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New Electron-Wave Amplitude Oscillation
due to an Initial Spatial Disturbance

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Abstract

When an initial disturbance in a plasma is so strong that $f_o(\lambda)$ ($f_o(v)$: unperturbed distribution, λ: phase velocity) and $g(\lambda) \cos kx$ ($g(v) \cos kx$: disturbance) are of the same order, the spatial inhomogeneity represented by the latter causes a new type of amplitude oscillation while the usual amplitude oscillation predicted by O'Neil is not sensitive whether $g(\lambda) = 0$ or not. This is due to the fact that the initial spatial disturbance survives around λ unsubjected to phase mixing for some time. The most remarkable feature of this oscillation is that the amplitude almost always remains larger than the initial one. The maximum amplitude of an electric field E is nearly $E_o(1 + 2\phi_o{}^2)$ where $\phi_o = \omega_D^2\omega_{pe}/\omega_B^3$, $\omega_D^2 = 4\pi e^2 g(\lambda) v_{TR}/m$, $v_{TR} = \omega_B/k$ and ω_B is the bouncing frequency of the electron.

§1. Introduction

It is well-known that the amplitude of monochromatic disturbance in a plasma will not Landau-damp, but will oscillate when the initial bouncing frequency of a particle (electron) ω_B is larger than the Landau damping coefficient γ_L [1,2,3]. Since the gradient of the initial unperturbed distribution $f_o(v)$ of the electrons at the phase velocity λ is not zero, i.e., $\partial f_o(\lambda)\partial v \equiv f_o'(\lambda) \neq 0$, there is a difference between the numbers of trapped electrons with velocities larger and smaller than λ, and there then occurs a population inversion around λ and an energy exchange between the trapped electrons and the wave. This is thought to be the reason of the amplitude oscillation predicted by O'Neil[1].

We will show here that another kind of amplitude oscillation can occur when $g(\lambda)$ is finite even if $f_o'(\lambda) = 0$, where $g(v)\cos kx$ is a deviation from the unperturbed distribution. When $g(\lambda) \neq 0$, the initial spatial disturbance survives around λ unsubjected to phase mixing for some time. Furthermore, the spatial inhomogeneity of this initial disturbance turns into one in velocity space, and together with a finite shift in phase velocity causes an amplitude oscillation. This means that the amplitude oscillation takes place independently of whether $f'(\lambda) = 0$ or not when $g(\lambda)$ is small but finite. And, when $f_o'(\lambda) \neq 0$, the effect of $g(\lambda)$ is still appreciable if $g(\lambda)$

and $f_o'(\lambda)$ are of the same order. We here formulate the problem only taking into account the effect of $g(v)$, although usually the effects of both $g(\lambda)$ and $f'(\lambda)$ coexist. The importance of the initial spatial disturbance was first emphasized by Taniuti[4], whereas O'Neil[1] has failed to estimate its effect.

§2. Calculation and Discussions

The method of treatment of the problem follows reference (5) where an asymptotic expansion of the Vlasov equation is developed. We first note that the exact Vlasov equation expresses only incompressible flow in phase space, that is,

$$f(x,v,t) = f(x(0),\ v(0),\ 0) = f_o(v(0)) - 2\varepsilon^2 g(v(0)) \cos kx(0) \tag{1}$$

or f is constant along the characteristics $dx/dt = v$, $dv/dt = -E$, and E is determined by the Poisson equation. Here $f_o(v(0))$, which may be considered to be a Maxwellian, stands for the unperturbed state of the plasma while $g(v(0))$ represents an initial disturbance, and all quantities are normalized appropriately[5]. The smallness parameter ε is taken to be the square root of the smallest decay constant of Landau damping, γ_L, that is, $\varepsilon^2 = \pi G(\lambda)/2k^2 \equiv \gamma_L$ where $G(\lambda)$ and λ are $\partial f_o(\lambda)/\partial v$ and phase velocity.

Suppose that $\underline{E}$ asymptotically takes the form

$$E = \varepsilon^2(E(\eta)e^{ik(x-\lambda t)} + \text{C.C.}) + \varepsilon^3 \Sigma E_\ell^{(1)}(\eta)e^{i\ell k(x-\lambda t)} + \ldots \tag{2}$$

where $\eta = \varepsilon t$ is a stretched time. Substituting eq.(2) into the equation of characteristics will allow one to determine the long time asymptotic behaviours of the characteristics. Results will be remarkably different for the two regions, the non-resonance region for which $|v - \lambda| >> \varepsilon$, and the resonance region where v becomes close to λ.

First we consider the non-resonance region. We assume, corresponding to eq.(2), that the deviation of the particle orbit from the straight line path is of the order of ε^2. We calculate the orbit up to $0(\varepsilon^3)$ and express $f(x(0), v(0))$ in terms of $(x(t), v(t))$ and the deviations. Finally, from the Poisson equation we get[5)]

$$2\varepsilon^3 k\frac{dE}{d\eta} + \text{C.C.} = \text{Terms of } O(\varepsilon^3) \text{ of } \frac{k}{2\pi}\int_{-\pi}^{\pi} e^{-ik(x-\lambda t)}dx\int_{res} f dv \tag{3}$$

Here we have used the fact that the contribution from the initial disturbance g(v) will phase mix to zero in this region; also, $\int_{res}$ means an integration over the resonance region.

In order to evaluate the integral on the RHS of (3), we have to specify the velocity range of the resonant particles. They will move with almost phase velocity, then we impose the requirment that the resonant particles do not appreciably feel the plasma oscillation. Still the boundary between non-resonance and resonance regions is quite diffusive. We introduce an artificial critical velocity δ such that the velocity range of the resonant particles is to be given by $|v-\lambda|<\delta$ and the δ satisfies the following assignments, i.e., i) $G(\lambda\pm\delta) = O(\varepsilon)$ and ii) $1>\delta>>\varepsilon$. If the integration over the resonance region is independent of an arbitrary δ which satisfies i) and ii), we consider that the integration is defined. More detailed discussion of these two regions is given in reference (5). Since the resonant particles do not feel the plasma oscillation, their orbits vary so slowly that $\tilde{x} = x - \lambda t$ and $\tilde{v} = v - \lambda$ are functions of η only. Assume $\tilde{x}(\eta) = X + \varepsilon x^1 + \cdots$ and $\tilde{v}(\eta) = \varepsilon V + \cdots$, and we have

$$\frac{dX}{d\eta} = V, \quad \frac{dV}{d\eta} = -E(\eta) \exp ikX + C.C.$$

Then $\int_{res} f(v(0))dv$ is expressed as

$$\int_{res} f dv = \int_{res} f(\lambda)dv + \varepsilon^2 \frac{\partial f}{\partial v}(\lambda) \int_{-\delta/\varepsilon}^{\delta/\varepsilon} V(0)dV \qquad (3')$$

$$-2\varepsilon^3 g(\lambda) \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \cos(kX(0))dV$$

It is shown that the second integral on the RHS converges independently of the value of $\delta/\varepsilon(>>1)$[3]. We, for a while, assume the third integral converges, too. Then considering the definition of ε, we know the second term is of the order of ε^4, hence the term in the above equation corresponding to the one on the RHS of eq.(3) is the last one. Equating this with the LHS of eq.(3), multiplying both sides by e^{-ikX} and integrating over X, we will have an equation governing $dE/d\eta$. We introduce the following variables: $y = kX$, $u = kV/(2kE_o)^{\frac{1}{2}}$, $\tau = (2kE_o)^{\frac{1}{2}}\eta$, $De^{i\phi-\frac{\pi}{2}i}=E/E_o$ (D: real and positive), E_o being $|E(0)|$. Noting that the phase volume dXdV or dydu is conserved[6], we finally get

$$\frac{du}{d\tau} = -D\sin(y+\phi) \tag{4}$$

$$\frac{dD}{d\tau} = -\frac{g(\lambda)}{2\pi k^2 E_o}\int_{-\pi}^{\pi}dy_o\int_{-\xi}^{\xi}du_o\ \sin(y+\phi)\cos y_o \tag{5}$$

$$\frac{d\phi}{d\tau} = -\frac{g(\lambda)}{2\pi k^2 D E_o}\int_{-\pi}^{\pi}dy_o\int_{-\xi}^{\xi}du_o\ \cos(y+\phi)\cos y_o \tag{6}$$

where $y_o = y(0)$ and $u_o = u(0)$. The only parameter is now $g(\lambda)/(k^2E_o)$ provided $\phi(0) = 0$. The boundary value of the integration ξ is of the order of δ/ε because k and E are O(1). It can be shown that the integrands of (5) and (6) are peaked around $u_o = 0$ and decrease rapidly for large $|u_o|$. Then we may put $\pm\xi = \pm\infty$ in the actual calculation, i.e., the third integral of (3') converges and our formulation to get (5) and (6) will be consistent.

We estimate $D(\tau)$ analytically. We assume that only deeply trapped electrons contribute to the calculation and that D does not change significantly during the time interval considered and may be put equal to unity as long as the time derivative is not important. Notice that for deeply trapped electrons $|y + \phi| << 1$, then the RHS of (6) is almost constant and $\phi \approx -\phi_o\tau$ where $\phi_o = g(\lambda)\Delta y_o \Delta u_o/(k^2 E_o)$, and $\Delta y_o \Delta u_o$ is the phase volume to which deeply trapped electrons initially belonged. We note that ϕ is given alternatively by $\phi \approx (\omega_D^2 \omega_{pe}/\omega_B^3)t$, $\omega_D^2 = 4\pi e^2 g(\lambda) v_{TR}/m$ and $v_{TR} = \omega_B/k$ in unnormalized quantities. The equation of motion (4) can be solved and gives us

$$y = \phi_o\tau + (u_o - \phi_o)\sin\tau + y_o \cos\tau$$

Then eq.(5) reduces to

$$\frac{dD}{d\tau} = \phi_o{}^2 \sin\tau \quad \text{or } D = 1 + 2\phi_o{}^2 \sin^2(\tau/2)$$

It is interesting to note that: (i) the amplitude grows periodically over the initial one, or an instability takes place, and (ii) the phase velocity increases as ϕ_0 does. We solve the set of eqs.(4), (5) and (6) numerically and find nearly the same results as predicted by the above analytical estimation. These are shown in Fig.1.

We shall attempt to give a more lucid explanation of the phase shift and the amplitude oscillation. In about half a cycle, or in π/ω_B, the spatial inhomogeneity turns into the velocity space inhomogeneity and there occurs a dip in the velocity distribution as may be seen from the particle trajectories in phase space which follow. A schematic diagram is given in Fig.2. Note that the distribution of untrapped electrons adjacent to the trapped regions must be, in an average sense, smoother than that of the trapped electrons. The dip causes a phase velocity shift. The dispersion relation for the distribution shown in Fig.2 is given by[7)]

$$1 = \omega^{-2} - \omega_D^2/[(\omega - k\lambda)^2 - \omega_B^2]$$

Putting $\omega = k\lambda + k\delta\lambda \equiv 1 + k\delta\lambda$ and expanding the first term on the RHS with respect to $\delta\lambda$, we get a cubic equation for $\delta\lambda$. Using $\omega_D^2/\omega_B^2 = O(\varepsilon)$, then the smallest root corresponding to the phase shift considered is given by $\delta\lambda = \omega^2{}_D/(2k\omega_B{}^2)$ and the phase shift $k(\delta\lambda)t$ is $g(\lambda)\Delta u_0\tau/(4k^2E_0) \simeq g(\lambda)\Delta u_0\Delta y_0\tau/(4k^2E_0) \simeq \phi_0\tau$ which is the one given before where we have used $\Delta y_o \simeq 1$. Since $\omega_D^2/\omega_B^2 = 0(\varepsilon)$, the dispersion relation above does not show any instability and in order to explain the growth in amplitude we must examine the configuration of the trapped region.

As soon as the phase velocity shifts, the trapped region will become as shown in Fig.2 and 3. The horizontally lined portion where the magnitude of the distribution is relatively small is almost in the lower half in the new potential trough in Fig.3, while the newly trapped portion is supposed to have the average density. Then there are more electrons in the upper half portion than in the lower half. Therefore in the next half cycle the wave will grow. This is the mechanism of this new type of amplitude oscillation. We numerically consider the situation in which the amplitude oscillation will appear. Suppose $g(\lambda)/(2\pi k^2 E_o) = 1/20$, $2\varepsilon^2 g(\lambda) = f_o(\lambda)$, $\lambda = k^{-1} = 4.5(=3.2v_{th,e})$ and $T_e = 10$ eV, so we have $\varepsilon^2 = 2.3 \times 10^{-3}$ and $0.4\sqrt{n(\mathrm{cm}^{-3})} \times 10^{-5}$V/cm for the electric field. We also estimate $q \equiv \gamma_L/\omega_B$ introduced in reference (3) to be 0.15. In this case an ordinary amplitude oscillation will also appear, although smaller than the "new" oscillation amplitude, and the profile of the amplitude will usually become a superposition of these two types of amplitude oscillations. It is interesting to note that since, as is seen in Fig.2 and 3, the magnitude of the distribution in the trapped region is small compared with that in the adjacent untrapped region, the side band instability which has not been discussed in this paper will be excited more easily that in those cases where the trapped region is flat or convex[8]. We note that in experiments[9] and computer simulations[10] of the two beam instability excitation, there sometimes occurs a strong amplitude oscillation of the excited wave. This may be explained as follows: due to the strong interaction between a plasma and a beam, the beam bunches spatially and since the phase velocity of excited wave is almost the same as the velocity of the beam, the spatial bunching turns into an inhomogeneity in velocity space and causes the amplitude oscillation.

§3. Conclusion

We have solved the Vlasov equation with an initial condition having a spatial inhomogeneity. Under the conditions $\omega_B \gg \gamma_L$ and $O(g(\lambda)) = O(f'(\lambda))$ we find a new type of amplitude oscillation which differs from the usual one in the following points: i) the initial spatial inhomogeneity is the source of the amplitude oscillation; ii) there appears a finite phase shift; and iii) the amplitude almost always remains larger than the initial one.

§4. Acknowledgements

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Figure Caption

Fig.1. The time dependence of the normalized amplitude D and the phase ϕ. $b \equiv g(\lambda)/(2\pi k^2 E_o)$ where $g(\lambda)$ and E_o represent the initial disturbance of the distribution at phase velocity and the initial field, respectively.

Fig.2. A schematic diagram of the distribution f(v) vs. velocity v near the phase velocity. ←-----→ Region of trapping for zero phase velocity shift. ⟷ Region of trapping for finite, nearly constant phase velocity shift.

Fig.3. A schematic diagram of trapped regions. The trapped region surrounded by the dotted line gradually transfers to the one surrounded by the solid line at about $t = \pi/\omega_B$. In the vertically lined, the blank and the horizontally lined area, the magnitude of the distribution function is relatively large, average-sized, and relatively small, respectively.

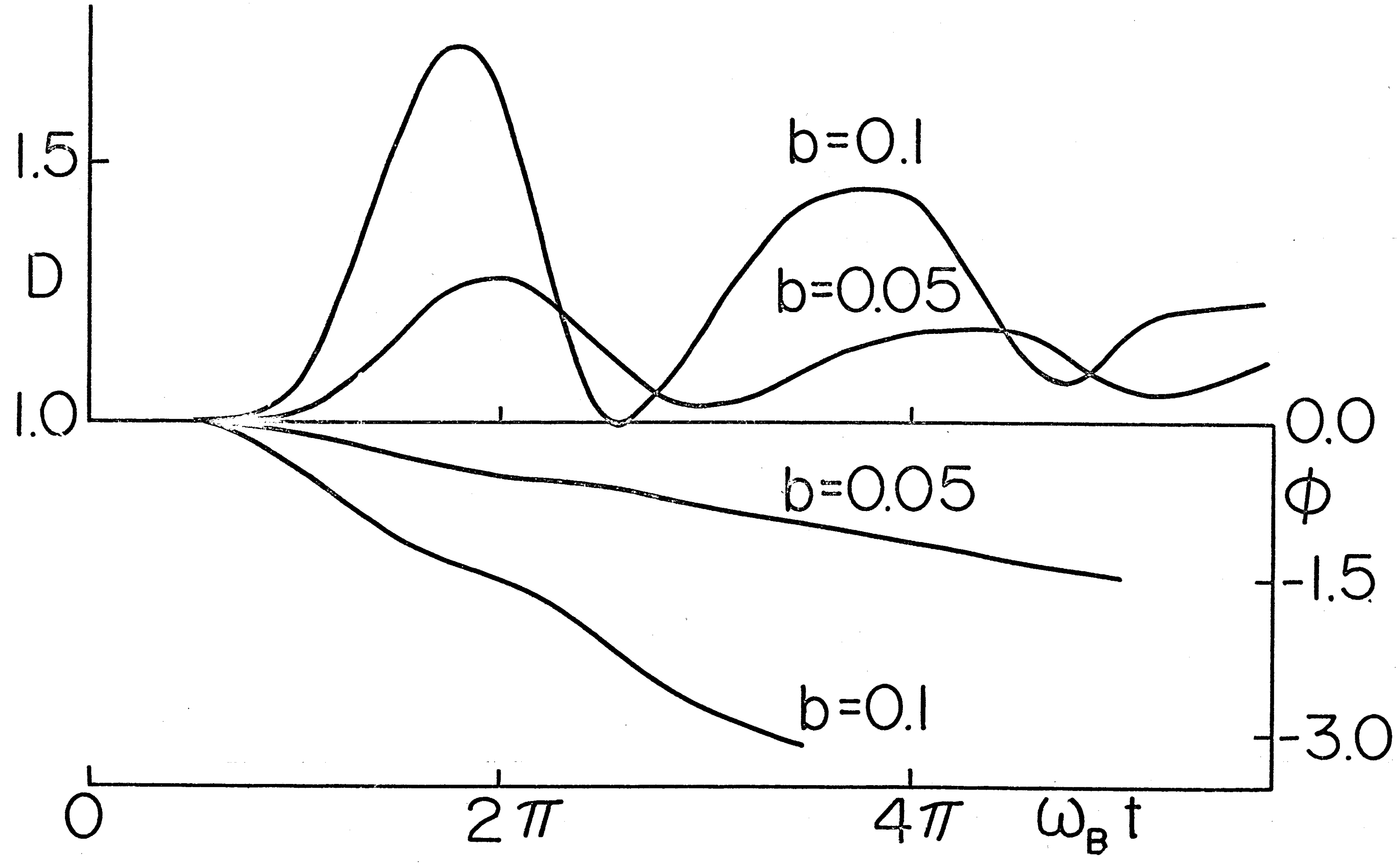

FIG. 1

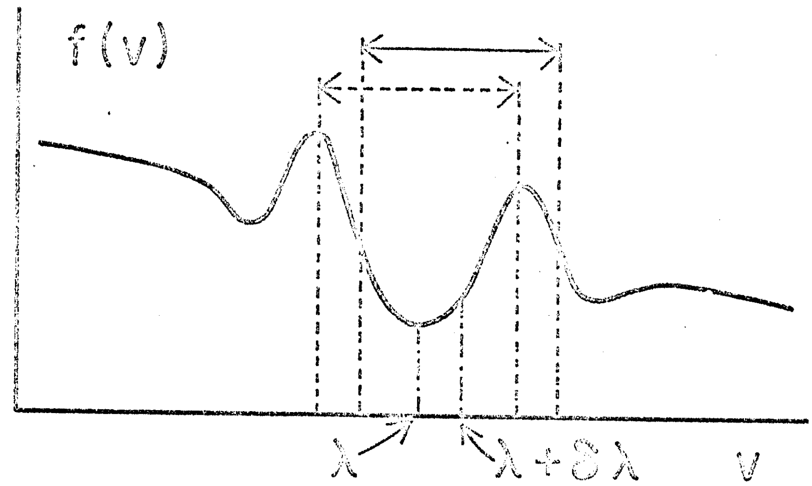

FIG. 2

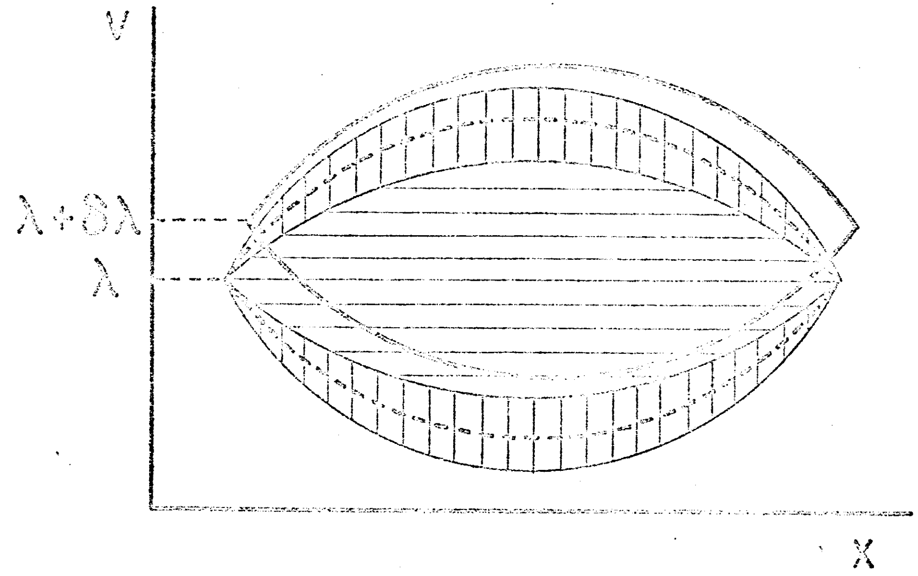

FIG. 3