepsilon[\phi,\ \bar{G}[p,\phi]] \quad ,$$
$$\delta p = p' - p = \varepsilon[p,\ \bar{G}[p,\phi]] \quad , \qquad \text{(A.11)}$$

up to the first order in ε. Note that ϕ' in the last members is replaced by ϕ. That is, a functional $\bar{G}[p,\phi]$ plays the role of the infinitesimal displacement operator on p and ϕ in the sense of P. B.

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