INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

RESEARCH REPORT

NAGOYA, JAPAN

Theoretical Analysis

of Ion Cyclotron Waves

Kiyokata Matsuura and Hiroshi Toyama

IPPJ-150 February 1973

Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, JAPAN.

Abstract

Dispersion and energy relations of ion cyclotron wave for a cold plasma are systematically studied retaining the electron mass. In this case, two wave modes having different perpendicular wave numbers can exist in a uniform plasma column. The dominant mode is a quasi-TE mode, in which the plasma current j_z is an important quantity. Because this current is carried by electron motion along the static magnetic field, the dispersion relation and energy relations are considerably modified for light mass ions compared with the customary one in which the electron mass is neglected.

Plasma loading for a coil excitation of the Stix type is also considered using this dispersion relation. The electric field, magnetic field, energy density, and energy flow are calculated based on the quasi-TE mode. The energy losses due to cyclotron damping and collisions are also summarized. Here a possible mechanism due to resistivity is pointed out.

1. INTRODUCTION

The theoretical analysis of ion cyclotron waves in a uniform cold plasma has been well established. For example, Stix has described the analysis in some detail in his book[1]. In such an analysis, it is usually assumed that the mass of the electron is quite small and has a negligible effect. The coupling between waves and the exciting field has been calculated by Stix, by combining the dispersion relation with the boundary conditions. Experimental results[2] are well understood in terms of these theoretical analyses.

However, there exists an effect due to finite mass of the electron. Hosea et al[3] studied this finite mass effect relating it to the experimental results of the Model C Sterallater at PPL. Princeton, and found that the modification in the dispersion relation and the coupling are considerable.

In a magnetized plasma, the major part of the plasma current along a static magnetic field is carried by the electrons. Therefore, when this plasma current plays an important role in a wave, the characteristics of the wave will be affected by the electron motion. As described in the following, such a situation exists for ion cyclotron waves of light mass ions.

In this report, dispersion relation, plasma loading, and related problems are systematically studied by taking the electron mass into consideration.

2. DISPERSION RELATION

The theoretical analysis of electric and magnetic fields in a cold plasma has been well established[4]. Therefore, we shall give only a straightforward analysis of the wave fields within a cold plasma without a detailed discussion.

In the following calculation, we shall use cylindrical coordinates in which the z axis is parallel to the direction of a static magnetic field. We may then use the dielectric tensor which gives the following relation between displacement $\mathbb{D}$ and $\mathbb{E}$;

$$\mathbb{D} = \begin{pmatrix} \varepsilon_t & \varepsilon_h & 0 \\ -\varepsilon_h & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_\ell \end{pmatrix} \begin{pmatrix} E_r \\ E_\theta \\ E_z \end{pmatrix} . \tag{1}$$

We are concerned here only with wave fields, and the field components are assumed to vary in time as $e^{-i\omega t}$. Then the Maxwell's equation may be written as

div D = 0:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\varepsilon_t E_r) + \frac{1}{r}\frac{\partial}{\partial r}(r\varepsilon_h E_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(-\varepsilon_h E_r)$$

$$+ \frac{1}{r}\frac{\partial}{\partial \theta}(\varepsilon_t E_\theta) + \frac{\partial D_z}{\partial z} = 0, \tag{2}$$

curl $\mathbb{E} = -\frac{1}{c}\frac{\partial \mathbb{B}}{\partial t}$:

$$\hat{r};\quad \frac{1}{r}\frac{\partial}{\partial\theta} E_z - \frac{\partial}{\partial z} E_\theta = i\frac{\omega}{c} B_r, \tag{3}$$

$$\hat{\theta};\quad \frac{\partial}{\partial z} E_r - \frac{\partial}{\partial r} E_z = i\frac{\omega}{c} B_\theta, \tag{4}$$

$$\hat{z};\quad \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta} E_r = i\frac{\omega}{c} B_z, \tag{5}$$

div$\mathbb{B} = 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial}{\partial\theta} B_\theta + \frac{\partial}{\partial z} B_z = 0, \tag{6}$$

curl $\mathbb{B} = \frac{1}{c}\frac{\partial}{\partial t}\mathbb{D}$:

$$\hat{r};\quad \frac{1}{r}\frac{\partial}{\partial\theta} B_z - \frac{\partial}{\partial z} B_\theta = -i\frac{\omega}{c} D_r, \tag{7}$$

$$\hat{\theta};\quad \frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z = -i\frac{\omega}{c} D_\theta, \tag{8}$$

$$\hat{z};\quad \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta} B_r = -i\frac{\omega}{c} D_z. \tag{9}$$

Combining equations (3), (4), (8), and (2), we obtain a relation between D_z and B_z;

$$\{\frac{\partial^2}{\partial z^2} + \frac{\varepsilon_t}{\varepsilon_\ell}\nabla_\perp^2 + \varepsilon_t\,(\frac{\omega}{c})^2\}\,\frac{i\omega}{c} D_z - \{\varepsilon_h(\frac{\omega}{c})^2\}\,\frac{\partial}{\partial z} B_z = 0, \tag{10}$$

where $\nabla_{\perp}^2 \equiv \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r}) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$.

Combining equations (7), (8), (5), and (6) we obtain another relation between B_z and D_z;

$$\{\frac{\partial^2}{\partial z^2}\} \frac{i\omega}{c} D_z - \frac{1}{\varepsilon_h} \{\varepsilon_t \nabla^2 + (\frac{\omega}{c})^2 (\varepsilon_t{}^2 + \varepsilon_h{}^2)\} \frac{\partial}{\partial z} B_z = 0, \tag{11}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial z^2} + \nabla_{\perp}^2$.

Here, we may note that equations (10) and (11) are valid also in cartesian coordinate if the differential operator $\nabla_{\perp}^2$ in the cylindrical coordinate is replaced by the corresponding operator in the cartesian coordinates.

In order to get a simultaneous solution for equations (10) and (11), we assume that the field components have the following forms.

$$\left.\begin{aligned} D_z &= aJ_m(\nu r)e^{i(m\theta + K_{\parallel} z - \omega t)}, \\ B_z &= bJ_m(\nu r)e^{i(m\theta + K_{\parallel} z - \omega t)}, \end{aligned}\right\} \tag{12}$$

where J_m is a Bessel function. The dispersion relation, which is obtained from the condition for a nontrivial solution of the set of field equations (10) and (11) is then

$$(\varepsilon_\ell k_\parallel^2 + \varepsilon_t{}^2\nu^2 - \varepsilon_\ell\varepsilon_t \frac{\omega^2}{c^2})(k_\parallel^2 + \nu^2 - \frac{\omega^2}{c^2}\varepsilon_t)$$

$$- \frac{\omega^2}{c^2}\varepsilon_h^2(\nu^2 - \varepsilon_\ell \frac{\omega^2}{c^2}) = 0, \tag{13}$$

or

$$(\varepsilon_t N_\perp^2 + \varepsilon_\ell N_\parallel^2 - \varepsilon_\ell\varepsilon_t)(N^2 - \varepsilon_t) - (N_\perp^2 - \varepsilon_\ell)\varepsilon_h^2 = 0, \tag{14}$$

where

$$N_\parallel^2 \equiv (\frac{k_\parallel c}{\omega})^2,\ N_\perp^2 \equiv (\frac{\nu c}{\omega})^2,\ N^2 \equiv N_\parallel^2 + N_\perp^2.$$

Here, we may note that the dispersion relation (14) is equivalent to the usual one, which is written in the following form[1];

$$AN^4 - BN^2 + C = 0.$$

From equations (10) and (11), the ratio of B_z to E_z is

$$\zeta \equiv \frac{B_z}{E_z} = - \frac{\varepsilon_\ell\varepsilon_h N_\parallel}{\varepsilon_t(\varepsilon_t - N^2) + \varepsilon_h^2}$$

$$= - \frac{N_\perp^2\varepsilon_h N_\parallel}{(N_\parallel^2 - \varepsilon_t)(N^2 - \varepsilon_t) + \varepsilon_h^2}. \tag{15}$$

It is clear from equation (14) that ν^2 has two values for a given k^2. Thus N^2, $N_\perp^2$ and ζ are also double valued which correspond to two wave modes.

3. PERPENDICULAR FIELD COMPONENTS

Here we shall introduce the following functions;

$$\left.\begin{aligned} B_{\pm} &\equiv B_r \pm iB_\theta, \\ E_{\pm} &\equiv E_r \pm iE_\theta, \\ \frac{\partial}{\partial s_{\pm}} &\equiv \frac{\partial}{\partial r} \pm i\frac{1}{r}\frac{\partial}{\partial\theta}. \end{aligned}\right\} \tag{16}$$

From equations (3) and (4), we have

$$\frac{\omega}{c} B_{\pm} = \mp \left[\frac{\partial}{\partial s_{\pm}} E_z - \frac{\partial}{\partial z} E_{\pm}\right]. \tag{17}$$

In the same way, from equations (7) and (8), we have

$$\frac{\omega}{c} E_{\pm} = \pm \frac{1}{\varepsilon_t \mp i\varepsilon_h} \left[\frac{\partial B_z}{\partial s_{\pm}} - \frac{\partial}{\partial z} B_{\pm}\right]. \tag{18}$$

Combining equations (17) and (18), $E_{\pm}$ and $B_{\pm}$ are expressed in the following forms;

$$E_{\pm} = A_0^{-1}\{(\varepsilon_t - N_\parallel^2) \pm i\varepsilon_h\}\left[\frac{c}{\omega}\frac{\partial}{\partial s_{\pm}}(iN_\parallel E_z \pm B_z)\right], \tag{19}$$

$$B_{\pm} = A_0^{-1}\{(\varepsilon_t - N_\parallel^2) \pm i\varepsilon_h\}\left[\frac{c}{\omega}\frac{\partial}{\partial s_{\pm}}\{iN_\parallel B_z \mp (\varepsilon_t \mp i\varepsilon_h)E_z\}\right], \tag{20}$$

where

$$A_0 \equiv [(\varepsilon_t - N_\parallel^2)^2 + \varepsilon_h^2].$$

Using the definition (16), we can easily obtain the perpendicular field components to be

$$\begin{aligned}
E_r &= \frac{c}{\omega}\frac{1}{A_0}[\frac{\partial}{\partial r}\{i(\varepsilon_t - N_\parallel^2)N_\parallel E_z + i\varepsilon_h B_z\} \\
&\quad - i\frac{1}{r}\frac{\partial}{\partial\theta}\{\varepsilon_h N_\parallel E_z - (\varepsilon_t - N_\parallel^2)B_z\}], \\
E_\theta &= \frac{c}{\omega}\frac{1}{A_0}[\frac{\partial}{\partial r}\{i\varepsilon_h N_\parallel E_z - i(\varepsilon_t - N_\parallel^2)B_z\} \\
&\quad + i\frac{1}{r}\frac{\partial}{\partial\theta}\{(\varepsilon_t - N_\parallel^2)N_\parallel E_z + \varepsilon_h B_z\}], \\
B_r &= \frac{c}{\omega}\frac{1}{A_0}[\frac{\partial}{\partial r}\{iN_\parallel(\varepsilon_t - N_\parallel^2)B_z - iN_\parallel^2\varepsilon_h E_z\} \\
&\quad + i\frac{1}{r}\frac{\partial}{\partial\theta}\{-N_\parallel\varepsilon_h B_z - (\varepsilon_t^2 + \varepsilon_h^2 - N_\parallel^2\varepsilon_t)E_z\}], \\
B_\theta &= \frac{c}{\omega}\frac{1}{A_0}[\frac{\partial}{\partial r}\{iN_\parallel\varepsilon_h B_z + i(\varepsilon_t^2 + \varepsilon_h^2 - N_\parallel^2\varepsilon_t)E_z\} \\
&\quad + i\frac{1}{r}\frac{\partial}{\partial\theta}\{N_\parallel(\varepsilon_t - N_\parallel^2)B_z - N_\parallel^2\varepsilon_h E_z\}].
\end{aligned} \tag{21}$$

Since there is the relation (15) between B_z and E_z, the field components may be expressed in terms of E_z (or also B_z) as follows;

$$\left.\begin{aligned}
E_r &= \frac{c}{\omega} N_{\parallel} A[i(\varepsilon_t - N^2) \frac{\partial}{\partial r} E_z - i\varepsilon_h \frac{1}{r} \frac{\partial}{\partial \theta} E_z],\\
E_\theta &= \frac{c}{\omega} N_{\parallel} A[i\varepsilon_h \frac{\partial}{\partial r} E_z + i(\varepsilon_t - N^2) \frac{1}{r} \frac{\partial}{\partial \theta} E_z],\\
B_r &= \frac{c}{\omega} A[-iN_{\parallel}^2 \varepsilon_h \frac{\partial}{\partial r} E_z - i\{\varepsilon_t(\varepsilon_t - N^2) + \varepsilon_h^2\} \frac{1}{r} \frac{\partial}{\partial \theta} E_z],\\
B_\theta &= \frac{c}{\omega} A[i\{\varepsilon_t(\varepsilon_t - N^2) + \varepsilon_h^2\} \frac{\partial}{\partial r} E_z - iN_{\parallel}^2 \varepsilon_h \frac{1}{r} \frac{\partial}{\partial \theta} E_z],
\end{aligned}\right\} \tag{22}$$

where

$$A \equiv [(N_{\parallel}^2 - \varepsilon_t)(N^2 - \varepsilon_t) + \varepsilon_h^2]^{-1}.$$

The equations (21) and (22) are also valid in cartesian coordinates if the differntial operatars $\frac{\partial}{\partial r}$ and $\frac{1}{r} \frac{\partial}{\partial \theta}$ are replaced by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively.

4. ELECTRIC AND MAGNETIC FIELDS IN A VACUUM

In a vacuum, the components of the dielectric tensor in equation (1) are

$$\left.\begin{aligned}
\varepsilon_t &= \varepsilon_\ell = 1,\\
\varepsilon_h &= 0.
\end{aligned}\right\} \tag{23}$$

Therefore, the corresponding equations to (10) and (11) become

$$\left.\begin{aligned} (\nabla^2 + \frac{\omega^2}{c^2})\ E_z &= 0, \\ (\nabla^2 + \frac{\omega^2}{c^2})\ B_z &= 0. \end{aligned}\right\} \qquad (24)$$

These are standard Bessel equations for E_z and B_z. For E_z, this is

$$\frac{\partial^2}{\partial r^2} E_z + \frac{1}{r}\frac{\partial}{\partial r} E_z - \{\frac{\omega^2}{c^2}(N_\parallel^2 - 1) + \frac{m^2}{r^2}\} E_z = 0.$$

Here, we again have assumed that the components vary as $\exp i(k_\parallel z + m\theta - \omega t)$. In the experimental conditions for low frequency waves, such as the ion cyclotron wave, it may be assumed that $N_\parallel^2 > 1$. Therefore the field components E_z and B_z may be written as

$$\left.\begin{aligned} E_z &= a' I_m(kr) + a'' K_m(kr), \\ B_z &= b' I_m(kr) + b'' K_m(kr), \end{aligned}\right\} \qquad (25)$$

where

$$(\frac{kc}{\omega})^2 = N_\parallel^2 - 1.$$

and, a', a", b' and b" are arbitrary coefficients, and I_m, K_m are modified Bessel functions.

In a plasma, the relation between E_z and B_z is given by equation (15). In a vacuum, however, E_z and B_z are independent of each other as is clear from equation (24). Equations (17) and (18) are also useful in vacuum. We then have

$$E_\pm = \frac{c}{\omega} \frac{1}{1 - N_\parallel^2} \frac{\partial}{\partial s_\pm} [iN_\parallel E_z \pm B_z], \tag{26}$$

$$B_\pm = \frac{c}{\omega} \frac{1}{1 - N_\parallel^2} \frac{\partial}{\partial s_\pm} [iN_\parallel B_z \mp E_z]. \tag{27}$$

Thus we obtain a set of perpendicular field components;

$$\left.\begin{aligned} E_r &= \alpha_1 \frac{\partial}{\partial r} E_z + \alpha_2 B_z, \\ E_\theta &= \alpha_3 \frac{\partial}{\partial r} B_z + \alpha_4 E_z, \\ B_r &= \alpha_1 \frac{\partial}{\partial r} B_z - \alpha_2 E_z, \\ B_\theta &= -\alpha_3 \frac{\partial}{\partial r} E_z + \alpha_4 B_z, \end{aligned}\right\} \tag{28}$$

where

$$\alpha_1 \equiv - i \frac{c}{\omega} \frac{N_\parallel}{N_\parallel^2 - 1},$$

$$\alpha_2 \equiv \frac{m}{r}\frac{c}{\omega}\frac{1}{N_\parallel^2 - 1},$$

$$\alpha_3 \equiv i\frac{c}{\omega}\frac{1}{N_\parallel^2 - 1},$$

$$\alpha_4 \equiv \frac{m}{r}\frac{c}{\omega}\frac{N_\parallel}{N_\parallel^2 - 1}.$$

5. SUMMARY OF THE FIELDS

In equation (22), the field components in a cold plasma are expressed in terms of E_z. In the vacuum, the field components are expressed in terms of E_z and B_z. Therefore, we can write the field components as follows with arbitrary coefficients, a and b:

A) In the vacuum between the sheet current and the plasma $(s > r > p)$;

$$E_z^v = a_1 I_m(kr) + a_2 K_m(kr),$$

$$B_z^v = a_3 I_m(kr) + a_4 K_m(kr),$$

$$E_r^v = a_1\alpha_1 k I_m'(kr) + a_2\alpha_1 k K_m'(kr) + a_3\alpha_2 I_m(kr) + a_4\alpha_2 K_m(kr),$$

$$E_\theta^v = a_1\alpha_4 I_m(kr) + a_2\alpha_4 K_m(kr) + a_3\alpha_3 k I_m'(kr) + a_4\alpha_3 k K_m'(kr),$$

$$B_r^v = a_3\alpha_1 k I_m'(kr) + a_4\alpha_1 k K_m'(kr) - a_1\alpha_2 I_m(kr) - a_2\alpha_2 K_m(kr),$$

$$B_\theta^v = -a_1\alpha_3 k I_m'(kr) - a_2\alpha_3 k K_m'(kr) + a_3\alpha_4 I_m(kr) + a_4\alpha_4 K_m(kr),$$

B) In the vacuum outside the sheet current $(r > s)$;

$$E_z^w = a_6 K_m(kr),$$

$$B_z^w = a_7 K_m(kr),$$

$$E_r^w = a_6\alpha_1 k K_m'(kr) + a_7\alpha_2 K_m(kr),$$

$$E_\theta^w = a_6\alpha_4 K_m(kr) + a_7\alpha_3 k K_m'(kr),$$

$$B_\theta^w = -a_6\alpha_3 k K_m'(kr) + a_7\alpha_4 K_m(kr), \qquad (29)$$

$$B_r^w = a_7\alpha_1 k K_m'(kr) - a_6\alpha_2 K_m(kr),$$

c) In the plasma $(r < p)$;

$$E_z^p = a_5 J_m(\nu_1 r) + b_5 J_m(\nu_2 r),$$

$$B_z^p = a_5\zeta_1 J_m(\nu_1 r) + b_5\zeta_2 J_m(\nu_2 r),$$

$$E_\theta^p = a_5\delta_m(\nu_1 r) + b_5\delta_m(\nu_2 r),$$

$$B_\theta^p = a_5\xi_m(\nu_1 r) + b_5\xi_m(\nu_2 r).$$

where

$$\delta_m(\nu_j r) \equiv \frac{N_\parallel}{\{(N_\parallel^2 - \varepsilon_t)(N_j^2 - \varepsilon_t) + \varepsilon_h^2\}} [i\varepsilon_h |N_{\perp j}| Z_m'(\nu_j r)$$

$$- \frac{m}{k_\parallel p} N_\parallel (\varepsilon_t - N_j^2) Z_m(\nu_j r)],$$

$$\xi_m(\nu_j r) \equiv \frac{1}{\{(N^2 - \varepsilon_t)(N_j^2 - \varepsilon_t) + \varepsilon_h^2\}} [i|N_{\perp j}|\{\varepsilon_t(\varepsilon_t - N_j^2)$$

$$+ \varepsilon_h^2\} Z_m'(\nu_j r)],$$

$$\zeta_j \equiv \frac{-N_\perp^2 N_\parallel \varepsilon_h}{\{N_\parallel^2 - \varepsilon_t)(N_j^2 - \varepsilon_t) + \varepsilon_h^2\}},$$

$$Z_m'(\nu r) \equiv \frac{dZ_m(\nu r)}{d(\nu r)},$$

$$K_m'(kr) \equiv \frac{dK_m(kr)}{d(kr)},$$

$$Z_m(\nu_j r) \equiv \begin{cases} J_m(\nu_j r) & \text{for } \nu_j^2 \geq 0, \\ I_m(|\nu_j| r) & \text{for } \nu_j^2 < 0. \end{cases}$$

In the above expression for the field components, the dependence of exp $i(m\theta + k_\parallel z - \omega t)$ are omitted for the sake of simplicity.

6. BOUNDARY CONDITIONS

We consider a cylinder of cold plasma, infinitely long, surrounded by a vacuum. The radius of the plasma is p. At the radius $r = s$, $(s \geq p)$, there is a sheet current of density $j^* \exp i(m\theta + k_\parallel z - \omega t)$.

At the plasma-vacuum interface, there may exist a surface-charge and surface-current. Here, however, for the sake of simplicity, we shall assume that the surface= current to be zero. Then we have the following linearized continuity relations across the plasma-vacuum surface ($r = p$), for small amplitude wave fields;

$$\left.\begin{aligned} & B_z^p - B_z^v = 0, \\ & E_\theta^p - E_\theta^v = 0, \\ & E_z^p - E_z^v = 0, \\ & B_\theta^p - B_\theta^v = 0, \end{aligned}\right\} \tag{30}$$

where indices p and v refer to plasma and vacuum respectively.

At the sheet surface ($r = s$), the linearized boundary conditions are

$$B_z^v - B_z^w = - \frac{4\pi}{c} j_{\theta w}^*,$$

$$
\left.\begin{aligned}
B_\theta^v - B_\theta^w &= \frac{4\pi}{c}\, j_{zw}^* \,, \\
E_z^v - E_z^w &= 0 \,, \\
E_\theta^v - E_\theta^w &= 0 \,,
\end{aligned}\right\} \tag{31}
$$

where index w denotes the outerside vacuum of the sheet, and $j_{\theta w}^*$ and j_{zw}^* are the sheet current along the θ and the z directions respectively.

7. DETERMINATION OF THE COEFFICIENTS

The arbitrary coefficients a and b in the expressions (29) can be fixed if the boundary conditions are taken into consideration. Combining equations (29) and (31), we obtain

$$
\left.\begin{aligned}
a_1 &= \frac{i\omega 4\pi s}{c^2}\,\{(N_\parallel^2 - 1)J_{zw}^* + \frac{m}{sk_\parallel} N_\parallel^2 J_{\theta w}^*\}\, K_m(ks) \,, \\
(a_2 - a_6) &= -\,\frac{i\omega 4\pi s}{c^2}\,\{(N_\parallel^2 - 1)J_{zw}^* + \frac{m}{sk_\parallel} N_\parallel^2 J_{\theta w}^*\}\, I_m(ks) \,, \\
a_3 &= \frac{4\pi sk}{c}\, K_m'(ks) J_{\theta w}^* \,,
\end{aligned}\right\}
$$

$$
(a_4 - a_7) = -\,\frac{4\pi sk}{c}\, I_m'(ks) J_{\theta w}^* . \tag{32}
$$

Using the plasma-vacuum boundary conditions of equation (30) together with equations (29) and (32), we obtain the remaining coefficients as follows:

$$
\left.
\begin{aligned}
a_2 &= -\frac{1}{\Delta}\begin{vmatrix} G_1 & \zeta_2 Z_m(\nu_2 p) & K_m(kp) & \zeta_1 Z_m(\nu_1 p) \\ G_2 & \delta_{mg}(\nu_2 p) & \alpha_3 k K_m'(kp) & \delta_{mg}(\nu_1 p) \\ G_3 & Z_m(\nu_2 p) & 0 & Z_m(\nu_1 p) \\ G_4 & \xi_{mg}(\nu_2 p) & 0 & \xi_{mg}(\nu_1 p) \end{vmatrix}, \\
a_4 &= -\frac{1}{\Delta}\begin{vmatrix} 0 & \zeta_2 Z_m(\nu_2 p) & G_1 & \zeta_1 Z_m(\nu_1 p) \\ 0 & \delta_{mg}(\nu_2 p) & G_2 & \delta_{mg}(\nu_1 p) \\ K_m(kp) & Z_m(\nu_2 p) & G_3 & Z_m(\nu_1 p) \\ -\alpha_3 k K_m'(kp) & \xi_{mg}(\nu_2 p) & G_4 & \xi_{mg}(\nu_1 p) \end{vmatrix}, \\
a_5 &= \frac{1}{\Delta}\begin{vmatrix} 0 & \zeta_2 Z_m(\nu_2 p) & K_m(kp) & G_1 \\ 0 & \delta_{mg}(\nu_2 p) & \alpha_3 k K_m'(kp) & G_2 \\ K_m(kp) & Z_m(\nu_2 p) & 0 & G_3 \\ -\alpha_3 k K_m'(kp) & \xi_{mg}(\nu_2 p) & 0 & G_4 \end{vmatrix}, \\
b_5 &= \frac{1}{\Delta}\begin{vmatrix} 0 & G_1 & K_m(kp) & \zeta_1 Z_m(\nu_1 p) \\ 0 & G_2 & \alpha_3 k K_m'(kp) & \delta_{mg}(\nu_1 p) \\ K_m(kp) & G_3 & 0 & Z_m(\nu_1 p) \\ -\alpha_3 k K_m'(kp) & G_4 & 0 & \xi_{mg}(\nu_1 p) \end{vmatrix},
\end{aligned}
\right\} \quad (33)
$$

where

$$\Delta \equiv \begin{vmatrix} 0 & \zeta_2 Z_m(\nu_2 p) & K_m(kp) & \zeta_1 Z_m(\nu_1 p) \\ 0 & \delta_{mg}(\nu_2 p) & \alpha_3 k K_m'(kp) & \delta_{mg}(\nu_1 p) \\ K_m(kp) & Z_m(\nu_2 p) & 0 & Z_m(\nu_1 p) \\ -\alpha_3 k K_m'(kp) & \xi_{mg}(\nu_2 p) & 0 & \xi_{mg}(\nu_1 p) \end{vmatrix},$$

$$G_1 \equiv a_3 I_m(kp),$$

$$G_2 \equiv a_3 \alpha_3 k I_m'(kp),$$

$$G_3 \equiv a_1 I_m(kp),$$

$$G_4 \equiv -a_1 \alpha_3 k I_m'(kp),$$

$$\delta_{mg}(\nu_j p) \equiv \delta_m(\nu_j p) - \alpha_4(p) Z_m(\nu_j p),$$

$$\xi_{mg}(\nu_j p) \equiv \xi_m(\nu_j p) - \zeta_j \alpha_4(p) Z_m(\nu_j p),$$

$$\alpha_4(p) \equiv \alpha_4(r = p).$$

8. DISPERSION RELATION FOR THE ION CYCLOTRON WAVE

Equation (14) can be written in the following form;

$$L^4 \left[\frac{\varepsilon_t}{N_\parallel^2 - 1}\right] + L^2 \left[\frac{1}{(N_\parallel^2 - 1)^2} \{(N_\parallel^2 - \varepsilon_t)(\varepsilon_t + \varepsilon_\ell) - \varepsilon_h^2\}\right]$$

$$+ \left[\frac{\varepsilon_\ell}{(N_\parallel^2 - 1)^3} \{(N_\parallel^2 - \varepsilon_t)^2 + \varepsilon_h^2\}\right] = 0, \qquad (34)$$

where

$$L^2 \equiv \frac{N_\perp^2}{N_\parallel^2 - 1} .$$

Using this equation, $N_\perp^2$ can easily be obtained. For a single ion species, the components of the dielectric tensor for an uniform cold plasma are

$$\varepsilon_t = 1 + \frac{\Pi^2(\Omega_i\Omega_e - \omega^2)}{(\Omega_i^2 - \omega^2)(\Omega_e^2 - \omega^2)} = 1 + \frac{\gamma\{1 - \mu\Omega^2\}}{(1 - \Omega^2)\{1 - \mu^2\Omega^2\}} ,$$

$$\varepsilon_h = i\,\frac{\Pi^2\omega(\Omega_e - \Omega_i)}{(\Omega_i^2 - \omega^2)(\Omega_e^2 - \omega^2)} = i\,\frac{\gamma\Omega(1 - \mu)}{(1 - \Omega^2)\{1 - \mu^2\Omega^2\}} ,$$

$$\varepsilon_\ell = 1 - \frac{\gamma}{\mu\Omega^2} , \qquad (35)$$

where

$$\Omega \equiv \frac{\omega}{\Omega_i} ,$$

$$\gamma \equiv \frac{\Pi^2}{\Omega_e\Omega_i} ,$$

$$\mu \equiv \frac{\Omega_i}{\Omega_e} ,$$

and Ω_i and Ω_e are the ion and the electron cyclotron frequencies respectively and Π is the plasma frequency.

Assuming the condition that, $\mu^2 << 1$, $\frac{\gamma}{\mu\Omega^2} >> 1$, and that $\gamma >> 1$ in the coefficient of L^4, equation (34) may be reduced to

$$L^4\{\mu\Omega^2\} - L^2\{\Omega^4(2\mu\Omega_0^2) - \Omega^2(1 + \mu + \Omega_0^2) + 1\}$$

$$- \{\Omega^4(\Omega_0^4 + 2\mu\Omega_0^2) - \Omega^2(1 + 2\Omega_0^2) + 1\} = 0, \tag{36}$$

or

$$\Omega^4\{\Omega_0^4 + 2\mu\Omega_0^2 + 2L^2\mu\Omega_0^2\} - \Omega^2\{\mu L^4 + L^2(1 + \mu + \Omega_0^2)$$

$$+ (1 + 2\Omega_0^2)\} + \{L^2 + 1\} = 0, \tag{37}$$

where

$$\Omega_0^2 \equiv \frac{\gamma}{(N_{\parallel}^2 - 1)\Omega^2}$$

It is clear from equation (36) that L^2 has two roots, namely, L_1 and L_2 corresponding to two wave modes. Here, we shall assume that $|L_1| \geq |L_2|$. For $L_1^2 >> 1$, the Ω^4 terms in equation (37) may be neglected when $\Omega^2 < 1$ and $\Omega_0^2 < 1$. The latter is the experimental condition for ion cyclotron waves. We then obtain the following approximate solution of Ω;

$$\Omega^2 \simeq \frac{L_1^2 + 1}{\mu L_1^4 + L_1^2(1 + \Omega_0^2) + (1 + 2\Omega_0^2)}\,. \tag{38}$$

L_2 also satisfies equation (37). Using this value of Ω^2 in equation (36) and neglecting the L^4 term, we have the following approximate relation between L_1 and L_2

$$L_2^2 \simeq \frac{\Omega_0^2 - \mu L_1^2 (1 + \mu L_1^2)}{\mu L_1^2 (\mu L_1^2 + 1 + \Omega_0^2)} . \tag{39}$$

9. NATURAL WAVE

The condition for a natural ion cyclotron wave in a cylinder of a cold uniform plasma can be determined from the condition $\Delta = 0$, where Δ is the determinant defined in equation (33);

$$\Delta = iK_m^2(kp) Z_m(L_1 kp) Z_m(L_2 kp) \{ (E_1 + \alpha\gamma_1)(F_2 + \alpha) - (E_2 + \alpha\gamma_2)(F_1 + \alpha) \} = 0, \tag{40}$$

where

$$\alpha \equiv \frac{1}{\sqrt{N^2 - 1}} \frac{K_m'(kp)}{K_m(kp)} ,$$

$$\gamma_j \equiv -i\zeta_j = -N \frac{\varepsilon_\ell (-i\varepsilon_h)}{(N^2 - 1)^2 \varepsilon_j} ,$$

$$E_j \equiv \frac{\delta_{mg}(\nu_j p)}{Z_m(\nu_j p)} = -\left(\frac{m}{kp}\right)\frac{1}{N_\parallel\sqrt{N_\parallel^2 - 1}}\left\{1 + \frac{\varepsilon_\ell}{L_j^2}\right\}$$

$$+ \gamma_j \frac{|L_j| \qquad Z_m'(L_j kp)}{\sqrt{N_\parallel^2 - 1}\; L_j^2 \; Z_m(L_j kp)},$$

$$F_j \equiv -\frac{i\xi_{mg}(\nu_j p)}{Z_m(\nu_j p)} = -\left(\frac{m}{kp}\right)\frac{N_\parallel}{\sqrt{N_\parallel^2 - 1}}\left\{1 + \frac{1}{L_j^2}\right\}\gamma_j$$

$$+ \frac{\varepsilon_\ell}{\sqrt{N_\parallel^2 - 1}} \frac{|L_j|}{L_j^2} \frac{Z_m'(L_j kp)}{Z_m(L_j kp)},$$

$$\varepsilon_j \equiv \frac{\varepsilon_t}{(N_\parallel^2 - 1)}\left\{\frac{(\varepsilon_t - 1)}{(N_\parallel^2 - 1)} - (1 + L^2{}_j)\right\} - \left(\frac{i\varepsilon_h}{N_\parallel^2 - 1}\right)^2.$$

In these expressions, the indices j = 1,2 denote the values corresponding to L_j. Here we may note that the condition $Z_m(L_j kp) = 0$ does not lead to the condition for a natural wave, since these factors will cancel out due to the numerators of the coefficients a and b given by equation (33). Thus the condition for a natural wave becomes

$$(E_1 + \alpha\gamma_1)(F_2 + \alpha) - (E_2 + \alpha\gamma_2)(F_1 + \alpha) = 0. \qquad (41)$$

In the case of m = 0, this condition may be reduced to a simple form assuming $F_j \gg \alpha$, and $N_\parallel^2 \gg 1$;

$$\frac{L_1^2}{|L_1|}\frac{Z_0(L_1kp)}{Z_0'(L_1kp)} = \frac{K_0(kp)}{K_0'(kp)}\left(\frac{\varepsilon_1}{\varepsilon_2} - 1\right) + \frac{L_2^2}{|L_2|}\frac{\varepsilon_1}{\varepsilon_2}\frac{Z_0(L_2kp)}{Z_0'(L_2kp)}. \quad (42)$$

The ratio $\varepsilon_1/\varepsilon_2$ for ion cyclotron waves can be obtained from equation (35) and the definition of ε_j given in equation (40). The approximate form of the ratio is

$$\frac{\varepsilon_1}{\varepsilon_2} \simeq \frac{\mu L_1^4}{\Omega^2\Omega_0^2} . \quad (43)$$

For the case in which $\mu = 0$, it can be shown that the condition for a natural wave given by equation (42) is reduced to the usual one [e.g. Stix ref. 1, p.88, eq. 25]. In this case, however, L_2 plays the role of L_1.

In the case of $L_1^2 >> 1$, the ratio, $\varepsilon_1/\varepsilon_2$, is sufficiently large, so that the second term on the right hand side of equation (42) becomes more important. A rough approximation from equation (42) is then (for moderate values of L_1kp),

$$J_1(L_1kp) \sim 0. \quad (44)$$

In Fig.1, the characteristics of the dispersion relation for $\mu = 0$, and $\mu \neq 0$ are schematically illustrated. Perpendicular resonance at which ν goes to infinity for a certain Ω, $0 < \Omega < 1$, appears in the case of $\mu = 0$. In the case of $\mu \neq 0$, however, this resonance disappears. In Fig. 2, the characteristics of the equation (36) is demonstrated.

In Fig.3, the values of the terms in equation (42) are plotted against kp to indicate the character of the equation.

10. STORED ENERGY IN EXCITING SYSTEM

Here we shall consider the stored energy in an exciting system without plasma. The stored energy density p in vacuum is generally given by

$$P = \frac{1}{8\pi} (\mathbb{E}\cdot\bar{\mathbb{E}} + \mathbb{H}\cdot\bar{\mathbb{H}}), \tag{45}$$

where $\bar{E}$ and $\bar{H}$ denote the complex conjugates of E and H respectively. Using the expressions for E and H given by equation (29), the stored energy density $p_i(r)$ for $0 \leq r \leq s$ is

$$P_i(r) = \frac{1}{8\pi} [\{1 + (\frac{N_\parallel^2 + 1}{N_\parallel^2 - 1})(\frac{m}{kr})^2\}(a_1\bar{a}_1 + a_3\bar{a}_3) I_m^2(kr)$$

$$+ (\frac{N_\parallel^2 + 1}{N_\parallel^2 - 1})(a_1\bar{a}_1 + a_3\bar{a}_3) \quad (I_m'(kr)^2$$

$$+ i(\frac{m}{kr})(\frac{4N_\parallel}{N_\parallel^2 - 1})(\bar{a}_1 a_3 - a_1\bar{a}_3) I_m(kr) I_m'(kr)]. \tag{46}$$

In the same way, the energy density $P_o(r)$ for $r \geq s$ is

$$P_0(r) = \frac{1}{8\pi}[\{1 + (\frac{N_\parallel^2 + 1}{N_\parallel^2 - 1})(\frac{m}{kr})^2\}(a_6\bar{a}_6 + a_7\bar{a}_7) \quad K_m^2(kr)$$

$$+ \left(\frac{N_\perp^2 + 1}{N_\parallel^2 - 1}\right)(a_6\bar{a}_6 + a_7\bar{a}_7)\ (K_m'(kr))^2$$

$$+ i\left(\frac{m}{kr}\right)\left(\frac{4N_\parallel}{N_\parallel^2 - 1}\right)(\bar{a}_6 a_7 - a_6\bar{a}_7)K_m(kr)K_m'(kr)]. \quad (47)$$

From the continuity relations for E_z and E_θ across the sheet surface, we obtain the following relations between the coefficients;

$$\left.\begin{aligned} a_6 &= \frac{I_m(ks)}{K_m(ks)}\, a_1, \\ a_7 &= \frac{I_m'(ks)}{K_m'(ks)}\, a_3. \end{aligned}\right\} \quad (48)$$

Then, the total stored energy P_m per unit length along the z direction is

$$P_m = \int_0^s 2\pi r P_i\, dr + \int_s^\infty 2\pi r P_o\, dr$$

$$= \frac{s^2}{4}\{a_1\bar{a}_1 Y_1 + a_3\bar{a}_3 Y_3 + (\bar{a}_1 a_3 - a_1\bar{a}_3)Y_2\}, \quad (49)$$

where

$$Y_1 \equiv \frac{1}{ksK_m(ks)}\left\{\left(\frac{N_\parallel^2 + 1}{N_\parallel^2 - 1}\right)\frac{I_m(ks)}{ks} + \frac{1}{N_\parallel^2 - 1}\ (2I_m'(ks)\right.$$

$$- \frac{1}{ksK_m(ks)})\},$$

$$Y_3 \equiv \frac{1}{ksK'_m(ks)} \{(\frac{N_\parallel^2 + 1}{N_\parallel^2 - 1})(\frac{-I'_m(ks)}{ks})$$

$$+ \frac{1}{N_\parallel^2 - 1} (1 + (\frac{m}{ks})^2)(2I_m(ks) + \frac{1}{ksK'_m(ks)})\},$$

$$Y_2 \equiv -i \frac{2N_\parallel}{N_\parallel^2 - 1} \frac{m}{(ks)^2} \{\frac{I_m(ks)}{K'_m(ks)} \frac{1}{ks}\}.$$

To obtain equation (40), the following mathematical identities are used;

$$\int r I_m^2(kr)\,dr = \frac{r^2}{2} \{I_m^2(kr)(1 + (\frac{m}{kr})^2) - (I'_m(kr))^2\},$$

$$\int \{r(I'_m(kr))^2 + rI_m^2(kr) + \frac{m^2}{k^2} \frac{1}{r} I_m^2(kr)\}dr = \frac{r}{k} I_m(kr) I'_m(kr). \tag{50}$$

In equation (50), $I_m(kr)$ and $I'_m(kr)$ can be replaced by $K_m(kr)$ and $K'_m(kr)$ respectively.

11. PLASMA LOADING

Energy transmission from the sheet current into a plasma can be calculated using Poynting's vector at the surface just inside the sheet. For an infinite length coil, the energy transmission W_o over the lengh ℓ and per unit time is

$$W_o = \text{Re}\,\frac{\omega}{4\pi}\int_0^{2\pi/\omega} dt \int_0^{2\pi} d\theta \int_0^{\ell} dz\,[\,\frac{c}{4\pi}(-\mathbf{n})\ [\mathbb{E} \times \bar{\mathbb{B}}], \qquad (51)$$

where $\mathbf{n}$ is the unit vector normal to the sheet surface, $\bar{\mathbb{B}}$ is the complex conjugate of the $\mathbb{B}$ and Re means the real part.

For a coil of finite length ($0 \leq z \leq \ell$), the end effect may be calculated by making use of the Fourier integral of the field components. But, here we shall use another method which leads to the same result.

Since there may be a natural wave of arbitrary amplitude, the wave in a plasma can be considered as a superposition of an excited wave and the natural wave. We are concerned with a wave which propagates in the Z direction so it will be quite natural to assume that the wave fields is zero at Z = 0. To the contrary, if we are concerned with a wave propagating in the -Z direction, the wave field should be zero at Z = ℓ. Thus we can determine the amplitude of the natural wave in this direction. The field components $\mathbb{E}$ and $\mathbb{B}$ just inside the sheet surface may then be expressed as

$$\mathbb{E} = \mathbb{E}_e e^{i(k_{\parallel} z + m\theta - \omega t)}$$

$$+ \mathbb{E}_1\{e^{i(k_{\parallel} z + m\theta - \omega t)} - e^{i(k_p z + m\theta - \omega t)}\},$$

$$\mathbb{B} = \mathbb{B}_e e^{i(k_\parallel z + m\theta - \omega t)}$$

$$+ \mathbb{B}_1\{e^{i(k_\parallel z + m\theta - \omega t)} - e^{i(k_p z + m\theta - \omega t)}\}, \quad (52)$$

where, $\mathbb{E}_e$ and $\mathbb{B}_e$ are the exciting fields, $\mathbb{E}_1$ and $\mathbb{B}_1$ are the fields due to the waves and k_p is the Z component of the wave number of the natural wave.

The power transmission W, from the sheet current of length ℓ to the plasma is then given by the following relation;

$$W = R_e \frac{\omega c}{(4\pi)^2} \int_0^{2\pi/\omega} dt \int_0^{2\pi} s d\theta \int_0^{\ell} dz$$

$$[-\{E_{e\theta}\bar{B}_{1z} - E_{ez}\bar{B}_{1\theta}\}(1 - e^{-i(k_p - k_\parallel)z})$$

$$- \{E_{1\theta}\bar{B}_{ez} - E_{1z}\bar{B}_{e\theta}\}(1 - e^{i(k_p - k_\parallel)z})$$

$$- \{E_{e\theta}\bar{B}_{ez} - E_{ez}\bar{B}_{e\theta}\}$$

$$- \{E_{1\theta}B_1z - E_1zB_{1\theta}\}(1 - e^{i(k_p - k_\parallel)z})(1 - e^{-i(k_p}$$

$$- k_\parallel)z)]. \quad (53)$$

In this equation, the third and the fourth terms in the integrand, are related to the self-energy of the exciting field and wave itself respectively. They are pure imaginary and do not make any contribution to W. Carrying out the integration of equation (53), we obtain

$$W = \frac{c}{4k} \frac{1}{\sqrt{N_{\parallel}^2 - 1}} (-a_1 a_2 + a_3 a_4) \{ \frac{1 - \cos (k_p - k_{\parallel}) \ell}{k_p - k_{\parallel}} \}. \tag{54}$$

Here, $a_1 a_2$ and $a_3 a_4$ are real quantities.

Coupling between the external field and the plasma wave may be expressed in terms of a "figure of merit", Q, of the exciting system, which is defined by the ratio of fractional energy loss per radian to stored energy in the exciting system. For the case where the stored energy is not affected very much by the pressence of plasma, the Q may be approximated as

$$Q \simeq \frac{\omega (P_m \ell)}{W}$$

$$= \frac{(ks)^2}{(\frac{\ell}{2}) S} \{ \frac{a_1 \bar{a}_1 Y_1 + a_3 \bar{a}_3 Y_3 + (\bar{a}_1 a_3 - a_1 \bar{a}_3) Y_2}{(k_{\parallel} - k_p)(a_1 a_2 - a_3 a_4)} \}, \tag{55}$$

where

$$S \equiv \frac{2\{1 - \cos (k_p - k_{\parallel}) \ell\}}{\{\ell (k_p - k_{\parallel})\}^2}.$$

Here, S is a shape factor of a coupling resonance and it is unity for $k_p - k_{\parallel} = 0$, where the coupling reaches its maximum value.

From equation (33), a_2 and a_4 are

$$a_2 = -\frac{1}{\Delta} Z_m(\nu_1 p) Z_m(\nu_2 p) \frac{1}{kp} \frac{1}{\sqrt{N_\parallel^2 - 1}} [a_3 A_{23} + i a_1 A_{21}]$$

$$- \frac{a_1 I_m(kp)}{k_m(kp)},$$

$$a_4 = \frac{1}{\Delta} Z_m(\nu_1 p) Z_m(\nu_2 p) \frac{1}{kp} \frac{1}{\sqrt{N_\parallel^2 - 1}} [i a_3 A_{43} + a_1 A_{41}]$$

$$- \frac{a_3 I'_m(kP)}{k'_m(kP)}, \tag{56}$$

where

$$A_{21} \equiv (E_1 - E_2) + \alpha(\gamma_1 - \gamma_2),$$

$$A_{23} \equiv F_2 - F_1,$$

$$A_{41} \equiv E_1\gamma_2 - E_2\gamma_1,$$

$$A_{43} \equiv \frac{1}{\alpha}(E_1F_2 - E_2F_1) + (E_1 - E_2).$$

Because the coupling or loading is efficient for the condition $k_p - k_\parallel \simeq 0$, a_2 and a_4 may be expanded around $k_p - k_\parallel = 0$, where $\Delta = 0$, (the last terms in equation (56) are independent of the Δ). Thus the Q is

$$Q \simeq \frac{\sqrt{N_{\parallel}^2 - 1}\,(ks)^2 kp K_m^2(kp)}{S(\ell/2)}$$

$$\times\ \Delta'a\left\{\frac{a_1\bar{a}_1Y_1 + a_3\bar{a}_3Y_3 + (\bar{a}_1a_3 - a_1\bar{a}_3)Y_2}{-a_1a_1A_{21} - a_3a_3A_{43} + ia_1a_3(A_{23} + A_{41})}\right\}, \qquad (57)$$

where

$$\Delta_a' \equiv \frac{d}{dk}\{(E_1 + \alpha\gamma_1)(F_2 + \alpha) - (E_2 + \alpha\gamma_2)(F_1 + \alpha)\}.$$

When an induction coil of the Stix type is used to generate an ion cyclotron wave of m = 0, it is clear that the sheet current j_θ is the main source of the external field. In this case a_1 may be neglected compared with a_3 which represents the j_θ current. Then, keeping only the main terms in Y_3, Q may be approximated to

$$Q \sim \frac{2}{S\ell A_{43}}\left\{(kp)\frac{I_m'(ks)}{K_m'(ks)}\sqrt{N_{\parallel}^2 - 1}\,K_m^2(kp)\ \Delta_a'\right\}. \qquad (58)$$

12. EVALUATION OF PLASMA LOADING FOR m = 0

The evaluation of Q given by equation (57) will still be troublesome, because the calculations of Δ_a' and A are necessary. Here we shall limit ourselves to the case of m = 0.

Because there is a relation between E_j and F_j for m = 0, ($E_j\varepsilon_\ell = \gamma_jF_j$), the Δ_a' which is defined in equation

(57) may be expressed in the form

$$\Delta_a' = \frac{d}{dk}\,[\alpha F_1 F_2 \gamma_1 \sqrt{N_\parallel^2 - 1}\,\frac{1}{\varepsilon_\ell}\,Ⓗ\,], \tag{59}$$

where

$$Ⓗ \equiv (1 - \frac{\varepsilon_1}{\varepsilon_2})\,\frac{K_0(kp)}{K_0'(kp)} + \frac{L_1^2}{|L_1|}\,\frac{Z_0(L_1kp)}{Z_0'(L_1kp)} - \frac{\varepsilon_1}{\varepsilon_2}\,\frac{L_2^2}{|L_2|}\,\frac{Z_0(L_2kp)}{Z_0'(L_2kp)}$$

From the boundary relation (42), it is clear that the Ⓗ is zero where $\Delta = 0$. In Fig.1, we can see that the second term in Ⓗ depends critically on K, while the remainder of the terms are rather insensitive to K. Therefore, we may write the Δ_a' as

$$\Delta_a' \sim \frac{1}{\varepsilon_\ell}\,\alpha F_1 F_2 \gamma_1 \sqrt{N_\parallel^2 - 1}\,\frac{d}{dk}\,\{L_1\,\frac{Z_0(L_1kp)}{Z_0'(L_1kp)}$$

$$\sim \frac{1}{\varepsilon_\ell}\,\alpha F_1 F_2 \gamma_1 \sqrt{N_\parallel^2 - 1}\,[p\,\frac{L_1^2(\mu L_1^2 + \Omega_0^2)}{(\mu L_1^4 - \Omega_0^2)}\,\{L_1^2 + (\frac{\varepsilon_1}{\varepsilon_2}\,\frac{2}{kp})^2\}]. \tag{60}$$

In the last relation, we used the following approximations;

$$\frac{k}{2}\,\frac{dL_1^2}{dk} \sim \frac{\Omega_0^2 L_1^4}{\mu L_1^4 - \Omega_0^2},$$

$$\left(\frac{\varepsilon_2}{\varepsilon_1}\right) L_1 \frac{Z_0(L_1kp)}{Z_0'(L_1kp)} \sim \frac{L_2^2}{|L_2|} \frac{Z_0(L_2kp)}{Z_0'(L_2kp)} \sim -\left(\frac{2}{kp}\right),$$

(at $\Delta = 0$, for $|L_2kp| << 1$).

Because A_{43} is a smooth function of $k_{\parallel}$, we may fix the value at $k_{\parallel} = k_p$ and use again the relations $\varepsilon_\ell E_j = \gamma_j F_j$ for $m = 0$, and $\textcircled{H} = 0$ at $\Delta = 0$. Then we have

$$A_{43} = \gamma_1 F_1 F_2 \frac{1}{\varepsilon_\ell^2} \sqrt{N_{\parallel}^2 - 1}\ \left[\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)\left\{\frac{L_1^2}{|L_1|} \frac{Z_0(L_1kp)}{Z_0'(L_1kp)}\right.\right.$$

$$\left.\left. + \frac{K_0(kp)}{K_0'(kp)}(1 + \varepsilon_\ell)\right\}\right] \sim \gamma_1 F_1 F_2 \frac{1}{\varepsilon_\ell} \sqrt{N_{\parallel}^2 - 1}$$

$$\times \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right) \frac{K_0(kp)}{K_0'(kp)}, \quad (\text{for } \varepsilon_\ell >> 1). \tag{61}$$

Using these equations (60) and (61), we have the following approximate expression of Q for moderate values of L_1kp;

$$Q \simeq \frac{2\ I_1(ks)}{S\ k\ell\ K_1(ks)} (kpK_1(kp))^2 L_1^4 M_e, \tag{62}$$

where

$$M_e = \frac{\Omega_0^2(\mu L_1^2 + \Omega_0^2)}{(\mu L_1^4 - \Omega_0^2)^2} \left\{1 + \left(\frac{2\mu L_1^4}{\Omega_0^2 \nu_1 p}\right)^2\right\}. \tag{63-a}$$

When μ in the M_e goes to zero, we see that M_e goes to unity.*
The Q given by equation (62) is essentialy the same to that of Stix's calculation, provided that $\mu = 0$ and $Z_0(L_1 k_p) \sim 0$, where the latter is the boundary condition for the natural wave in that case.

The munimum value of equation (62) against L_1 is obtained at

$$\mu L_1^4 \sim 3\Omega_0^2 .$$

Corresponding values of $L_1^4 M_e$ and Ω^2 are

$$(L_1^4 M_e)_{min} \sim \frac{9c_0^2}{(k_p)^4} ,$$

$$\Omega^2 \sim \frac{1}{1 + \Omega_0^2} ,$$

where c_0 is the value which satisfies $J_1(c_0) = 0$. Therefore, plasma loading will be considerably modified by the finite mass of the electron.

In the above calculation of Q, we have taken the derivative of only the second term in H of equation (59). For $L_1 \sim L_2$, hcwever, the last term can not be ignored. For this case, keeping the last term we obtain the expression of $L_1{}^4 M_e$ in equation (62) to be

* Note: mathematically, for μ=0, L_1 should be replaced by L_2. Then L_2 plays the role of L_1 in the analysis.

$$L_1^4 M_e \sim \frac{2L_1^4}{(\frac{\varepsilon_1}{\varepsilon_2} - 1)^2}, \text{ (for } L_1 \sim L_2). \tag{63-a}$$

Here we have used the following approximations;

$$\frac{dL_2^2}{dk} \sim - \frac{\varepsilon_2}{\varepsilon_1}\frac{dL_1^2}{dk},$$

$$\frac{L_1^2}{L_2^2} \sim \frac{\varepsilon_1}{\varepsilon_2}.$$

13. WAVE MODE

There are two equations (10) and (11), in which the components of wave fields D_z and B_z are related to each other. When the D_z and B_z are nearly independent, we may have two isolated dispersion relations;

$$\varepsilon_\ell(\varepsilon_t - N_\parallel^2) - \varepsilon_t N_\perp^2 \sim 0, \text{ (for } B_z = 0), \tag{64}$$

$$\varepsilon_t(\varepsilon_t - N^2) + \varepsilon_h^2 \sim 0, \text{ (for } D_z = 0). \tag{65}$$

These two dispersion relations are schematically indicated in Fig.4. We see that the dispersion relation, which corresponds to equation (14), is almost equivalent to those of (64) and (65) except in the crossing region of these modes. Since equation (14) has two roots of L, say L_1 and L_2, we shall here call these two modes the L_1 - mode and the L_2 - mode. Because we have assumed that $|L_1| \geq |L_2|$, the L_1 - mode may be considered to be a quasi - TM mode, and the L_2 - mode a quasi - TE mode.

The distribution of components of magnetic field and plasma current for these two modes are schematically shown in Fig. 5. The analytical forms of these components obtained from equation (21) are summarized in Table 1.

For the L_1 - mode, the plasma current J_z which flows along the static magnetic field is important. The current J_z and J_r form a divergence-free flow within a plasma, when the approximate boundary condition given by equation (44) is satisfied.

The other current J_θ is also divergence-free and is related to J_r as $|J_\theta| \sim |\Omega\, J_r|$, for the case of m = 0. The magnetic field component B_θ for m = 0 is directly related to the current J_z by equation (9).

When ε_ℓ is large enough, the dispersion relation (64) may be approximated as

$$\varepsilon_t - N_{\parallel}^2 \sim 0, \text{ (for } L_1\text{-mode)}. \tag{66}$$

From equation (65) the dispersion relation of the L_1 mode is

$$\Omega^2 = \frac{1}{\mu L_1^2 + (1 + \Omega_0^2)} . \tag{67}$$

We see that equation (67) is an asymptotic form of equation (38) for $L_1^2 >> 1$. The difference between the two equations appears only for the condition that $\mu L_1^4 \leq \Omega_0^2$

For the L_2-mode, the approximate dispersion relation is

$$\Omega^2 \simeq \frac{L_2^2 + 1}{\Omega_0^2} . \tag{68}$$

Because Ω_0^2 is a quantity which roughly indicates the frequency shift of the ion cyclotron wave from the ion cyclotron frequency, we may assume that Ω_0^2 is considerably smaller than unity based on usual experimental condition. Therefore, L_2^2 is usually a nagative quantity.

The group velocity of a wave V_g along the z direction can be obtained by a calculation of $\partial\omega/\partial k$. We obtain

$$V_g \sim V_p(1 - \Omega^2), \text{ (for } L_1\text{-mode)}, \tag{69}$$

$$V_g \sim V_p \frac{1}{\Omega^2 \Omega_0^2}, \text{ (for } L_2\text{-mode)}. \tag{70}$$

where, V_p is the phase velocity of the wave along the Z direction.

14. FIELDS OF THE MODES

From equation (29), we see that the coefficients a_5 and b_5 are associated with the L_1-mode and the L_2-mode respectively. The ratio b_5/a_5 may be obtained from equation (33). For the natural wave, the ratio is

$$\frac{b_5}{a_5} = - \frac{Z_m(L_1 kp)}{Z_m(L_2 kp)} \frac{F_1 + \alpha}{F_2 + \alpha}$$

$$\sim - \frac{Z_m(L_1 kp)}{Z_m(L_2 kp)} \frac{F_1}{F_2} . \tag{71}$$

For the case m = 0 and assuming that kp << 1 and $\varepsilon_1/\varepsilon_2$ >> 1, equation (71) may be approximated to be

$$\frac{b_5}{a_5} \sim - \frac{L_2^2 |L_1| Z_0'(L_1 kp)}{|L_2| \; L_1^2 Z_0'(L_2 kp)} . \tag{72}$$

Thus, we can obtain the wave field components of the L_2-mode in terms of the field components of the L_1-mode. The results are summarized in Table 2.

The field components in the vacuum outside of the plasma are also obtained directly from the continuity relations across the plasma-vacuum surface, using the expressions for the field components of the L_1 and L_2-mode. The results are also given in Table 2.

15. WAVE ENERGY

The wave energy in a plasma can be calculated by the following relation as described in the Stix's book (p.48, eq.7).

$$W = \frac{1}{16}\left[\overline{\mathbf{B}}\cdot\mathbf{B} + \overline{\mathbf{E}}\left\{\frac{\partial}{\partial\omega}(\mathbb{K}(\omega_r) + \mathbb{K}^{+}(\omega_r))\right\}\mathbf{E}\right] \quad \exp 2\int_{-\infty}^{t}\omega_i dt', \tag{73}$$

where ω_r and ω_i are the real and imaginary parts of the wave frequency ω respectively, $\mathbb{K}$ is the dielectric tensor and $\mathbb{K}^{+}$ is the Hermitian conjugate of $\mathbb{K}$. For a loss-free plasma, $\mathbb{K}$ is a Hermitian, that is, $\mathbb{K}(\omega_r) = \mathbb{K}^{+}(\omega_r)$. Using the dielectric tenser components given by expression (35), we have the following derivatives of the tensor components with respect to ω:

$$\frac{\partial}{\partial\omega}(\omega\varepsilon_t) \simeq 1 + \varepsilon_t \frac{1+\Omega^2}{1-\Omega^2},$$

$$\frac{\partial}{\partial\omega}(\omega\varepsilon_h) \simeq 2\varepsilon_h \frac{1}{1-\Omega^2}, \tag{74}$$

$$\frac{\partial}{\partial\omega}(\omega\varepsilon_\ell) \simeq 1 - \varepsilon_\ell,$$

$$W = \frac{1}{16} [\bar{\mathbb{B}} \cdot \mathbb{B} + \bar{\mathbb{E}} \cdot \mathbb{E} + E_k] , \tag{75}$$

where

$$E_k \equiv \varepsilon_t \frac{1 + \Omega^2}{1 - \Omega^2} (\bar{E}_r E_r + \bar{E}_\theta E_\theta) - \varepsilon_\ell \bar{E}_z E_z \; .$$

In this equation, the first term is the magnetic energy, the second is the electric energy, and the third is the kinetic energy associated with the coherent wave motion of the charged particles.

The velocity of charged particles in a wave field is given by

$$v_x^i = i \left| \frac{Ze}{m_i} \right| \frac{1}{\omega^2 - \Omega_i^2} (\omega E_x + i \Omega_i E_y) ,$$

$$v_y^i = \left| \frac{Ze}{m_i} \right| \frac{1}{\omega^2 - \Omega_i^2} (\Omega_i E_y + i \omega E_x) ,$$

$$v_z^i = \left| \frac{Ze}{m_i} \right| \frac{1}{\omega} E_z ,$$

$$v_z^e = \left| \frac{e}{m_e} \right| \frac{1}{\omega} E_z ,$$

where indces i and e refer to the ion and electron. The kinetic energy of the charged particles is

$$\sum_{i\,e} \frac{1}{2} n_i\, m_i v_i^2 \sim \frac{1}{8\pi} [\frac{1 + \Omega^2}{1 - \Omega^2} \varepsilon_t |E_\perp^2| - \varepsilon_\ell |E_z^2|]. \tag{76}$$

Thus we see that the kinetic energy E_k consists of ion Larmor motion and electron motion which is parallel to the z direction. Since,

$$\mathbb{K}\mathbf{D} = \mathbb{E} + i\frac{4\pi}{\omega} \dot{\mathbf{J}} \ ,$$

the E_k may be written as

$$E_k = i \frac{8\pi}{\omega} \overline{\mathbb{E}}\cdot\mathbb{j} \frac{1}{1 - \Omega^2} . \tag{77}$$

Here we have used the relations that

$$\mathbb{j}_k = n_k Z_k e\, \varepsilon_k \mathbf{v}_k,$$

and

$$\frac{\partial}{\partial\omega} \mathbf{v}_i \sim \mathbf{v}_i \frac{2}{(1 - \Omega^2)\Omega_i} ,$$

where the suffix k refers to the species of the charged particles, and ε_k to the signs of the charges.

In equation (75), the E_k is the dominant term and the electric energy $\overline{E}$ E is at most negligible provided that ε_t has a large value. The ratio of magnetic energy to the kinetic energy E_k is roughly equal to $\Omega^2\Omega_0^2/(1+\Omega^2)$. The total energy W, in terms of $E_{\theta 1}$, is

$$W \sim \left(\frac{E_{\theta 1}^2 N_\parallel^2 L_1^4}{16\pi J_1(\nu_1 r)}\right)\left[\frac{1+\Omega^2}{\Omega_0^2\Omega^4} J_1^2(\nu_1 r) + \frac{\mu L_1^2}{\Omega^2\Omega_0^2} J_0^2(\nu_1 r)\right.$$

$$\left. + \frac{1}{\Omega^2} J_1^2(\nu_1 r)\right], \qquad (87)$$

for $L_1^2 \gg 1$. In this equation, the first, the second, and the third terms are associated with the field components E_r, E_z, and B_θ, respectively.

16. ENERGY FLOW

The density of energy flow F(r) along the z direction can be obtained by multipling W with the group velocity V_g, where F(r) is a function of radial position r. This energy flow can also be obtained more directly from a Poynting's vector calculation;

$$F(r) = R_e \frac{c}{8\pi} \mathbb{E} \times \mathbb{B}$$

$$\sim \left(\frac{E_{\theta 1}^2 c N_\parallel}{8\pi J_1(\nu_1 r)}\right)\left[\frac{(1-\Omega^2)(1+L_1^2)}{\Omega_0^2\Omega^4} J_1^2(\nu_1 r)\right.$$

$$\left. + \frac{L_1^2}{|L_2|}\frac{J_0(\nu_1 r)}{Z_0(\nu_2 r)} Z_1^2(\nu_2 r)\right]. \qquad (79)$$

In this expression, the first and the second terms are associated with the energy of the L_1-mode and the L_2-mode respectively. For a large value of L_1, the first term is dominant and the second is negligible.

The energy flow calculated using the two methods, one from W. V_g and the other from Poynting's vector, coincide reasonably well. The difference is due to approximations for the field components. In the vacuum outside of the plasma, there is also an energy flow, mainly associated with the wave field of the L_2-mode, though it is not important usually.

The total energy flow F, along the z direction over the entire cross-section of the plasma column is obtained by integrating F(r) with respect to the radial position r. Taking only the L_1-mode field and neglecting the energy which flows through the outside vacuum, the energy flow F is approximatly given by

$$F \simeq R_e \int_0^p \frac{c}{8\pi} (\mathbb{E} \times \mathbb{B}) 2\pi r dr$$

$$\sim \left|\frac{B_{z_1}}{J_0(\nu_1 r)}\right| \frac{(1 - \Omega^2)P^2 L_1^2 V_p}{8\Omega^2\Omega_0^2} J_0^2(c_0), \tag{80}$$

where the boundary relation $J_1(L_1 kp) \sim 0$ has been used. From this relation, the field components of the L_1-mode may be expressed in terms of the energy flow F. The results are summarized in Table 4. We see, from the results, that the electric field E_r and the magnetic field B_θ are important in the L_1-mode. The plasma currents j_z, j_r, and j_θ can be obtained from the electric fields and the dielectric tensor. The results are also given in the Table 4. Among the currents, j_z has the largest value as far as $L_1 >> 1$.

17. ENERGY LOSSES

The dispersion relations and energy relations described above are based on a dielectric tensor for a cold plasma, so there is no loss except energy loss which is associated with wave propagation. In a warm plasma, however, there are energy losses due to ion cyclotron damping and collisions. Here we shall consider these damping mechanisms of the wave.

Cyclotron Damping

In a case of finite ion temperature, each ion in a plasma feels its own perturbing frequency because of a Doppler shift. The ion, whose perturbing frequency is nearly equal to the ion gyration frequency, will resonantly absorb wave energy, and cause a damping of the wave. This damping mechanism, known as cyclotron damping, will, therefore, be large if

$$v^{i}_{th} \gtrsim (1 - \Omega)V_p. \tag{81}$$

where v^i_{th} is the ion thermal velocity. From this relation, we can see that there is a limiting value of $k_\parallel$ for wave propagation since relation (81) can be easily satisfied by a large $k_\parallel$ and there appears a strong damping of the wave. Using equation (67) and the definition of Ω_0 which is given in equation (37), the limiting value of $k_\parallel$ is obtained as

$$K_\parallel^3 \sim \frac{\Pi_i^2 \Omega}{2c^2 v^i_{th}}, \tag{82}$$

where we have assumed that $\mu_1 L_1^2 \ll \Omega_0^2$. This relation may be compared with Stix's expression for the maximum k (ref.1, p.196, eq.27).

To obtain a more quantitative expression, we may use the spitzer's attenuation distance of the wave d, (reference 5, p.88, eq.3-59), together with the relation $\omega_i/k_i = V_g$, where k_i is the imaginary part of wave number k, and has the value of 1/d. The result is

$$\frac{\omega_i}{\omega} \sim 2\sqrt{\pi}\Omega_0^2 \Gamma e^{-\Gamma^2}, \tag{83}$$

where

$$\Gamma \equiv \frac{\Omega_i}{k v^i_{th}} (1 - \Omega) \sim \frac{1}{2} \frac{V_g}{v^i_{th}}. \tag{84}$$

This equation is the same as that described by Stix (ref.1, p.195, eq.23), if $(1 - \Omega)$ is replaced by $\frac{1}{2}\,\Omega_0^2$, and that $\omega \sim \Omega_i$. In the usual case, the value of Γ is not very small, so this damping may be neglected. This mechanism is, however, essential in a "beach field".

Ion-Electron Collision

The time constant for ion-electron collisions τ_{ie}, which is given by Spitzer (reference 5, p.135,eq. 5-29) is

$$\tau_{ie} = 11.7 \, \frac{A^2 T_e^{3/2}(^\circ k)}{(A + A_e) A_e^{1/2} n_e Z^2 \ln\Lambda} \, , \tag{85}$$

where A is the atomic mass of the ion, and A_e is that of the electron. This collision will causes an additional broadening of the coupling resonance whose half-width in frequency is f_{ie}, where $f_{ie} = 1/2\pi\tau_{ie}$.
The numerical evaluation of f_{ie} is given by

$$f_{ie} \simeq 2.54 \times 10^{-10} \, \frac{n_e \ln\Lambda}{A T_e^{3/2}(ev)} \, , \quad (H_z) \, . \tag{86}$$

This damping mechanism will be important when the electron temperature of a plasma is low.

Ion-Ion Collision

Since we are concerned with the averaged ion motion, collisions between the same ion species is not important. Here we consider the collisions with different ion species. The collision time of i-th ion with j-th ion given by Spitzer is

$$\tau_{ij} = 11.4 \frac{T^{3/2}(^\circ k)}{n_j Z_i^2 Z_j^2 \ell n \Lambda} \left(\frac{A_i A_j}{A_i + A_j} \right)^{\frac{1}{2}}.$$

The corresponding broadening, f_{ij}, is therefore

$$f_{ij} \sim 1.14 \times 10^{-8} \frac{Z_j^2 \ell n \Lambda n_j}{T_i^{3/2}(ev)} \left(\frac{A_i + A_j}{A_i A_j} \right)^{\frac{1}{2}}, \quad (H_z). \qquad (87)$$

The collision time for ion-neutral collisions τ_n is given by

$$\tau_n = \frac{\lambda}{v_{th}^i} = \frac{1}{n_n \sigma_n v_{th}^i},$$

where λ is the mean free path, n_n is the neutral particle density, and σ_n is the collision cross-section for this encounter. Because σ_n is of the order of 10^{-15} cm^2, this collision frequency is ususlly low and may be neglected.

Resistivity

Because, the plasma current j_z in a wave is high, the energy loss due to resistivity may be considerable. To estimate this effect, we consider that the mass of electron is a complex quantity as can be deduced from a generalized Ohm's law neglecting the pressure terms. So, here, we shall introduce a complex quantity μ^* which is defined by

$$\mu^* = \mu(1 + i\frac{\alpha}{\Omega}), \tag{88}$$

where

$$\alpha \equiv \frac{\nu_{ei}}{\Omega_i} = \frac{\Pi_e^2 \eta}{4\pi\omega c^2} . \tag{89}$$

In the equation, ν_{ei} is the electron-ion collsion frequency and η is the corresponding resistivity. Using this μ^* for μ in equation (38), we have the following approximate solution of Ω in complex form for $L_1 >> 1$;

$$\Omega \sim \Omega_p[\{1 - \frac{1}{8}(\alpha\Omega_p\mu L_1^2)^2\} - i\frac{\Omega_p^2}{2}\alpha\mu L_1^2]$$

$$\sim \{1 - \frac{1}{2}(\Omega_0^2 + \mu L_1^2)\} - i\frac{1}{2}\alpha\mu L_1^2(1 - \Omega_0^2), \tag{90}$$

where Ω_p is the value of Ω for real μ which has been given by equation (38). In the last relation, we assumed that

$$\frac{1}{8}(\alpha\Omega_p \mu L_1^2) << 1.$$

The ratio of the imaginary part to the real part in Ω is then

$$\left|\frac{\Omega_i}{\Omega_r}\right| \simeq \frac{1}{2}\alpha\mu L_1^2$$

$$= \frac{\Pi_e^2 \eta}{4\pi c^2} \left(\frac{\mu L_1^2}{2}\right). \tag{91}$$

Thus the loss will be considerable when both Π_e^2 and L_1^2 are large. In this mechanism, wave energy will be transmitted directly to the electrons.

18. DISCUSSION AND CONCLUSION

The dispersion relation of an ion cyclotron wave is analytically calculated retaining the electron mass. In this case we have two wave modes in a plasma, the L_1-mode (quasi-TM) and the L_2-mode (quasi-TE). Under most conditions of ion cyclotron wave experiments, the L_1-mode is essential. The plasma current j_z is the important component in the L_1-mode, so the feature of electron motion associated with the j_z plays a role in the dispersion relation and energy relations of the ion cyclotron wave. The modifications in these relations will be considerable if the ratio ν/k has a large value. The negligible electric field E_z is due to a high conductivity along a static magnetic field and does not mean a low plasma current j_z. In fact, the j_z has a finite value that is independent of μ, when μ goes to zero.

The plasma current j_z for $\mu = 0$, which has been calculated by Stix, has a distribution profile of $J_o(\nu r)$ and is nearly zero at the plasma surface. In this case, the surface current flows to form a closed circuit for the body plasma current. In the analysis described here, however, the boundary relation is $J_1(\nu_1 p) \sim 0$, rather than $J_o(\nu_1 p) \sim 0$, at the plasma surface. Then the plasma current has a divergence-free flow in the plasma without a surface current.

It is shown that, the field components of the L_1-mode are nearly canceled out by the components of the L_2-mode at the plasma-vacuum surface. As a result, coupling between the exciting field and the wave field is somewhat reduced. The wave energy is calculated based on the field components of the L_1-mode. We see that most of the wave energy is kinetic energy of coherent ion Larmor motion. The wave energy flows with a group velocity, $V_g = V_p(1 - \Omega^2)$. The field components, therefore, can be obtained as a function of energy flow and are summarized in Table 3. It is shown that the important field components are B_θ and E_r, for a large value of ν/k.

The damping of waves is presented for cyclotron damping, collision loss, and resistive loss. The third is due to electron-ion collisions, so there is a possibility of direct electron heating by the wave field of E_z. This heating will be considerable if L_1^2 is large and the plasma density is high.

Thus we have a nearly complete understanding of ion cyclotron wave together with a basic physical picture of these waves.

In the analysis, however, we have used a dielcctric tensor for a cold plasma. Energy losses are included as corrections to it. Moreover, the density profile with a sharply bounded uniform distribution is assumed. Therefore, the analysis described above should, more or less, be considered to be a semi-quantitative one for a practical plasma in which plasma has a diffused profile with a finite temperature.

It should be noted here that our considerations are concerned mainly with ion cyclotron waves of light mass ions. In case of heavy ion mass, effects of electron mass will not be important. In that case, L_2-mode plays the role of L_1-mode, because L_1 tends to infinity and L_2 should satisfy the boundary condition as the dominant mode. Some cautions seem to be necessary on this problem, though a little is payed in the analyses.

ACKNOWLEDGEMENTS

The authers wish to thank Drs. G. Horikoshi, T. Kuroda, M. Masuzaki, and Y. Mizuno for helpful discussions.

REFERENCES

(1) Stix, T. H., The Theory of Plasma Waves (McGraw-Hill Book Co., Inc., New York, 1972).

(2) Stix, T. H. and Palladino, R. W., Phys. Fluids 3 (1960) 641.

Hooke, W. M., Tenney, F. H., Brennan, M. H., Hill Jr., H. M. and Stix, T. H., Phys. Fluids 4 (1961) 1131.

Hooke, W. M., Avivi, P., Brennan, M., Rothman, M. A. and Stix, T. H., Nuclear Fusion Pt. 3 (1962) 1083.

(3) Hosea, J. C. and Sinclair, R. M., Phys. Rev. Letters 23 (1969) 3.

Hosea, J. C. and Sinclair, R. M., Phys. Fluids 13 (1970) 701.

(4) Allis, W.P., Buchsbaum, S.J., and Bers, A., Waves in Plasmas (John Wiley Sons, Inc., New York, 1962).

(5) Spitzer, L., Jr, Physics of Fully Ionized Gasas (Interscience Publishers, Inc., New York, 1962), 2nd ed.

L_1 - mode	L_2 - mode
$\dfrac{E_{z1}}{E_{\theta 1}} \sim \dfrac{\mu L_1 (1+L_1^2)}{\Omega\Omega_0^2} \dfrac{Z_0(\nu_1 r)}{Z_0'(\nu_1 r)}$	$\dfrac{E_{z2}}{E_{\theta 2}} \sim - \dfrac{L_2^2\ \Omega\ Z_0(\nu_2 r)}{\lvert L_2 \rvert\ L_1^2\ Z_0'(\nu_2 r)}$
$\dfrac{E_{\theta 1}}{E_{\theta 1}} \equiv 1$	$\dfrac{E_{\theta 2}}{E_{\theta 2}} \equiv 1$
$\dfrac{E_{r1}}{E_{\theta 1}} \sim i \dfrac{(1-\Omega^2)(1+L_1^2)}{\Omega^3 \Omega_0^2}$	$\dfrac{E_{r2}}{E_{\theta 2}} \sim i \dfrac{\Omega}{\mu L_1^2} (\mu L_1^2 - \Omega_0^2)$
$\dfrac{B_{z1}}{E_{\theta 1}} = i N_{\parallel} L_1 \dfrac{Z_0(\nu_1 r)}{Z_0'(\nu_1 r)}$	$\dfrac{B_{z2}}{E_{\theta 2}} = i N_{\parallel} \dfrac{L_2^2}{\lvert L_2 \rvert} \dfrac{Z_0(\nu_2 r)}{Z_0'(\nu_2 r)}$
$\dfrac{B_{\theta 1}}{E_{\theta 1}} \sim i N_{\parallel} \dfrac{1}{\Omega}(1+L_1^2)$	$\dfrac{B_{\theta 2}}{E_{\theta 2}} \sim i N_{\parallel} \Omega\Omega_0^2 \dfrac{1}{\mu L_1^2}$
$\dfrac{B_{r1}}{E_{\theta 1}} = -N_{\parallel}$	$\dfrac{B_{r2}}{E_{\theta 2}} = -N_{\parallel}$

Table 1.

Wave fields of the L_1-mode and L_2-mode.

components	formulas (in units of $E_{\theta 1}$ at r=p)	components	ratios at r=p
E_{r2}	$i\frac{(1-\Omega^2)(1+L_1^2)}{\Omega_0^2\Omega^3}\left(\frac{L_1^2\lvert L_2\rvert}{\lvert L_1\rvert L_2^2}\right)\frac{\Omega_0^2\Omega^2(\mu^2L_1^2-\Omega_0^2)}{\mu L_1^2(\mu L_1^4+L_1^2\Omega_0^2+\Omega_0^2)}\frac{Z_0(\nu_1 p)}{Z_0(\nu_2 p)}Z_0'(\nu_2 r)$	$\frac{E_{r2}}{E_{r1}}$	$\sim \frac{\varepsilon_1}{\varepsilon_2}\left[\frac{\Omega_0^2\Omega^2(\mu^2L_1^2-\Omega_0^2)}{\mu L_1^2\{\mu L_1^4+L_1^2\Omega_0^2+\Omega_0^2\}}\right]$
$E_{\theta 2}$	$-\left(\frac{L_1^2\lvert L_2\rvert}{\lvert L_1\rvert L_2^2}\right)\frac{Z_0(\nu_1 p)}{Z_0(\nu_2 p)}\ Z_0'(\nu_2 r)$	$\frac{E_{\theta 2}}{E_{\theta 1}}$	$-\frac{\varepsilon_1}{\varepsilon_2}$
E_{z2}	$-\frac{\varepsilon_2}{\varepsilon_1}\frac{\mu L_1^2(1+L_1^2)}{\Omega\ \Omega_0^2}\frac{Z_0(\nu_1 p)}{Z_0(\nu_2 p)}\ Z_0(\nu_2 r)$	$\frac{E_{z2}}{E_{z1}}$	$-\frac{\varepsilon_2}{\varepsilon_1}$
B_{r2}	$N_{\parallel}\left(\frac{L_1^2\lvert L_2\rvert}{\lvert L_1\rvert L_2^2}\right)\frac{Z_0(\nu_1 p)}{Z_0(\nu_2 p)}\ Z_0'(\nu_2 r)$	$\frac{B_{r2}}{B_{r1}}$	$-\frac{\varepsilon_1}{\varepsilon_2}$
$B_{\theta 2}$	$-iN_{\parallel}\frac{\varepsilon_2}{\varepsilon_1}\left(\frac{L_1^2\lvert L_2\rvert}{\lvert L_1\rvert L_2^2}\right)\frac{Z_0(\nu_1 p)}{Z_0(\nu_2 p)}\ Z_0'(\nu_2 r)$	$\frac{B_{\theta 2}}{B_{\theta 1}}$	-1
B_{z2}	$-iN_{\parallel}L_1\ \frac{Z_0(\nu_1 p)}{Z_0(\nu_2 p)}\ Z_0(\nu_2 r)$	$\frac{B_{z2}}{B_{z1}}$	-1

Table 2. Field components of the L_2-mode in terms of the L_1-mode.

Field components	Numerical values	Units
$\left\lvert\frac{B_{z1}}{J_0(\nu_1 r)}\right\rvert$	$\sim 1.15\ d\sqrt{F}$	gauss
$\left\lvert\frac{B_{\theta 1}}{J_1(\nu_1 r)}\right\rvert$	$\sim 35.1\ \frac{d}{\Omega p}\ \sqrt{F}$	gauss
$\left\lvert\frac{B_{r1}}{J_1(\nu_1 r)}\right\rvert$	$\sim 3.77 \times 10^{-2}\ pd\ \sqrt{F}$	gauss
$\left\lvert\frac{E_{z1}}{J_0(\nu_1 r)}\right\rvert$	$\sim 9.57\ \frac{d}{p^2}\ (\frac{10^{12}}{n_e})\ \sqrt{F}$	volt/cm
$\left\lvert\frac{E_{\theta 1}}{J_1(\nu_1 r)}\right\rvert$	$\sim 7.55 \times 10^{-2}\Omega pd\sqrt{F}$	volt/cm
$\left\lvert\frac{E_{r1}}{J_1(\nu_1 r)}\right\rvert$	$\sim 70.2\ \frac{1}{pd}\ \sqrt{F}$	volt/cm
$\left\lvert\frac{j_{z1}}{J_0(\nu_1 r)}\right\rvert$	$\sim 1.07 \times 10^2\ \frac{d}{p^2}\ \sqrt{F}$	Amp/cm^2
$\left\lvert\frac{j_{r1}}{J_1(\nu_1 r)}\right\rvert$	$\sim 3.51\ \frac{d}{p}\ \sqrt{F}$	Amp/cm^2
$\left\lvert\frac{j_{\theta 1}}{J_1(\nu_1 r)}\right\rvert$	$\sim 3.51\Omega\ \frac{d}{p}\ \sqrt{F}$	Amp/cm^2

F; energy flow (kW)

p; plasma radius (cm)

n_e; electron density (cm^{-3})

$$d^2 \equiv \frac{\Omega^2 \Omega_0^4}{1 - \Omega^2}\ \frac{10^6}{f\lambda^3}$$

f; wave frequency (MH_z)

$\lambda \equiv \frac{2\pi}{k}$; wave length (cm)

Table 3. Values of field components in terms of energy flow F.

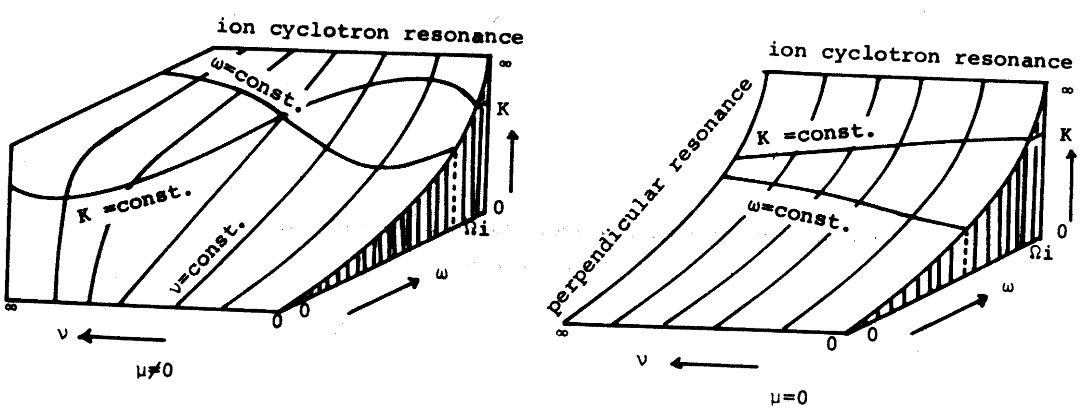

Fig. 1.

Basic chracteristics of the dispersion relation of ion cyclotron wave for $\mu=0$ and $\mu\neq0$.

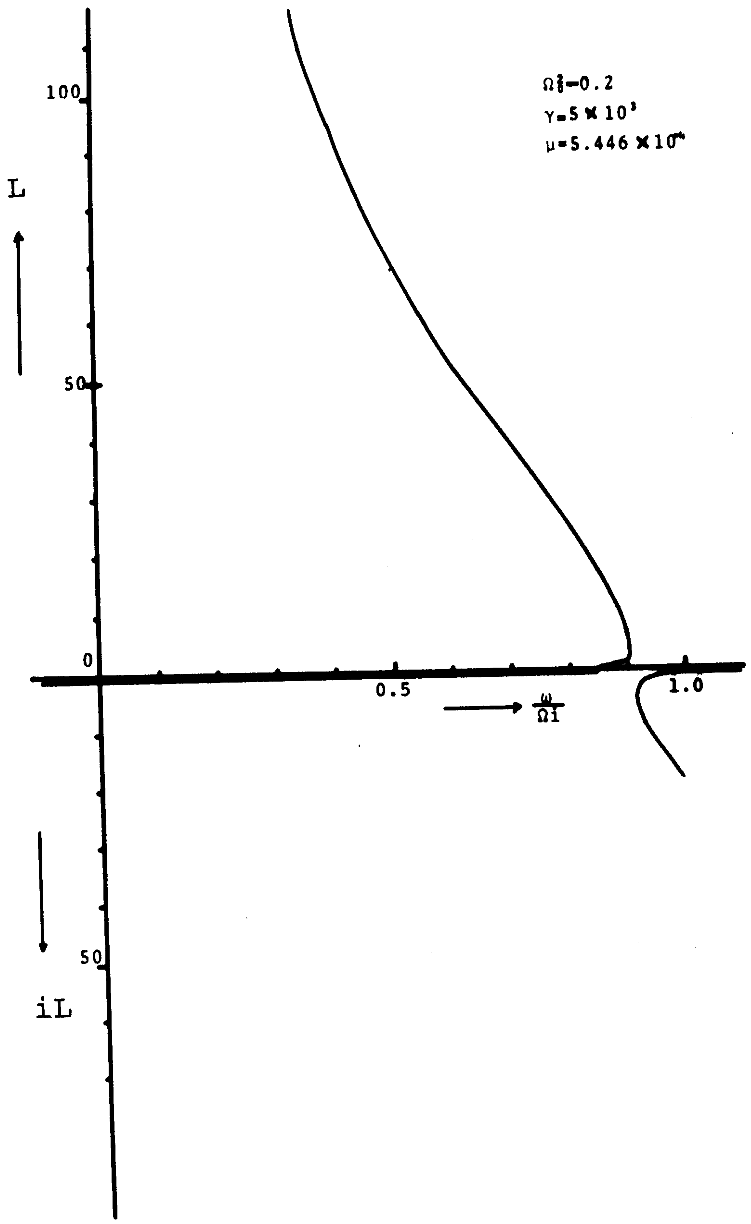

Fig. 2.

The values of L against ω.

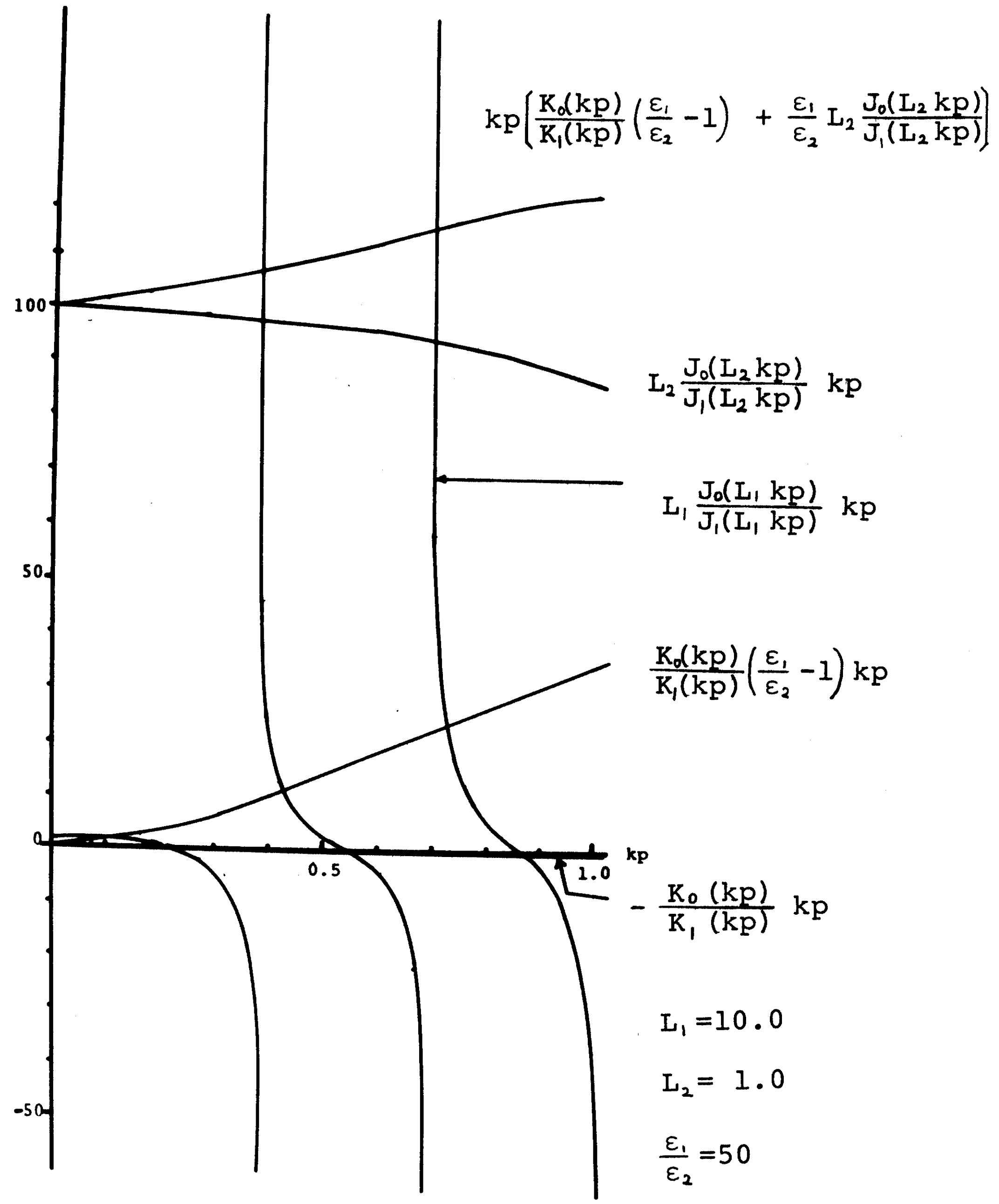

Fig. 3.

The values of the terms in equation (42) against kp.

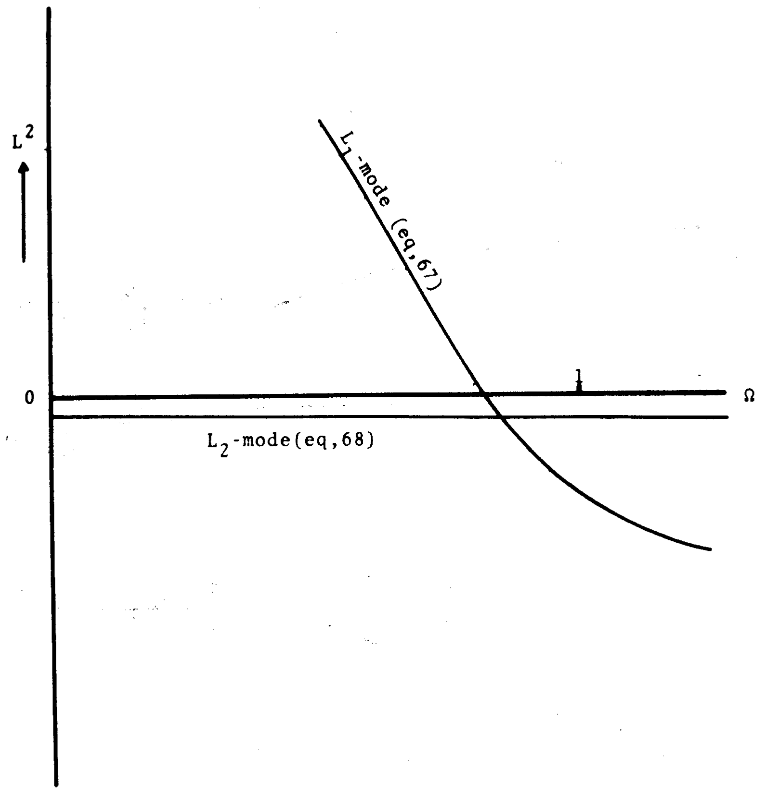

Fig. 4.

Schematic drawing of the dispersion relations of the L_1 and the L_2 modes. Interaction between the two modes are ignored.

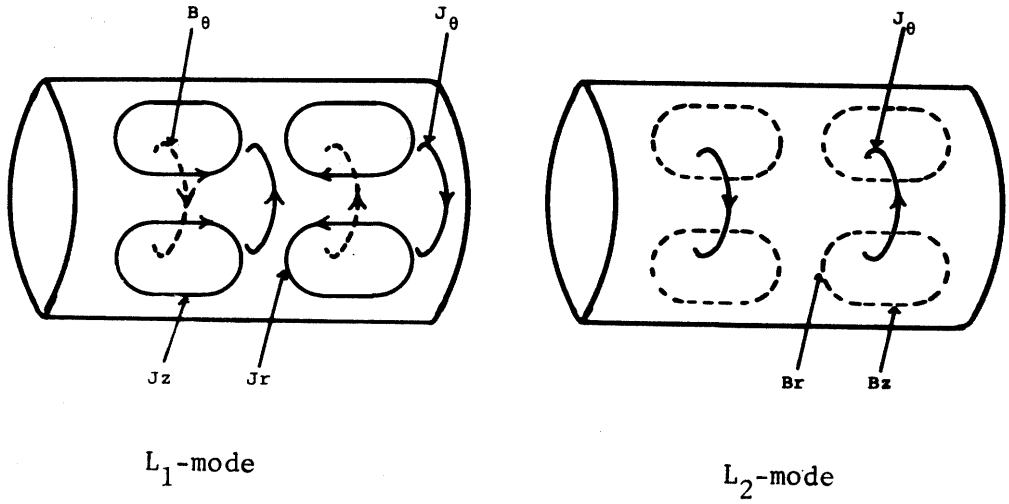

Fig. 5.

Schematic drawings of wave field distributions for the L_1 and L_2 modes, where m=0.