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Electrostatic Oscillations and Feedback Control
of a Lossfree Plasma

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ABSTRACT

The general features of the electrostatic oscillations in a lossfree plasma are discussed from the dispersion relation and energy equation. It is shown that the electrostatic waves in a lossfree plasma have always zero energy density, and negative kinetic energy density, the magnitude of which is equal to the electrostatic energy density. The feedback stabilization for reactive instabilities is reconsidered by a simple model, and a cold electron plasma-beam system is studied as a simple example.

§1. Introduction

Electrostatic instabilities in plasmas are classified into a dissipative and a reactive type.[1)2)] Such a classification was made first by Hasegawa[1)] according to the electric conductivity of the plasma. However, the classification by the dielectric constant[2)] is simpler and more instructive, as shown in the present paper. As is well known, whether the plasma is lossfree or not should be determined according to whether the dielectric constant is Hermitain or not. It should be noted, however, that if we intend to determine form the so-called Joule loss whether the plasma is lossfree or not, we have to take both the frequency and the wavevector as real because otherwise the Joule loss may be finite even in a lossfree plasma (see §3). This point is fundamental but may be sometimes misunderstood.

The instabilities in dissipative plasma are caused by energy exchange between the waves and the plasma medium, due to collisions for a collisional plasma, and due to the Landau damping or cyclotron damping for a collisionless plasma. In this case, the dielectric constant has an anti-Hermitian term and the instabilities are called dissipative. On the other hand, the reactive instabilities occur in a dissipation-free plasma, whose dielectric constant has no anti-Hermitian term, and the growing wave is always accompanied with the damping wave having the complex conjugate frequency and gains energy only from the latter within the linear theory.

By making use of a simple model, Taylor and Lashmore-Davies[2] studied the general features of feedback statilization for the both types of instability. Feedback stabilization can be achieved by reversing the direction of the energy transfer or by introducing an energy dissipation. For the reactive type without feedback, the energy transfer is only possible between a stable and an unstable mode in a pair, whose frequency is complex conjugate each other, so that the reversal of the direction of energy transfer by feedback leads to at most neutral stability. In ref.2, for reactive instabilities, only the conditions for neutral stability were considered and the phase shift of the feedback system had to be precisely specified for stabilization. However, as shown in ref.3, the analysis of the feedback stabilization by introducing energy dissipation is usually too difficult for reactive instabilities to obtain trustworthy results.

In the next section, the general properties of electrostatic oscillations in a lossfree plasma are discussed by making use of the dielectric constant. In §3, the energy flow is studied to make clearer the properties of a lossfree plasma, and the negative kinetic energy density of the wave is introduced to apply consistently the formula of the wave energy density defined for the dissipative case. In §4.1, the general features of feedback stabilization for reactive instabilities are reconsidered by using a simpler model than in ref.2, and the same stability criteria are derived. In

§4.2, as a simple example, feedback stabilization is considered for a cold electron plasma-beam system.

§2. Electrostatic Oscillations in a Lossfree Plasma

It is well known that the behaviour of small electrostatic perturbations in a plasma, varying with time as exp $(-i\omega t)$, is described in terms of the dielectric constant of the plasma, $\varepsilon(\omega) = \varepsilon_1(\omega) + i\varepsilon_2(\omega)$, where $\varepsilon_1(\omega)$ and $\varepsilon_2(\omega)$ are the real and imaginary parts at real frequency, respectively. In the present discussions, the spatial dispersion is not important and omitted for simplicity.

As is well known, the plasma with no $\varepsilon_2(\omega)$ is lossfree and the dielectric constant is Hermitian or reactive.[4) The instabilities in such a plasma are called reactive following to Hasegawa.[1) First, for simplicity, we assume $|\gamma_0| << |\omega_0|$, where $\omega = \omega_0 + i\gamma_0$, ω_0 and γ_0 being real, is the solution of the dispersion relation $\varepsilon(\omega) = 0$. Then we can write approximately the dispersion relation of a lossfree plasma as follows;

$$\varepsilon_1(\omega_0) + i\gamma_0\frac{\partial\varepsilon_1}{\partial\omega}\bigg|_{\omega_0} + \frac{1}{2}(i\gamma_0)^2\frac{\partial^2\varepsilon_1}{\partial\omega^2}\bigg|_{\omega_0} = 0 \qquad (2.1)$$

where, differently from the dissipative plasmas, the last term is retained for the case of $\partial\varepsilon_1/\partial\omega|_{\omega_0} = 0$. From eq.(2.1) we have

$$\varepsilon_1(\omega_0) - \frac{1}{2}\gamma^2{}_0 \frac{\partial^2\varepsilon_1}{\partial\omega^2}\bigg|_{\omega_0} = 0, \tag{2.2}$$

$$\gamma_0 \frac{\partial\varepsilon_1}{\partial\omega}\bigg|_{\omega_0} = 0. \tag{2.3}$$

When $\partial\varepsilon_1/\partial\omega|_{\omega_0} \neq 0$, always we have $\gamma_0 = 0$ i.e. a neutrally stable, negative or positive energy wave according to the sign of $\partial\varepsilon_1/\partial\omega|_{\omega_0}$, and the frequency ω_0 is determined by the equation $\varepsilon_1(\omega_0) = 0$. Then the wave energy density is expressed in the same form as in the dissipative case:[5)]

$$W = \frac{|\vec{E}|^2}{16\pi} \frac{\partial}{\partial\omega} (\omega\varepsilon_1(\omega))\big|_{\omega_0}. \tag{2.4}$$

where $\vec{E}$ is the electric field of the perturbation.

When $\partial\varepsilon_1/\partial\omega|_{\omega_0} = 0$, we have either $\gamma_0 = 0$ or $\gamma_0 \neq 0$. For $\gamma_0 = 0$, the frequency ω_0 is determined from $\varepsilon_1(\omega_0) = 0$ or $\partial\varepsilon_1/\partial\omega|_{\omega_0} = 0$, and this case corresponds to the threshold of reactive instability as a result of the degeneracy of a negative and a positive energy wave. For $\gamma_0 \neq 0$, the frequency ω_0 is determined by $\partial\varepsilon_1/\partial\omega|_{\omega_0} = 0$ and the growth (or damping) rate is given by

$$\gamma^2{}_0 = \frac{2\varepsilon_1(\omega_0)}{\frac{\partial^2\varepsilon_1}{\partial\omega^2}\Big|_{\omega_0}} > 0. \tag{2.5}$$

For the reactive instability, therefore, the unstable wave is always accompanied with the stable wave having the complex conjugate frequency. The instabilities which appear under the linearized treatment of plasmas as an ideal fluid belong to this type.

Although we have assumed $|\gamma_o| << |\omega_o|$ in the above discussions, it is easily shown that generally in a lossfree plasma the modes with complex frequency can exist only as a complex conjugate pair. That is because the dispersion relation for the lossfree plasma can be written as an algebraic equation only with the real coefficients, even if the equation may become of infinite order by making use of the series expansions of special functions. When we have an interest in the complex conjugate modes $\omega = \omega_o \pm i\gamma_o$, the dispersion relation is written as follows:

$$\varepsilon_1(\omega) = \{(\omega - \omega_o)^2 + \gamma_o^2\}^p F(\omega) = 0, \qquad (2.6)$$

where $F(\omega)$ is generally a rational function and the poles nerest to ω_o determine the location of the branch to which the complex conjugate pair belongs. Since $F(\omega)$ near ω_o is regarded as almost constant, the above discussion for $|\gamma_o| << |\omega_o|$ is also valid for arbitrary values of complex frequencies, e.g. for $|\gamma_o| \gtrsim |\omega_o|$, except that if the multiplicity p of the roots is larger than unity eq.(2.5) is replaced by

$$\gamma_0^2 = \frac{2\ p\varepsilon_1(\omega_0)}{\left.\frac{\partial^2\varepsilon_1}{\partial\omega^2}\right|_{\omega_0}} > 0. \tag{2.7}$$

§3. Energy Flow

In this section, by comparing with the dissipative plasma we consider the energy balance equation aiming at the clear understnading of the characteristic features of electrostatic oscillations in a lossfree plasma. Let us start with the following relation[6)]

$$\frac{\partial\vec{D}}{\partial t} = \frac{\partial\vec{E}}{\partial t} + 4\pi\vec{j}\ , \tag{3.1}$$

Where D is the electric displacement and for the monochromatic mode $\vec{D}(\omega) = \varepsilon(\omega)\ \vec{E}(\omega)$, and $\vec{j}$ is the total current density and we assume no source current. We make the scalar product of eq.(3.1) with E to derive the energy equation. From the left-hand side of eq.(3.1) we obtain easily

$$\frac{1}{4\pi}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{i}{16\pi}\ |\vec{E}|^2\ \{\ \omega^*\varepsilon^*(\omega)\ -\omega\varepsilon(\omega)\}\ = 0, \tag{3.2}$$

where the bar means the average over the wave period and the asterisk the complex conjugate quantity.

As is well known, for dissipative plasmas, eq. (3.2) is reduced to[6,7,8]

$$\frac{|\vec{E}|^2}{8\pi}\{\gamma_0 \frac{\partial}{\partial\omega}(\omega\varepsilon_1(\omega))|_{\omega_0} + \omega_0\varepsilon_2(\omega_0)\} = 0, \tag{3.3}$$

where we have assumed $|\gamma_0| << |\omega_0|$ and $|\varepsilon_2| << |\varepsilon_1|$. The first term is the rate of change of the total energy density of the wave, W, which is composed of the electrostatic and kinetic energy densities, and the second term is the rate of energy absorption by the plasma medium.

By making use of the results obtained in the last section, we consider the lossfree plasma. It is easily seen that for $\partial\varepsilon_1/\partial\omega|_{\omega_0} \neq 0$, eq. (3.3) is also valid but the two terms are always zero. The zero of the second term means no energy exchange between the wave and the plasma medium, and the zero of the first term no change of the wave energy density. On the other hand, the wave energy density itself is finite and also given by eq. (2.4). Thus there exist neutrally stable, positive or negative energy waves and/or the pair of a positive and a negative energy wave.

For $\partial\varepsilon_1/\partial\omega|_{\omega_0} = 0$, eq. (3.2) is reduced to

$$\frac{|\vec{E}|^2}{8\pi}\{\gamma_0\varepsilon_1(\omega_0 + i\gamma_0) + \omega_0\varepsilon_2(\omega_0)\} = 0, \tag{3.4}$$

where each term is exactly zero. In this case the wave energy density is always zero and defined by

$$W = \frac{|\vec{E}|^2}{16\pi} \varepsilon_1(\omega_0 + i\gamma_0) = 0, \tag{3.5}$$

instead of which the definition by eq.(2.4) is also applicable. Although both the stable and unstable modes in a pair have the vanishing value of the wave energy density, the electrostatic energy density $|\vec{E}|^2/16\pi$ is not zero. Therefore, the both modes should have a negative kinetic energy density, the magnitude of which is equal to the electrostatic energy density.

Next, let us examine the scalar product of the right-hand side of eq.(3.1) with $\vec{E}$. Then we have

$$\overline{\vec{j}\cdot\vec{E}} = -\frac{1}{4\pi}\overline{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}} = -\frac{\gamma_0}{8\pi}|\vec{E}|^2 . \tag{3.6}$$

From this relation, we see that if γ_0 is not zero, the so-called Joule loss is finite even for the lossfree plasma. However, this does not always mean the finite value of energy exchange between the wave and the plasma. It should be noted that whether the energy exchange between them exists or not must be determined by $\varepsilon_2(\omega_0) \neq 0$ or $\varepsilon_2(\omega_0) = 0$. Especially we should be careful for the lossfree plasma, for which we have $\overline{\vec{j}\cdot\vec{E}} \neq 0$ for the waves forming a complex conjugate pair. For $\overline{\vec{j}\cdot\vec{E}} \neq 0$, even if we imagine apparent energy exchange between the wave and the lossfree plasma, the energy absorbed by the plasma from the stable mode is exactly given to the unstable mode so that the energy exchange vanishes between the complex conjugate pair and the plasma.[3)]

Because only if the frequency (and also the wavevector in the presence of spatial dispersion) is real, the term $\overline{\vec{j}\cdot\vec{E}}$ gives the absorbing power $\omega_0\ \varepsilon_2(\omega_0)|\vec{E}|^2/8\pi$, whether the dielectric constant is Hermitian or not should be examined for real frequency (and also for real wavevector at the same time).

§4. Feedback Stabilization

4.1 General features

The dispersion relation, $\varepsilon(\omega) = 0$, in the absence of feedback is modified by the application of feedback. The signal sensed by a sensor probe is amplified, phase-shifted and injected into the plasma as an appropriate suppressor signal. The change density ρ_f introduced by the suppressor is assumed to be propotional to the electric potential ϕ sensed at the sensor. That is, we assume $\rho_f = g\phi$, where $|g|$ is the feedback gain and arg(g) the feedback phase. For simplicity, the spatial correlation between the sensor and the suppressor is neglected. Then we have the basic equations relevant to the present problem as follows:

$$\mathrm{div}\, \vec{D}(\omega) = 4\pi g(\omega)\phi \ , \tag{4.1}$$

$$\vec{D}(\omega) = -\varepsilon(\omega)\nabla\phi \ , \tag{4.2}$$

from which we obtain

$$\varepsilon(\omega) + \frac{4\pi g(\omega)\phi}{\Delta\phi} = 0,$$

where the spatial dispersion is also neglected.

Regarding g as a small perturbation, and using the results in §2, we can rewrite eq.(4.3) for the lossfree plasma as follows:

$$\{(\omega - \omega_0)^2 + \gamma_0^2\}^p = G\, e^{i\alpha}, \tag{4.4}$$

$$G\, e^{i\alpha} = - \frac{4\pi g(\omega)\phi}{F(\omega)\Delta\phi} \ , \tag{4.5}$$

where G and α are real and dependent on the frequency. If we put p = 1, then eq.(4.4) is reduced to eq.(11) of ref.2. In what follows we assume p = 1. If we denote the frequency in the presence of weak feedback as $\omega_1 + i\gamma_1$, where ω_1 and γ_1 are real, eq.(4.4) is reduced to

$$(\omega_1 - \omega_0)^2 = \gamma_1^2 - \gamma_0^2 + G\cos\alpha \ , \tag{4.6}$$

$$2\gamma_1(\omega_1 - \omega_0) = G \sin\alpha, \tag{4.7}$$

from which we have

$$\gamma_1^2 = \frac{1}{2}\{\gamma_0^2 - G\cos\alpha \pm \sqrt{(\gamma_0^2 - G\cos\alpha)^2 + G^2\sin^2\alpha}\}, \tag{4.8}$$

where the lower sign is allowed only for the case of $\sin\alpha=0$.

For the feedback phase such as $\sin\alpha\neq 0$, if the otherwise unstable mode is stabilized the otherwise stable mode is destabilized. Therefore, we have the severe conditions for feedback stabilization

$$\begin{aligned} G\cos\alpha &\geqq \gamma_0^2 \\ G\sin\alpha &= 0. \end{aligned} \tag{4.9}$$

In eqs. (4.6) - (4.9), we may take approximately the values at the frequency without feedback for G and α. Equation (4.9) is the same as eq.(13) in ref.2, where $|\gamma_0| \ll |\omega_0|$ was assumed and the different conditions for $|\gamma_0| \gtrsim |\omega_0|$ were obtained. However, the stability conditions (4.9) are also valid for $|\gamma_0| \gtrsim |\omega_0|$ because we have made no assumption about the magnitudes of ω_0 and γ_0.

In the above discussions, only a certain pair of modes has been considered. However, if the other stable and/or unstable modes exist, these modes may be destabilized or remain unstable when the relevant pair of modes is stabilized.

4.2 A simple example

As a simple example of reactive instabilities, we consider the system which consists of a cold electron plasma and a cold thin electron beam moving with a constant velocity u. As is well known, under some conditions, the electrostatic instabilities occur in this system.[8)9)]

The fundamental equations for each component are

$$m\left(\frac{\partial}{\partial t} + v\cdot\nabla\right)v = e\,\nabla\phi, \tag{4.10}$$

$$\frac{\partial n}{\partial t} + \mathrm{div}\,(n\,v) = S, \tag{4.11}$$

where ϕ is the potential of the electric field of perturbations, S is the source term introduced by the feedback system and assumed to be propotional to the electron density perturbation n', and we put $S = -i\,n'\omega_f$. The absolute value of ω_f is the feedback gain and $\arg(\omega_f) = \theta$ the feedback phase.

Linearizing eqs.(4.10) and (4.11), and using the Poisson equation, we obtain the dispersion relation in the presence of feedback:

$$1 - \frac{\omega^2{}_p}{\omega(\omega - \omega_f)} - \frac{\beta\ \omega^2{}_p}{(\omega - \vec{k}\cdot\vec{u})(\omega - \vec{k}\cdot\vec{u} - \omega_f)} = 0, \qquad (4.12)$$

where $\omega^2{}_p = 4\pi n_p e^2/m$, and $\beta = n_b/n_p << 1$, n_p and n_b being the electron densities of the plasma and the beam respectively. Equation (4.12) without feedback shows that several unstable modes of oscillations are possible for some ranges of plasma parameters. For simplicity, we consider only the case where $\omega^2{}_p \neq (\vec{k}\cdot\vec{u})^2$. Then, as the four roots of eq.(4.12) without feedback, we obtain $\omega = \pm\ \omega_p$ and

$$\omega = \vec{k}\cdot\vec{u} \pm \frac{\sqrt{\beta}(\vec{k}\cdot\vec{u})\ \omega_p}{\sqrt{(\vec{k}\cdot\vec{u}) \quad - \omega_p^2}} \qquad (4.13)$$

If $(\vec{k}\cdot\vec{u})^2 < \omega^2{}_p$, the two roots given by eq.(4.13) are complex conjugate each other, one of which is unstable.

Assuming weak feedback such that $|\omega_f|^2 << (\vec{k}\cdot\vec{u})^2$, we put the frequency in the presence of feedback as $\omega = k\cdot u + \delta$. Then solving eq.(12) we obtain approximately

$$\delta = \frac{1}{2}\ \omega_f \pm \frac{1}{2}\ \omega_f \sqrt{1 - \frac{4\beta\omega^2_p}{A\omega^2_f}}, \qquad (4.14)$$

$$A = \frac{\omega^2{}_p}{(\vec{k}\cdot\vec{u})^2} - 1. \qquad (4.15)$$

To obtain the trustworthy results, we confine our calculation to the case where $4\beta\omega_p^2/A < |\omega_f|^2$. Then from eq.(4.14) we have

$$
\mathrm{Im}\ \delta = \begin{cases} |\omega_f| \quad (1 + \dfrac{\beta\omega_p^2}{A|\omega_f|^2}) \sin\theta, & \text{(otherwise unstable mode)} \\ \\ -\dfrac{\beta\omega_p^2}{A|\omega_f|^2} \sin\theta. & \text{(otherwise stable mode)} \end{cases} \tag{4.16}
$$

Therefore, for $\theta \neq 0$, if the otherwise unstable mode is stabilized, then the otherwise stable mode is destabilized. Thus the stability condition for feedback phase is $\theta = 0$, consistent with the result obtained in §4 and in ref.2. Whether or not the other two modes $\omega = \pm\omega_p$ remain stable at the same time is difficult to state because the correct calculation is very hard.

In conclusion, we can say that the feedback stabilization of reactive instabilities in lossfree plasma is very difficult because the stability condition for the feedback phase is stringent and also the other stable or unstable modes may be destabilized or remain unstable.

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