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Control of Plasma Current Distribution by Relativistic Electron Components

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Abstract

Possible method of controlling plasma current distribution in the diffuse pinch configuration is proposed. The equilibrium configuration is self-consistently constructed from a combination effect between the plasma force free current and the relativistic beam one. The parameters are found, which give an absolute minimum-B configuration.

§1. Introduction

The recent encouraging results obtained from the tokamak approaches to the controlled thermonuclear reactor program have stimulated interest in axisymmetric plasma containment devices where currents in the plasma provides the necessary rotational transform. The equilibrium of plasma in the devices is often described by the ideal magnetohydrostatic equations as follows (M.K.S. units are used throughout).

$$\vec{J}_p \times \vec{B} = \nabla P , \tag{1-a}$$

$$\vec{J}_p = \frac{1}{\mu_0} \nabla \times \vec{B} , \tag{2-a}$$

$$\nabla \cdot \vec{B} = 0 , \tag{3}$$

where $\vec{B}$ is the magnetic field, P the plasma pressure, $\vec{J}_p$ the plasma current density, and μ_0 the magnetic permeability of the vacuum. In the fully collisionless regime the equilibrium described by the set of the equations (1-a), (2-a) and (3) has an inherent difficulty of trapped particle instabilities, since it is impossible to design a force free, or vacuum magnetic confinement configuration in which some field lines or flux surfaces are located in a finite volume with field strength increasing everywhere outwards.[1)]

In a certain tokamak device more than 50% of longitudinal current is transported by the relativistic electrons.[2] The relativistic electron current behaves quite differently from the plasma current. Hence, it is realistic that the general diffuse pinch configurations are described by introducing another component of current produced by the relativistic electron beam.

As Christofilos has proposed earlier,[3] the presence of absolute minimum-B configuration of closed magnetic lines is possible by placing relativistic electron beam coils in a toroidal geometry. This configuration becomes a sheared system by an addition of a toroidal field.[4] This type of diffuse pinches with minimum-B configuration are superior to others in that a inherent difficulty of magnetohydrostatic equilibria to the trapped particle instabilities would be avoidable in the fully collisionless regime.

Recently, Yoshikawa and Christofilos[5] have given a self-consistent solution of the absolute minimum-B configuration produced by the relativistic electron beam for a given toroidal field distribution in a large aspect ratio limit. The question then arises whether or not we could control the toroidal field distribution. It is, therefore, important to find out the equilibria with absolute minimum-B by use of given controllable parameters without expecting help from toroidal field distribution.

The purpose of the present investigation is to show the equilibria with a controlled current distribution realized by controllable parameters and to clarify the presence of parameters which give the absolute minimum-B configurations. We see that the equilibrium considered is self-consistently constructed from a combination effect between the plasma force free current and the relativistic beam one.

The basic equations of the plasma equilibrium including the relativistic electron current is given in §2. Section 3 is devoted to consider the cylindrical solution. The parameters are found, which give the absolute minimum-B configuration. In §4 we consider the toroidal configuration. The equilibrium equation including the relativistic electron beam is reduced to a nonlinear differential equation of elliptic type for the stream function Ψ.

§2. Basic Equations

In the case of plasma equilibrium including a relativistic electron beam we use the following equation,

$$\vec{J}_e + \vec{J}_p = \frac{1}{\mu_o} \nabla \times \vec{B} , \qquad (2\text{-b})$$

in place of (2-a), where $\vec{J}_e$ is the relativistic beam current density.

The stationary equation of motion of the relativistic electron beam is governed by the following equation.

$$m_o \gamma (\vec{v}_e \cdot \nabla) \vec{v}_e = e\vec{v}_e \times \vec{B} , \qquad (4)$$

where $\vec{v}_e$ is the electron beam velocity and m_o and e are the electron rest mass and the charge of single electron, respectively, and $\gamma = (1 - \beta)^{-1/2}$ and $\beta = |\vec{v}_e|/c$. The quantity c represents the light velocity. The relation between the beam current density $\vec{J}_e$ and the beam velocity $\vec{v}_e$ is assumed to be

$$\vec{J}_e = ne\,\vec{v}_e\ , \tag{5}$$

where n is the uniform beam density. Equation (1-a) is rewritten as

$$\vec{J}_p = \frac{\alpha}{\mu_o}\,\vec{B} + \frac{\vec{B} \times \nabla P}{B^2}\ , \tag{1-b}$$

where

$$\alpha = \mu_o \frac{\vec{J}_p \cdot \vec{B}}{B^2}$$

and (6)

$$B = |\,\vec{B}|\ .$$

From Eqs. (2-b) and (1-b) we have

$$\frac{m_0\gamma}{ne^2}(\,\vec{J}_e\cdot\nabla)\,\vec{J}_e + \frac{1}{\mu_0}\,\vec{B} \times \nabla \times \vec{B} + \nabla P = 0\ , \tag{7}$$

where

$$\vec{J}_e = \frac{1}{\mu_0}(\nabla \times \vec{B} - \alpha\vec{B}) - \frac{\vec{B} \times \nabla P}{B^2} \tag{8}$$

The equations (3), (7) and (8) are used throughout as the basic equations of plasma equilibrium including the relativistic electron current.

§3. Cylindrical Equilibria

In this section we shall consider in detail about the cylindrical configuration. We use a cylindrical coordinate system (r,θ,z). In the cylindrical systems the equilibrium quantities are considered as functions of the distance r from the axis of the cylinder. For $\vec{B} = (0, B_\theta, B_z)$ equations (7) and (8) becomes

$$\frac{m_0\gamma}{ne^2} \frac{J_{e\theta}^2}{r} = \frac{1}{\mu_0}\left[B_z \frac{dB_z}{dr} + \frac{B_\theta}{r}\frac{d}{dr}(rB_\theta)\right] + \frac{dP}{dr}, \tag{9}$$

$$J_{e\theta} = -\frac{1}{\mu_0}\left[\frac{dB_z}{dr} - \alpha\right] - \frac{B_z}{B^2}\frac{dP}{dr} . \tag{10}$$

Here we made an assumption that the beam consists of the monoenergetic electrons, i.e. γ is constant in space. We introduce dimensionless quantities b_z, b_θ, Π and x as follows.

$$B_z = B_0 b_z \ ,$$

$$B_\theta = B_0 b_\theta \ , \qquad (11)$$

$$P = P_0 \Pi$$

and

$$r = ax \ ,$$

where B_0 and P_0 are the magnetic field intensity and the plasma pressure on the z axis, and a is the radius of the plasma. Equation (9) becomes

$$b_z \frac{db_z}{dx} + \frac{b_\theta}{x} \frac{d}{dx}(xb_\theta) + \tilde{\beta} \frac{d\Pi}{dx} = \frac{\Theta}{x}\left(\frac{db_z}{dx} + \delta b_\theta + \tilde{\beta}\frac{b_z}{b^2} \frac{d\Pi}{dx}\right)^2 \qquad (12)$$

where $\delta = \alpha a$, $\tilde{\beta} = \frac{\mu_0 P_0}{B^2}$, $\Theta = \frac{m_0 \gamma}{ne^2 \mu_0 a^2}$ and $b^2 = b_\theta^2 + b_z^2$.

It should be noted that α is a measure of plasma force free current which is an arbitrary function of x. The assumption is made that the quantity α is a constant throughout in this paper (See Appendix).

The configuration may be directly affected by the distribution of b_z. To see this we choose

$$b_z = 1 + \varepsilon x^s , \tag{13}$$

where ε is an arbitrary parameter and $s \geqq 2$.

The equilibrium parameters are ordered so that

$$\tilde{\beta} << \varepsilon^2 << \delta \approx \Theta \simeq 1 . \tag{14}$$

Then the solution of equation (12) becomes

$$b_\theta^2 = C^2 x^{2(\Theta\delta^2 - 1)}$$

$$+ \varepsilon \left[\frac{4\delta\Theta C s}{s - \Theta\delta^2} x^{s-2+\Theta\delta^2} - \frac{s}{s + 3 - 2\Theta\delta^2} x^s \right]$$

$$+ \varepsilon^2 \left[\Theta s^2 \frac{s + 3\Theta\delta^2}{(s - \Theta\delta^2)^2} x^{2s-2} - \frac{s}{2s + 3 - 2\Theta\delta^2} x^{2s+1} \right.$$

$$\left. - \frac{2\delta\Theta s^2}{C(s + 3 - 2\Theta\delta^2)(2s + 2 - 3\Theta\delta^2)} x^{2s-\Theta\delta^2} \right] + 0 (\varepsilon^3),$$

(15)

where C represents the total current along z axis. The condition of no singularity at $x = 0$ is

$$\Theta\delta^2 > 1 . \tag{16}$$

An interesting conclusion is drawn out if we consider the plasma stability in the configuration given by Eq.(15). The relativistic electron beam is assumed to have no interaction with any perturbation in the plasma. Then, after Johnson, Kulsrud and Weimer,[6)] the sufficient condition for the low $\tilde{\beta}$ stability against interchange can be expressed as

$$Q = b_z \frac{db_z}{dx} + xb_\theta \frac{d}{dx}\left(\frac{b_\theta}{x}\right) > 0 \ . \tag{17-a}$$

For the configuration given by Eq.(15) the quantity Q becomes

$$Q = C^2(\theta\delta^2 - 2)x^{2\theta\delta^2-3} + \varepsilon\left[\frac{2\delta\theta C s}{s - \theta\delta^2}(s + \theta\delta^2 - 4)x^{s-3+\theta\delta^2} + \frac{s}{s + 3 - 2\theta\delta^2}\left(\frac{s}{2} + 4 - 2\theta\delta^2\right)x^{s-1}\right] + 0(\varepsilon^2). \tag{18}$$

In the case of

$$\theta\delta^2 - 2 \gg |\varepsilon| \ , \tag{19}$$

the stability of plasma is independent of the distribution of b_z. Since $\pi a^2 nec\beta$ is approximately the total beam current I the parameter θ is rewritten as

$$\theta = I_C/I \ , \tag{20}$$

where $I_C = 4.2628 \times 10^3 \beta\gamma = I_A/4$, and I_A is the Alfvén critical current. We see from the expression (20) that, even if the total beam current is far beyond the Alfvén critical current an absolute minimum-B configuration can be realized by a careful choice of the plasma force free current.

For the case of $\delta^2\Theta - 2 \to 0$, the condition of the stability takes the form

$$\varepsilon \left\{\frac{4C}{\delta} + \frac{s}{2(s - 1)}\right\} > 0. \qquad (21)$$

The plasma stability is strongly affected by the distribution of b_z and the sign of δ in this case.

A sufficient condition of the plasma column with finite pressure for the flute instability[6] is

$$Q^* = b_z \frac{db_z}{dx} + x\ b_\theta \frac{d}{dx}\left(\frac{b_\theta}{x}\right) + \tilde{\beta}\frac{d\Pi}{dx} > 0\ . \qquad (17\text{-b})$$

We have numerically integrated the equation (12) for a given pressure profile to determine the sign of the quantity Q^* everywhere in the plasma region, i.e. $0 < x < 1$. It is not certain, however, which profile of plasma pressure should be chosen. Here, for simplicity, we take a simple pressure profile of the form

$$\Pi = 1 - x^2\ .$$

The resulting stability diagrams for $\tilde{\beta} = 0.05$ is shown in Fig.1, where the horizontal line represents the quantity $\delta^2\Theta$ and the vertical line does ε. If there is no region of negative Q* everywhere in the plasma column we define that the plasma is stable. The shaded areas are the regions of stability. For given $\delta^2\Theta$ and ε, the areas are present where b_θ^2 becomes negative. Whence, we have no physical solution in these areas. The cross hatched areas are the region of no physical solution. It is confirmed by the numerical computations that the boundary of the stable areas shifts to the larger values of $\delta^2\Theta$ for higher $\tilde{\beta}$. The typical field structure is shown in Fig.2 for the parameters $s = 2.0$, $\varepsilon = 0.05$ and $\tilde{\beta} = 0.05$.

§4. Toroidal Configurations

In this section we shall consider the axisymmetric toroidal configurations in a low $\tilde{\beta}$ limit. In the case given by the inequality (19) the deviation of the axial field from the vacuum is not important for the plasma stability. And for simplicity the toroidal magnetic field is assumed to remain a vacuum one even if $\nabla \times \vec{B} \neq 0$. This means there is no net poloidal current in the plasma.

The purpose of the present section is to find out what kind of the differential equation governs the equilibrium plasma system with relativistic beam. We will find out that the basic equations (3), (7) and (8) are reduced into a certain nonlinear partial differential equation with three parameters, i.e.. the beam energy, the intensity of plasma force free current and the toroidal field strength.

It is fairly easy to describe the axisymmetric toroidal equilibria with relativistic beam. This is most easily described in a cylindrical polar system centered on the axis of symmetry, (r, θ, z). The condition (3) is satisfied by introducing a stream function Ψ such that

$$rB_r = -\frac{\partial\Psi}{\partial z}, \quad rB_z = \frac{\partial\Psi}{\partial r} \text{ and } rB_\theta = \frac{\mu_0 I}{2\pi}, \tag{22}$$

where I represents the total current through a conductor along z axis.

We also introduce in this section the dimensionless quantities b_r, b_θ, b_z, x, y and δ as follows.

$$\begin{aligned} B_r &= B_p b_r, \\ B_\theta &= B_p b_\theta, \\ B_z &= B_p b_z, \\ r &= Rx, \\ z &= Ry, \text{ and } \delta = \alpha R, \end{aligned} \tag{23}$$

where R is the major radius of the torus and B_p is the characteristic poloidal magnetic field. The components of equation (7) then become

$$\frac{\tilde{\Theta}}{2}\frac{\partial G}{\partial x} = [\tilde{\Theta}(xF - \eta)\frac{d}{d\Phi}(xF) + xF(1 - \delta^2\tilde{\Theta})]\frac{1}{x^2}\frac{\partial \Phi}{\partial x} \quad , \tag{24}$$

$$\frac{1}{x}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}(xF) - \frac{1}{x}\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x}(xF) = 0 \; , \tag{25}$$

$$\frac{\tilde{\Theta}}{2}\frac{\partial G}{\partial y} = [\tilde{\Theta}(xF - \eta)\frac{d}{d\Phi}(xF) + xF(1 - \delta^2\tilde{\Theta})]\frac{1}{x^2}\frac{\partial \Phi}{\partial y} \; , \tag{26}$$

where

$$\Phi = \frac{\Psi}{B_p R^2} \; , \quad \tilde{\Theta} = \frac{m_0 \gamma}{ne^2 \mu_0 R^2} \; , \quad \eta = \frac{\delta \mu_0 I}{2\pi R B_p} \; ,$$

$$F = \frac{\partial b_r}{\partial y} - \frac{\partial b_z}{\partial x} \quad \text{and} \quad G = \delta^2(b_r^2 + b_z^2) + (\delta b_\theta - F)^2 \; . \tag{27}$$

Equation (25) implies that xF is a function of Φ alone;

$$xF = f(\Phi) \quad . \tag{28}$$

Eliminating the square bracket between the remaining Eqs. (24) and (26) gives

$$\frac{\partial G}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial G}{\partial y}\frac{\partial \Phi}{\partial x} = 0 \; . \tag{29}$$

Thus G is also a function of Φ alone;

$$G = g(\Phi) \quad . \tag{30}$$

The surface Φ = constant can then be identified with the magnetic surfaces, and the basic equation of the equilibrium of plasma including a relativistic beam becomes

$$\frac{\tilde{\Theta}}{2} x^2 \frac{dg}{d\Phi} - \tilde{\Theta}(f - \eta)\frac{df}{d\Phi} - f(1 - \delta^2\tilde{\Theta}) = 0 \ . \tag{31}$$

If the beam energy is monochromatic in the whole plasma region

$$g = \text{constant} \quad . \tag{32}$$

In this case

$$\Phi = \frac{\tilde{\Theta}}{\delta^2\tilde{\Theta} - 1} \int (1 - \frac{\eta}{f}) df \quad . \tag{33}$$

From eqs.(28) and (33) we have the following non-linear differential equation.

$$x \frac{\partial}{\partial x} (\frac{1}{x} \frac{\partial \Phi}{\partial x}) + \frac{\partial^2 \Phi}{\partial y^2} + f = 0 \quad , \tag{34}$$

where f should be determined by

$$\frac{\exp f}{f^{\eta}} = A \exp (\frac{\delta^2\tilde{\Theta} - 1}{\tilde{\Theta}} \Phi) \quad , \tag{35}$$

and A is an integral constant which is to be chosen by given boundary conditions. We could find out the most desirable confinement configuration by choosing the given parameters δ, $\tilde{\Theta}$, η and A.

A simple toroidal equilibrium is obtained when parameter η is small; i.e. the case of weak toroidal field.

$$|\eta| << 1 \quad . \tag{36}$$

Hence, Eqs.(34) and (35) are reduced to the form

$$x \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \Phi}{\partial x}\right) + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\delta^2 \tilde{\Theta} - 1}{\tilde{\Theta}} \Phi = 0 \tag{37}$$

with solutions

$$\Phi = \sum_k A(k)\ xB_1(\lambda x)\exp(iky) \quad , \tag{38}$$

where B_1 is an arbitrary first-order Bessel function and A(k) is an arbitrary constant, and $\lambda = \left(\frac{\delta^2 \tilde{\Theta} - 1}{\tilde{\Theta}} - k^2\right)^{1/2}$.

§5. Discussions and Conclusions

We have indicated that the parameters δ, $\tilde{\Theta}$ and η can uniquely give the plasma current distribution for a given boundary condition. From experimental point of view it is not difficult to evaluate δ and $\tilde{\Theta}$. If the plasma temperature and its density are known the plasma current δ can be controlled by the solenoidal electric field along a line of force. The electric field is produced inductively by a transformer. From Eqs.(20) and (27) we see that the parameter $\tilde{\Theta} = (a^2 I_C)/(R^2 I)$ is approximately proportional to the beam energy $m_0\gamma$. The energy of injected beam is a function of time of flight of the electron beam in the toroidal electric field. Thus, the parameter $\tilde{\Theta}$ is also controllable by the solenoidal electric field.

The quantity G is constant on the magnetic surface. This suggests that the strength of the magnetic field is not constant on the toroidal magnetic surfaces. By a careful choice of the boundary shape the deviation of the magnetic isobars from the magnetic surfaces should be minimized.

With regard to the toroidal equilibrium, we have derived an nonlinear differential equation in the case of no net poloidal current. The poloidal current can flow in the general configuration. In the configuration with poloidal current the equilibrium quantities are not the function of Φ alone.[7)]

Finally we note that the combination of plasma force free current and the relativistic electron beam can lead to realizing confinement configuration with a controlled current distribution. Under a certain controlled parameters the system becomes an absolute minimum-B configuration at least in the large aspect ratio limit.

Appendix

When δ is not constant we make an assumption that δ^2 can be expanded into a power series in x.

$$\delta^2 = \sum_{n=0}^{\infty} a_n x^n , \tag{A-1}$$

where a_n are the expansion coefficients. For $\varepsilon = 0$ and $\tilde{\beta} = 0$ equation (12) is reduced to

$$\frac{1}{r\, b_\theta} \frac{d(rb_\theta)}{dx} = \frac{\Theta}{x} \sum_{n=0}^{\infty} a_n x^n \tag{A-2}$$

with solution

$$b_\theta = C_A x^{a_0\Theta-1} \exp[\Theta \sum_{n=1}^{\infty} \frac{a_n}{n} x^n] \tag{A-3}$$

where C_A is a constant. The condition of no singularity at $x = 0$ is

$$a_0\Theta > 1 \tag{A-4}$$

Therefore the expansion coefficient a_0 should not vanish in order to have physically resonable solution.

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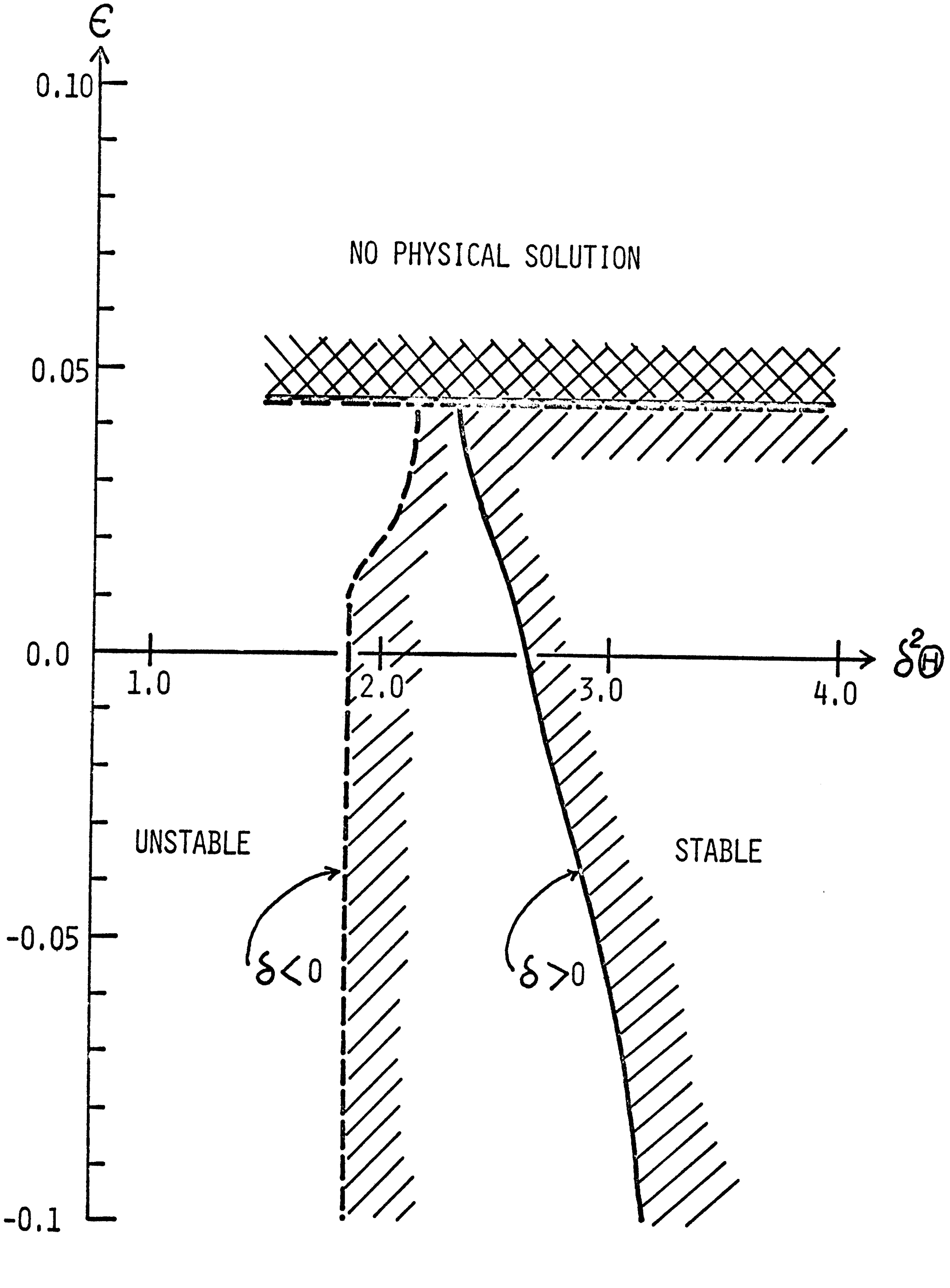
Figure Captions

Fig.1 Regions of the parameters ε and $\delta^2\Theta$ for the stability of plasma, where $\tilde{\beta} = 0.05$. The marginal curves drawn by the real and the dotted lines correspond to the cases $\delta > 0$ and $\delta < 0$, respectively.

Fig.2 Radial distributions of b_θ for different values of $\delta^2\Theta$, where $\tilde{\beta} = 0.05$, $s = 2.0$, $\Theta = 1.0$ and $\varepsilon = -0.05$. The sign of δ is chosen to be positive. The case $\delta^2\Theta = 3.0$ satisfies the stability condition for the interchange instability, and the case $\delta^2\Theta = 1.08$ does not.



ϵ
0.10
0.05
0.0
-0.05
-0.1
NO PHYSICAL SOLUTION
1.0
2.0
3.0
4.0
δ²Θ
UNSTABLE
STABLE
δ<0
δ>0

Fig. 1

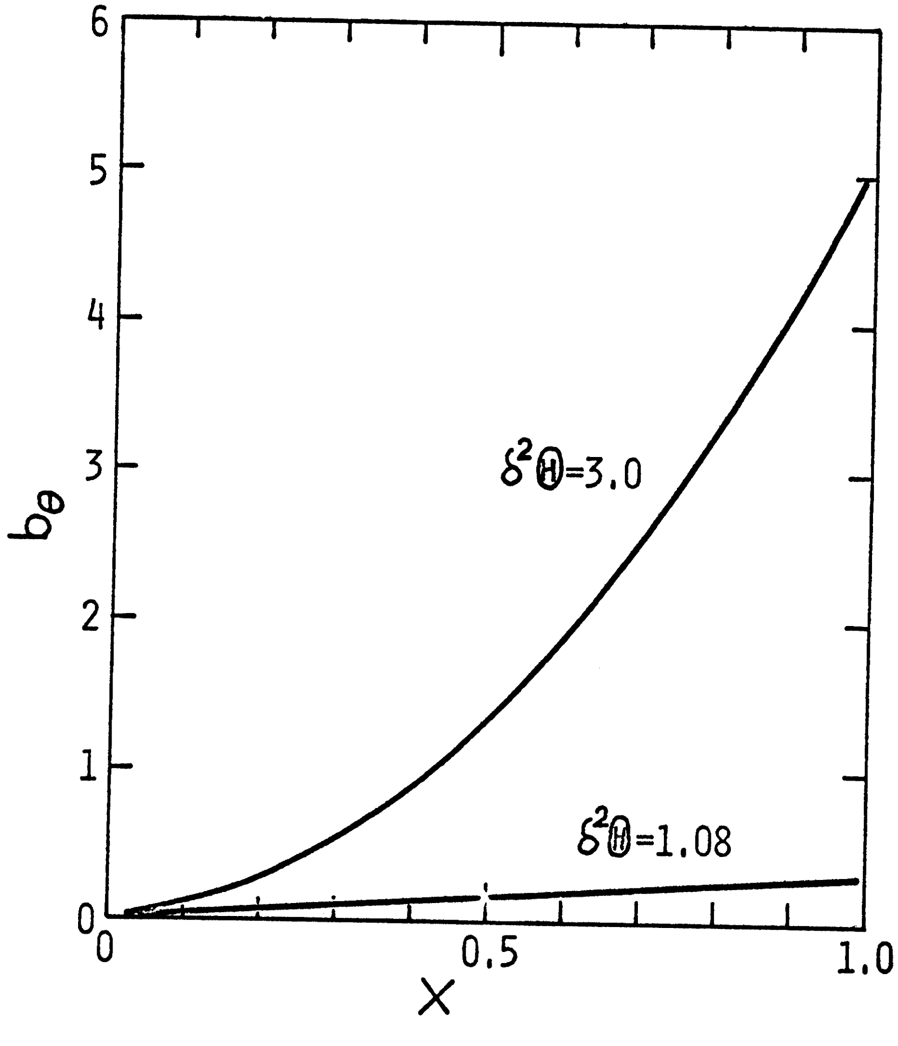


Fig. 2