INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

# RESEARCH REPORT

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# A Critical $\beta_p$ in a Torus with Rectangular Cross Section

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan.

Abstract

A simple diffused-boundary model is considered for magneto-hydrodynamic equilibrium in an axisymmetric torus with rectangular cross section. The pressure balance equations are reduced from the expressions for eigenvalues of the equiliblium equation. The parameter $\beta_p$ (ratio of plasma pressure to poloidal magnetic pressure) is closely related to the paramagnetism of plasma and geometrical factors. The condition of no reversed toroidal current gives the limitation on the paramagnetism and $\beta_p$. The critical $\beta_p$ becomes largest for the square plasma cross section.

It is of great interest to study the theoretical aspect of magnetohydrodynamic equilibria of the axisymmetric tori like Tokamaks[1] and the belt pinch[2] experiments. A number of researchers[3-6] have pointed our the existence of critical $\beta_p$, studying the magnetohydrodynamic equilibria. In the references [3] and [4], it is insisted that $\beta_p$ is limited to the order of unity by the appearance of a second magnetic axis. Recently it has been shown[5] that a plasma with arbitrary $\beta_p$ can be contained in a torus with rectangular cross section. However, there is a critical $\beta_p$ (of the order of unity) above which the toroidal reversed current appears in the plasma. On the other hand, numerical studies[6] have pointed out that there is no limitations of $\beta_p$ even when $\beta_p \sim (R_c/a)^2$ in the diffused toroidal pinches with circular cross section. In both pieces of work[5,6], it is assumed that the pressure vanishes at the wall but the pressure gradient is finite there. Therefore, there will be a finite current flowing in the vicinity of the wall which is undesirable feature for the nuclear fusion reactor.

In the present paper we study the case of a diffused pressure distribution in a torus with rectangular cross section in which both the pressure and the pressure gradient vanish at the wall and also the toroidal current does not flow in the vicinity of the wall. We treat the equilibrium equation in the toroidal geometry employing the perturbation method. We obtain the pressure balance equation which is equivalent to the expression for the eigenvalues of the equilibrium equation.

The pressure balance equation describes the relation between $\beta_p$ and M (plasma paramagnetism) for a given geometry.

We start from the ideal MHD equations,

$$\vec{j} \times \vec{B} = c\nabla p, \tag{1}$$

$$\nabla\vec{B} = \nabla\vec{j} = 0, \tag{2}$$

and

$$\nabla \times \vec{B} = 4\pi\vec{j}/c. \tag{3}$$

As is well known[1] the introduction of a stream function $\psi$ leads to the equations for the axisymmetric toroidal configuration in a cylindrical coordinate system ( R, $\phi$, Z );

$$\vec{B} = (2I/cR)\vec{\phi} + \nabla\psi \times \vec{\phi}/2\pi R, \tag{4}$$

$$\vec{j} = j_\phi\vec{\phi} + \nabla I \times \vec{\phi}/2\pi R, \tag{5}$$

and

$$R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = -\frac{8\pi^2}{c}(2\pi c R^2\frac{dp}{d\psi} + \frac{1}{c}\frac{dI^2}{d\psi})$$

$$= -\frac{8\pi^2}{c}Rj_\phi, \tag{6}$$

where p and I, which are the pressure and the current stream function respectively, are arbitrary functions of $\psi$ only.

We choose trial expressions

$$P(\psi) = C^{(1)} + C^{(2)}\psi + C^{(3)}\psi^2,$$

and

$$I^2(\psi) = d^{(1)} + d^{(2)}\psi + d^{(3)}\psi^2,$$

where $C^{(i)}$ and $d^{(i)}$ are arbitrary constants. We assume that the pressure and the pressure gradient vanish at the surrouding conducting wall and we denote the value of $\psi$ at the wall by $\psi_1$. Then we obtain

$$P(\psi) = C^{(3)}(\psi_1 - \psi)^2. \tag{7}$$

Substituting Eq.(7) and the expression for $I^2(\psi)$ into Eq.(6), we have

$$j_\phi = -4\pi cRC^{(3)}(\psi_1 - \psi) + (d^{(2)} + 2d^{(3)}\psi)/cR. \tag{8}$$

We assume also that the toroidal current vanishes at the wall. Then we obtain the following equation from Eq.(8),

$$d^{(2)} + 2d^{(3)}\psi_1 = 0.$$

By introducing the parameters $\lambda$ and $\mu$, we can express $p(\psi)$ and $I^2(\psi)$, which satisfy the above assumptions, as follows:

$$P(\psi) = P_0\psi^2/\psi_0^2 = \lambda^2\psi^2/(32\pi^3R_0^4), \tag{9}$$

$$I^2(\psi) = I_1^2 + c^2\mu\psi^2/(16\pi^2R_0^2), \tag{10}$$

and

$$\psi_1 = 0,$$

where suffixes 0 and 1 indicate the quantities on the magnetic axis and on the inner edge of the plasma boundary respectively. By integrating the expression for $B_z$-component of Eq.(4) $(B_z=(1/2\pi R)\partial\psi/\partial R)$, we can write $\psi_0$ (the value of $\psi$ on the magnetic axis) as

$$\psi_0 = 2\pi\bar{B}_z(R_0^2 - R_1^2), \tag{11}$$

where $\bar{B}_z$ is a mean value of the poloidal field between $R_1$ and $R_0$ on the median plane $Z = 0$. The physical meanings of the parameters $\lambda^2$ and $\mu$ will be understood from

$$\lambda^2 = \beta_p\pi^2/(1-R_1^2/R_0^2)^2, \tag{12}$$

and

$$\mu = M\pi^2/\theta^2(1-R_1^2/R_0^2)^2, \tag{13}$$

which are derived from Eqs.(9), (10) and (11). Here, $\beta_p = 8\pi P_0/(\pi\bar{B}_z)^2$, $\theta = \pi\bar{B}_z/B_{\phi 0 v}$, $M = (B_{\phi 0}{}^2/B^2{}_{\phi 0 v})-1$ and $B_{\phi 0 v}$ is the vacuum toroidal field on the magnetic axis. We find that plasma is paramagnetic (diamagnetic) when M is positive (negative).

We rewrite equation (6) using dimensionless coordinates (r,z) defined by $R = R_0 r$ and $Z = R_0 z$,

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r} + (\lambda^2 r^2+\mu)\psi = 0, \tag{14}$$

and

$$(\lambda^2 r^2+\mu)\psi = \frac{8\pi^2}{c}R_0^3 r j_\phi. \tag{15}$$

We can write the differential equation for u(t), setting $\psi = u(t)\cos\nu z$ and introducing a new variable $t = r^2/2$,

$$\frac{d^2u(t)}{dt^2} + (f^2 - \gamma\frac{t-t_0}{t})\ u(t) = 0, \tag{16}$$

where

$$f^2=\lambda^2+\gamma,\ \gamma=(\mu-\nu^2)/2t_0 \text{ and } t_0=R_0^2/2R_0^2=1/2.$$

Now we consider a toroidal plasma enclosed by a rectangular ideal conducting wall with width a, height b, and the distance from axis to the center of the rectangle $R_c$ (see Fig.1). In Fig.1, $R_2$ indicates the outer edge of the plasma boundary.

From the boundary condition that $\psi = 0$ at the wall $(Z = \pm b/2)$, $\nu$ is determined as

$$\nu = \nu_m = \pi R_0 (2m-1)/b, \quad m = 1, 2, 3, \cdots . \qquad (17)$$

Solutions of Eq.(16) are linear combinations of the Coulomb wave functions. However, we use the perturbation method in order to solve Eq.(16), which give us solutions much more convenient for the following discussions as compared with the Coulomb wave function. We consider the term proportional to $(t - t_0)/t$ as a perturbation and expand $u(t)$ in terms of the normalized and orthogonal functions $\xi_n(t)$, which are the solutions of the equation $d^2\xi_n(t)/dt^2 = -f^2\xi_n(t)$ and

$$f = f_n = n\pi/(t_2-t_1), \qquad n = 1, 2, 3, \cdots, \qquad (18)$$

$$\xi_n(t) = \{2/t_1-t_2\}^{1/2}\sin f_n(t-t_1). \qquad (19)$$

The functions $\xi_n(t)$ satisfy the boundary condition $\xi_n(t) = 0$ at $t = t_1$ $(=R_1^2/2R_0^2)$ and $t = t_2$ $(=R_2^2/2R_0^2)$. We replace $\gamma(t-t_0)/t$ by $\delta\gamma(t-t_0)$ to express the perturbation term explicitly and express $u(t)$ and $f^2$ as power series in $\delta$,

$$u = u_0 + \delta u_1 + \delta^2 u_2 + \cdots, \qquad (20)$$

$$f^2 = F_0 + \delta F_1 + \delta^2 F_2 + \cdots . \qquad (21)$$

we assume that these two series are analytic for $\delta$ between zero and one, and in the final results we set $\delta$ equal to one. After we substitute Eqs.(20) and (21) into Eq.(16), we have a set of equations:

$$d^2u_0/dt^2 + F_0u_0 = 0, \tag{22}$$

$$d^2u_1/dt^2 + \{F_1-\gamma(t-t_0)/t\}u_0 + F_0u_1 = 0, \tag{23}$$

$$d^2u_2/dt^2 + F_0u_2 + \{F_1-\gamma(t-t_0)/t\}u_1 + F_2u_0 = 0, \text{ etc.} \tag{24}$$

The solutions of the zeroth-order equation (22) are unperturbed eigenfunctions $\xi_n(t)$, i.e.,

$$u_0 = \xi_n(t) \quad , \quad F_0 = f_n^2.$$

According to the usual perturbation method[7] we expand $u_1$ and $u_2$ in terms of $\xi_i$ as $u_1 = \sum_{i=1}^{\infty} a_i\xi_i(t)$ and $u_2 = \sum_{i=1}^{\infty} b_i\xi_i(t)$. By using the orthogonality of $\xi_i(t)$, we obtain the eigenfunctions and eigenvalues up to the second order-perturbation as following,

$$u(t) = \xi_n(t) + \sum_{i\neq n}\frac{F_{in}\xi_i(t)}{f_n^2-f_i^2} + \sum_{i\neq n}[\{ \sum_{j\neq n}\frac{F_{ij}F_{in}}{(f_n^2-f_i^2)(f_n^2-f_j^2)}$$

$$- \frac{F_{in}F_{nn}}{(f_n^2-f_i^2)^2}\} \xi_i(t) - \frac{1}{2}\frac{F_{in}^2}{(f_n^2-f_i^2)^2}\xi_n(t)], \tag{25}$$

and

$$f^2 = f_n^2 + F_{nn} + \sum_{i\neq n} \frac{F_{ni}^{\ 2}}{f_n^2 - f_i^2}, \tag{26}$$

where $F_{in} = \gamma \int_{t_1}^{t_2} \{(t-t_0)/t\} \xi_i(t)\xi_n(t)dt$.

After some calculations[8], $F_{in}$ becomes

$$F_{in} = -\frac{4\gamma t_0 (t_2-t_1) in}{\pi^2 (i^2-n^2)^2} \{\frac{1}{t_1^2} - \frac{(-1)^{i+n}}{t_2^2}\} + 0\{(\frac{t_2-t_1}{t_1})^2\},\ i\neq n,$$

and

$$F_{nn} = \gamma[\ 1- \frac{t_0}{t_1} \{1- \frac{t_2-t_1}{2t_1} + \frac{(t_2-t_1)^2}{3t_1^2}\} + \frac{t_0 (t_1+t_2)(t_2-t_1)^2}{t_1^2 t_2^2 (2n\pi)^2}]$$
$$+ 0\{(\frac{t_2-t_1}{t_1})^3\} \tag{27}$$

We rewrite the final expressions (25), (26) and (27) in the real coordinate system $(R, \phi, Z)$, using the original notations $\beta_p$ and $M/\theta^2$ instead of $f^2$ and $\gamma$,

$$\psi_{nm} = \psi_0 \sin \frac{n\pi (R^2-R_1^2)}{(R_2^2-R_1^2)} \cos \frac{\pi(2m-1)Z}{b}, \tag{28}$$

and, from eigenvalues,

$$\frac{\beta_p}{(1-R_1^2/R_0^2)^2} = \frac{4n^2 R_0^4}{(R_2^2-R_1^2)^2} - \{\frac{M}{\theta^2 (1-R_1^2/R_0^2)^2} - \frac{(2m-1)^2 R_0^2}{b^2}\}$$
$$\times \{\frac{(3R_1^2-R_2^2)R}{2R_1^4} + \frac{(R_2^2-R_1^2)^2 R_0^2}{3R_1^6}(\ 1- \frac{1}{2n^2\pi^2})\}, \tag{29}$$

We have neglected the terms higher than the second-order of $(R_2^2-R_1^2)/R_1^2$ in the above equations. Equation (28) is equivalent to the pressure balance equations for each mode corresponding to each pair of integers (m,n).

Now we consider the condition under which the toroidal current density does not reverse in the plasma cross section. As $j_\phi$ is proportional to $(\lambda^2 r^2+\mu)\psi$, it is sufficient that $(\lambda^2 r^2+\mu)\psi \geqq 0$. This condition means that m=n=1 and $\lambda^2 r^2+\mu \geqq 0$, since $\psi_{m,n}$ changes sign in the plasma cross section when $m,n > 1$. From the inequality of $(r^2\lambda^2+\mu) \geqq 0$, we obtain

$$\beta_p R_1^2/R_0^2 + M/\theta^2 \geqq 0 \quad \text{or} \quad 8\pi P_0 R_1^2/R_0^2 \geqq B_{\phi 0 V}^{\ 2} - B_{\phi 0}^{\ 2}. \tag{30}$$

This means that, neglecting the toroidal effect, the plasma pressure on the magnetic axis must not be less than the difference of the magnetic pressure caused by the plasma diamagnetism. When the plasma pressure becomes small, the reversed toroidal current is needed to reduce the poloidal magnetic pressure.

In the case of the fundamental mode $\psi_{11}$ and $\beta_p R_1^2/R_0^2+M/\theta^2 \geqq 0$, the toroidal current is not reversed in the whole area of the plasma cross section and the magnetic surfaces enclose a single axis (see Fig.2). From the definition of magnetic axis ($\partial\psi/\partial R= \partial\psi/\partial z = 0$) and Eq.(28), the position of the magnetic axis is found to be fixed at the point $R_0 = (R_c^{\ 2} + a^2/4)^{1/2}$ and $Z = 0$, regardless of the plasma parameters. The effect of the plasma parameters on the position of the magnetic axis appears from the third-order terms of $(R_2^2-R_1^2)/R_1^2$ in $\psi$. Substituting $R_0$ into Eq.(29) we obtain the pressure balance equation for the fundamental mode:

$$\beta_p + \{1 + \frac{(1-8/\pi^2)\varepsilon^2}{3(1-4\varepsilon+6\varepsilon^2)}\} \frac{M}{\theta^2} = 1 + \frac{a^2}{b^2}\{1- \frac{2}{3} \frac{(1+4/\pi^2)\varepsilon^2}{(1-4\varepsilon+6\varepsilon^2)}\}, \tag{31}$$

where $\varepsilon = a/2R_c$ (inverse aspect ratio). We can calculate the poloidal field by using the poloidal flux function $\psi_{11}$ and obtain $B_{p\ max} = \pi\bar{B}_z$, where $B_{p\ max}$ is the maximum poloidal field on the median plane. Therefore the definition of $\beta_p$ given below Eq.(13) is the same one defined in reference[5].

However, it is more general and convenient to use $\beta_I$[9] in order to see the relation between the geometrical factors and poloidal beta. The parameter $\beta_I$ is defined by

$$\beta_I = 8\pi\{\int pdV/\int dV\}\{\oint_c B_p d\ell/\oint_c d\ell\}^{-2}$$

where $\int dV$ is the volume integral over the plasma region and $\oint_c d\ell$ is the line integral along the contour of plasma boundary. Then we obtain

$$\beta_I = \frac{\pi^2}{16} \beta_p (1 + \frac{a}{b})^2 \{1 + \frac{a^2}{b^2} \frac{1+\varepsilon^2}{(1-\varepsilon^2)^2}\}^{-2}. \tag{32}$$

From Eqs.(30),(31) and (32), we obtain the critical value of $\beta_I$, neglecting order of $\varepsilon^2$, such as

$$\beta_I \leqq \frac{\pi^2}{32} \frac{(1+a/b)^2}{1+a^2/b^2} .$$

As a function of a/b, the critical value of $\beta_I$ is between $\pi^2/16\varepsilon$ (a/b=1) and $\pi^2/32\varepsilon$ (a/b $\rightarrow$ 0 or a/b $\rightarrow \infty$).

We have calculated the poloidal flux function $\psi$ and the critical value of $\beta_I$ under the fully prescribed boundary condition, which is different from the case of reference [5]. The critical $\beta_I$ is obtained from the condition of the appearance not of the second magnetic axis but of no reversed toroidal current.

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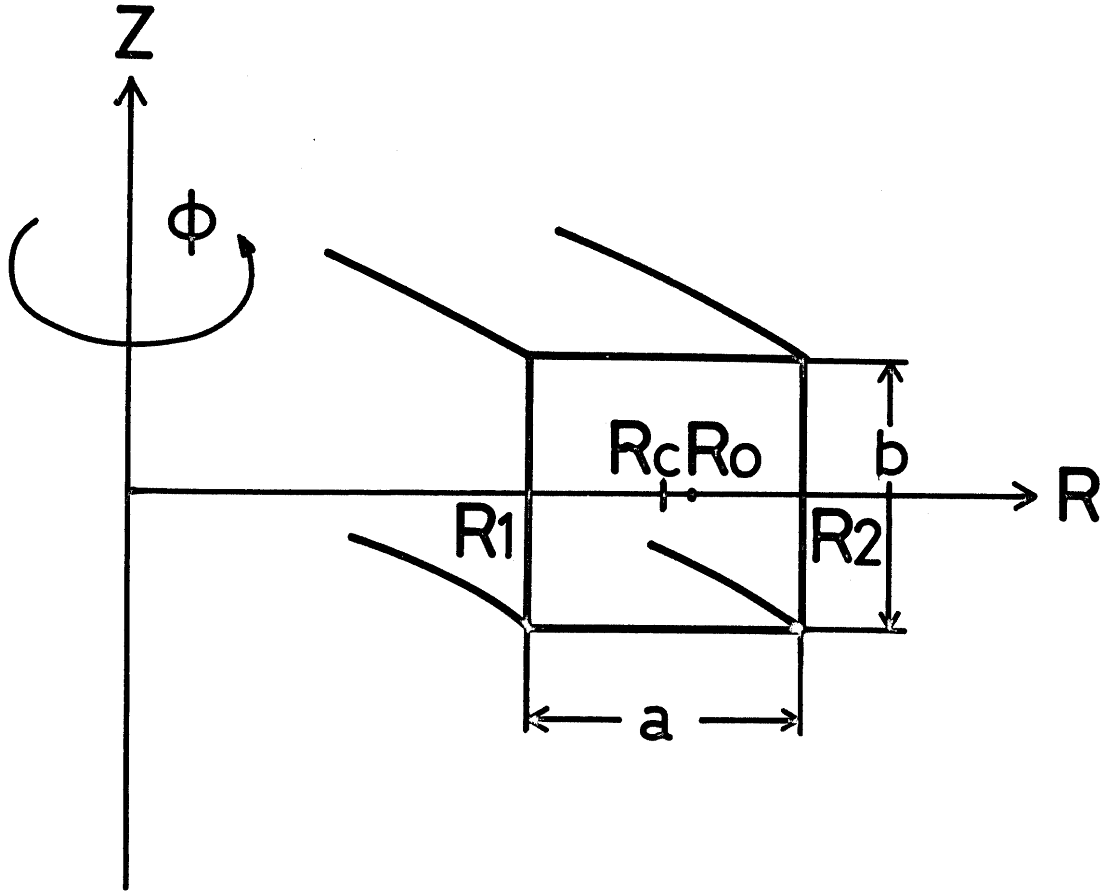
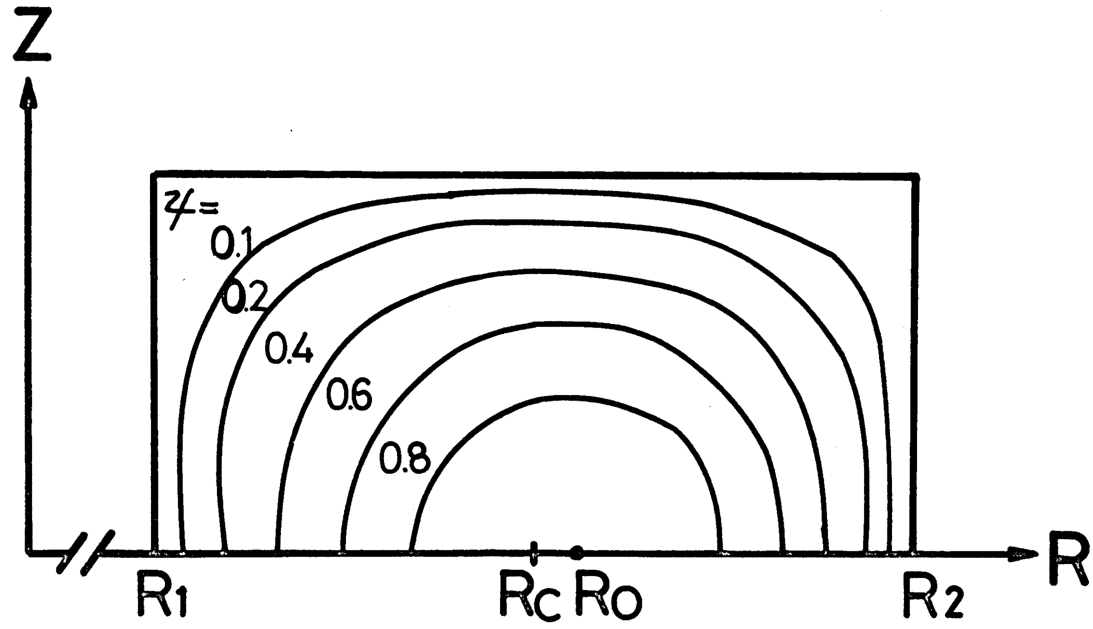
## Figure Captions

Fig.1. Coordinate system and dimensions of the conducting wall. $R_c = (1/2)(R_1+R_2)$. $a = R_2-R_1$.

Fig.2. Typical magnetic surfaces for $a/b = 1$, $a/2R_c = 1/4$ and $\psi_0 = 1$.



Z
φ
RcRo
b
R
R1
R2
a


Z
ψ=
0.1
0.2
0.4
0.6
0.8
R
R1
Rc
Ro
R2