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Equivalent Circuit for Ion Cyclotron Waves

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Abstract

Ion cyclotron waves are treated with the use of equivalent circuits, whose resonant frequency represents the dispersion relation. Convective energy losses and damping of the waves are represented in terms of the circuit resistances. The relation between the value of maximum loading and the resonance width in coupling is discussed from the equivalent circuit model.

1. Introduction

Theoretical analyses of the dispersion relation and energy relations for the ion cyclotron waves have been systematically made in a previous report.[1] In the analyses, effects of finite electron mass were taken into consideration. The energy flow and the damping of the wave were also analyzed.

In order to summarize these relations conveniently, here an idea of equivalent circuit is introduced. In the circuit, plasma parameters are expressed by circuit constants which are the inductance, capacitance, and resistance.

In the following consideration, we shall use the dispersion relation of ion cyclotron wave in a cold plasma of single ion species. Energy losses due to collisions are included as a correction to it.

In this report, we are concerning with the wave excitation by an induction coil of Stix type, where the wave is the axial symmetry mode. An equivalent circuit with the excitation coil system is also introduced to give the plasma loadings. In this case, it is necessary to consider the coupling coefficient between the exciting system and the wave.

Basic characteristics of the coupling resonance obtained from the equivalent circuit model will be compared with the experimental ones of the QP device.

2. Basic Concept of the Equivalent Circuit

Using the dielectric tensor, the electric induction $\mathbb{D}$ in a plasma can be expressed in terms of the electric field $\mathbb{E}$:

$$\mathbb{D} = \begin{pmatrix} \varepsilon_t & \varepsilon_h & 0 \\ -\varepsilon_h & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_\ell \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} . \tag{1}$$

where the static magnetic field $\mathbb{B}_0$ is assumed to be in the z direction and the wave vector $\mathbb{k}$ to be in the xz plane. The electric induction $\mathbb{D}$ is expressed as

$$\mathbb{D} = \mathbb{E} + i\frac{4\pi}{\omega}\mathbb{j} , \tag{2}$$

where $\mathbb{j}$ is the plasma current associated with the motion of the charged particles in the plasma and may be given by

$$j_x = \{\frac{\omega}{4\pi i}(\varepsilon_t - 1)\}E_x + \{\frac{\omega}{4\pi i}(\varepsilon_h)\}E_y ,$$

$$j_y = \{\frac{\omega}{4\pi i}(-\varepsilon_h)\}E_x + \{\frac{\omega}{4\pi i}(\varepsilon_h - 1)\}E_y , \tag{3}$$

$$j_z = \{\frac{\omega}{4\pi i}(\varepsilon_\ell - 1)\}E_z .$$

Thus, $\frac{\omega}{4\pi i}\varepsilon$ may be regarded as a specific conductivity of the plasma.

An approximated dispersion relation for the L_1-mode, which has been introduced in the previous report is

$$\varepsilon_t - N_\parallel^2 = 0, \tag{4}$$

where
$$\varepsilon_t \equiv \frac{\gamma}{1-\Omega^2},$$

$$N_\parallel^2 \equiv \left(\frac{k_\parallel c}{\omega}\right)^2,$$

$$\gamma \equiv \frac{\Pi_i^2}{\Omega_i^2},$$

$$\Omega \equiv \frac{\omega}{\Omega_i},$$

and Π_i is the ion plasma frequency, Ω_i is the ion cyclotron frequency, $k_\parallel$ is the wave number along the static magnetic field, and c is the light speed. The equation (4) may be rewritten in the following form with an introduction of an artifical factor $\frac{i4\pi\alpha}{\omega}j$:

$$j\left[\frac{i4\pi\alpha}{\omega}\left(\frac{1}{\varepsilon_t} - \frac{1}{N_\parallel^2}\right)\right] = 0, \tag{5}$$

where α^{-1} is an arbitrary constant which has the dimension of length. From equation (5), we have

$$-j\left[i\omega\left\{\left(\frac{4\pi\alpha}{\Omega_i^2\gamma}\right) + \left(\frac{4\pi\alpha}{k_\parallel^2 c^2}\right)\right\} - \frac{1}{i\omega\left(\frac{\gamma}{4\pi\alpha}\right)}\right] = 0. \tag{6}$$

This equation may be written as

$$i\omega(L + L_k) - \frac{1}{i\omega C} = 0, \tag{7}$$

where

$$L \equiv \frac{4\pi\alpha}{\Pi_i^2} ,$$

$$L_k \equiv \frac{4\pi\alpha}{k_{\parallel}^2 c^2} ,$$

$$C \equiv \frac{\gamma}{4\pi\alpha} .$$

The equation (7) is the resonance condition of the circuit shown in Fig.1. Thus the dispersion relation can be represented by the equivalent circuit. Here we may note that Eq.(6) is equivalent to the wellknown Kirchhoff's second law in an electric circuit which has no electro-motive force in it.

The arbitrary constant α may be so chosen that the voltage drops of circuit elements relate to electric fields in the plasma in a simple way. Here we shall choose α as

$$\alpha = \frac{\pi \ell}{\lambda^2} , \tag{8}$$

where λ is the wavelength along the static magnetic field, and ℓ is the length of the plasma column to which an equivalent circuit is concerned. Then L, L_k, and C are

$$L = \frac{k_{\parallel}^2 \ell}{\Pi_i^2} ,$$

$$L_k = \frac{\ell}{c^2} , \tag{9}$$

$$C = \frac{\gamma}{k_{\parallel}^2 \ell} .$$

Thus we have obtained some basic parameters of the equivalent circuit. For a dissipative plasma, some resistances should be added in the circuit.

3. Equivalent Circuit for Cold Plasma

More exactly, the dispersion relation for the L_1-mode is given by

$$\varepsilon_{\ell}(\varepsilon_t - N_{\parallel}^2) - \varepsilon_t N^2 = 0, \tag{10}$$

and is written as

$$\frac{1}{\varepsilon_t} - \frac{1}{N_{\parallel}^2} + \frac{1}{\varepsilon_{\ell}}\left(\frac{N_{\perp}^2}{N_{\parallel}^2}\right) = 0. \tag{11}$$

This equation is equivalent to the following relation:

$$i\omega(L + L_k + L_z) + \frac{1}{i\omega C} = 0, \tag{12}$$

where $\quad L_z \equiv \mu L_1^2 L, \quad L_1^2 \equiv \dfrac{N_{\perp}^2}{N_{\parallel}^2 - 1} .$

The resonance frequency ω of the equivalent circuit is

$$\Omega^2 = \frac{1}{\mu L_1^2 + 1 + \Omega_0^2} \tag{13}$$

where $\Omega_0^2 \equiv \frac{\Upsilon}{(N_\parallel^2 - 1)\Omega^2}$

This is also the frequency of resonace coupling of the L_1-mode for $L_1 >> 1$.

For the case that $k_\parallel \to \infty$ and $\mu = 0$, the resonance condition becomes $\omega^2 = (LC)^{-1}$, then L and C are associated with the cyclotron motion of the ions. For $\mu = 0$ and finite value of $k_\parallel$, the inductance L_k appears, which corresponds to the shift in coupling frequency of the wave from the cyclotron frequency. L_k is then associated with the wave propagation. Because L_z include μ, this is associated with electron motion along the z direction.

Here we shall consider energy relations in the case that an induction coil of Stix type of length ℓ is used to generate an ion cyclotron wave. The excited wave propagates away along the z direction, then, the stored energy in the excitation region is lost by this energy flow whose group velocity is V_g. The time constant τ of this energy loss is given by

$$\tau_\ell = \frac{\text{stored energy}}{\text{energy flow rate}} = \frac{\int_0^\ell z^2 dz}{|z^2 V_g|_{z=\ell}} = \frac{\ell}{3V_g}, \qquad (14)$$

where we have assumed that the excited wave amplitude is proprtional to z under the induction coil. This energy loss will result in some broadening of the coupling resonance. The broadening Δf_ℓ in frequency will be

$$2\Delta f_{\ell} = \frac{1}{2\pi\tau} = \frac{3V_g}{2\pi\ell} , \tag{15}$$

where $$V_g \simeq (1 - \Omega^2)\left(\frac{\omega}{k_{\shortparallel}}\right).$$

A more rigorous treatment of the broadening in coupling resonance leads to a shape factor S which is given by (reference 1, equation, 55)

$$S = \frac{2\{1 - \cos\Delta k_{\shortparallel}\ell\}}{\{\Delta k_{\shortparallel}\ell\}^2}.$$

The half-width of the coupling resonance is given by

$$\Delta k_{\shortparallel}\ell \simeq 2.78. \tag{16}$$

The corresponding broadening in frequency is

$$\Delta\omega \simeq \frac{d\omega}{dk_{\shortparallel}}\Delta k_{\shortparallel} = V_g\Delta k_{\shortparallel} . \tag{17}$$

Equation (17) may be written in the form:

$$\Delta f_{\ell} = \frac{2.78}{2\pi\ell}V_g . \tag{18}$$

We see that the both broadening coincide with a numerical factor of about two.

To give this broadening, a resistance R_k should be added in the equivalent circuit as shown in Fig.2. From equation (18), the value of R_k should be

$$R_k \sim \frac{2\times 2.78}{\Omega^2 c^2 k_{\parallel}} \frac{1}{d^2} , \tag{19}$$

where $$d^2 \equiv \frac{\Omega_0^2}{1 - \Omega^2} \sim \frac{\Omega_0^2}{\Omega^2(\mu L_1^2 + \Omega_0^2)} .$$

4. Equivalent Circuit for Warm Plasma.

In case of a warm plasma, there are energy losses due to cyclotron damping and collisional dampings. In order to represent these energy losses, certain resistances should be added in the equivalent circuit as is shown in Fig.3.

R_c; cyclotron damping.

This damping will be essential if the wave frequency is very close to the ion cyclotron frequency. This is the case of a magnetic beach. In a usual condition of wave excitation, this damping may be ignored because the wave frequency is not so close to the cyclotron frequency.

R; damping due to ion-ion collisions.

The ion motions associated with the cyclotron wave are disturbed by collisions with different ion species.

R_z; damping due to electron-ion collision.

Because there is a large plasma current along the z direction, disturbance of the electron motion causes wave damping. This may be regarded as a joule loss due to the plasma current j_z.

5. Equivalent Circuit with Exciting System

We shall here consider an equivalent circuit with exciting system in which a Stix type induction coil is used, being combined with a resonant capacitor. The equivalent circuit in this case is shown in Fig.4. Where L_e, C_e, and R_e refer respectively to the inductance, capacitance, and resistance of the exciting system. L_p, C_p, and R_p are the circuit constants associated with the ion cyclotron wave and

$$L_p = L + L_k + L_z \;,$$

$$C_p = C \;, \tag{20}$$

$$R_p = R_k + R + R_z + R_c \;.$$

The plasma loading of the exciting system may be expressed by a figure of merit Q which is defined by

$$Q \equiv \frac{\omega L_e I_e^2}{R_p I_p^2} \;. \tag{21}$$

In the case of cold plasmas, this Q has the following value (reference 1, equation 62);

$$Q \simeq \frac{2I_1(k_{\parallel}s)}{k_{\parallel}\ell K_1(k_{\parallel}s)} \{k_{\parallel}PK_1(kP)\}^2 L_1^2 Me \tag{22}$$

$$\sim \frac{k_{\parallel} s^2 L_1^2}{\ell} \left(\frac{4}{C_0^2 d^2}\right) , \qquad (\text{for } \mu L_1^4 >> \Omega_0^2) ,$$

where I_1 (x) and K_1 (x) are modified Bessel functions, p is the plasma radius, L_1 is the ratio $k_\perp$ to $k_\parallel$, M_e is a factor which depends on μ, L_1, and Ω_0^2, and is unity for $\mu = 0$. Here we have used the boundary condition of $J_1(c_0) = 0$, where $c_0 = k_\perp p$.

6. Summary and Characteristics of the Equivalent Circuit

Relation between plasma parameters and circuit parameters are summarized in Table 1. Numerical values for the QP plasma are also given in the table.

In the following, we shall consider some basic characteristics of the equivalent circuit. The broadening Δf_t given by the total resistance R_p is

$$\frac{\Delta f_t}{f} = \frac{1}{2} \omega C R_p$$

$$= \frac{1}{2} \frac{\Omega}{\Omega_i L_k} \left[\frac{R+R_z}{\Omega^2} (\Omega_m^2 - \Omega^2) + R_k d^2 (1-\Omega^2)\right] , \qquad (23)$$

where $\qquad \Omega_m^2 \equiv 1 - \mu L_1^2 \Omega^2$.

Thus there are two components, one is proportional to $(1-\Omega^2)$ and the other is proportional to $(\Omega_m^2 - \Omega^2)$ or plasma density. In case of cold plasmas $\mu=0$, we see that Δf_t is proportional to the plasma density, while the maximum loading is independent of the density.

The integral intensity of the coupling resonance is

$$\int_{-\infty}^{\infty} P_2(\omega)\,d\omega = Q_1 P_1 \int_{-\infty}^{\infty} K_c^2 Q_2 \frac{\left(\frac{1}{2Q_2}\right)^2}{(1-\Omega^2)^2 + \left(\frac{1}{2Q_2}\right)^2}\, d\Omega \sim \pi Q_1 P_1 K_c^2, \quad (24)$$

where Q_1, Q_2 are Q factors of the primary and the secondary circuits, P_1 is the power dissipation in the primary circuit, and K_c is the coupling constant between the primary and the secondary circuits. Here we have assumed that the capacitance in the exciting circuit is always so adjusted as to satisfy the resonance condition with a given P_1. Thus the integral is independent of the value of Q_2 and is proportional to the plasma density. The concept of integral intensity is quite similar to that of gain-bandwidth product[2] in an amplifier circuit, or spectroscopic stability[3] of spectral line in optics.

Assuming the invariance of the integral intensity, we obtain the maximum loading P_2 for a generarized case. That is

$$P_2 \simeq \frac{\int P_2(\omega)\,d\omega}{\left(\frac{\Delta f_t}{f}\right)}$$

$$\sim P_2^c \left\{ \frac{d^2}{1 + \left(\frac{R + R_z}{R_k}\right)} \right\} , \tag{25}$$

where P_2^c is the maximum loading for a cold plasma with $d^2=1$. The dependence of Δf_ℓ and P_2 on Ω is shown in Fig.5, where Δf_ℓ is the dominant term in Δf_t. Experimental loading characteristics obtained by using the QP plasma are shown in Fig.6. We see that experimental results coincide with the theoretical prediction qualitatively. Quantitatively, however, there are some discrepancies between the two. In actual plasmas, there exists some inhomogeneity in the density profile across the plasma column, though we have assumed the homogeneous profile with a sharp boundary in the denstiy profile at the convenience of the theoretical analyses. This inhomogeneity will cause broadening whose order of magnitude is comparable with the original one. If this inhomogeneous effect is taken into consideration, observed characteristics of the loading is considered to be reasonable.

7. Conclusion

The dispersion relation of ion cyclotron waves are represented by the resonance frequency of an equivalent circuit, in which ion motions are expressed by a combination of inductance and capacitance. Effect of wave propagation on resonance frequency is represented by an inductance L_k.

Effect of finite electron mass is also taken into consideration and is represented by L_z in the equivalent circuit. The energy relations such as wave energy being propagated away along the z direction are considered in association with resistances in the circuit. The values of the risistances are calculated from the coupling effeciency of the exciting system and damping of the wave. Though, the equivalent circuit, thus obtained is an approximate one, it sill be usefull to understand the behavior of plasma loading in experiments. As an example, loading characteristics obtained in the QP experiments are presented with the connection of the equivalent circuit.

Figure Captions

Fig.1. Equivalent circuit for an ion cyclotron wave in a homogeneous cold plasma whose dispersion relation is approximated to $N_{\parallel}^{2} - \varepsilon_{t} = 0$.

Fig.2. Equivalent circuit for an ion cyclotron wave in a cold plasma which is excited by an induction coil of length ℓ. The resistance R_k represents the energy loss due to wave propagation.

Fig.3. Equivalent circuit for a warm plasma. The resistances R_c, R, and R_z represent the energy losses due to cyclotron damping, ion-ion collision damping, and electron-ion collision damping of the wave respectively.

Fig.4. Equivalent circuit for an exciting system of ion cyclotron waves.

Fig.5. The dependences of the broadening Δf_{ℓ} and the value of maximum loading against Ω. Where Δf_{ℓ} associated with the wave propagation may be regarded as the total broadening Δf_t, because this is the dominante broadening.

Fig.6. Experimental results obtained by the QP plasma. The broadening and the maximum loading are protted against Ω.

References

1) K. Matsuura and H. Toyama, Research Report of the Institute of Plasma Physics, Nagoya University, IPPJ-1973).

2) G. H. Valley, Jr. and H. Wallman, Vacuum Tube Amplifiers (McGraw-Hill, New York, 1948).

3) J. H. Van Vleck, The Theory of Electric and Magnetic Susceptibilities (Clarendon Press, Oxford, 1932), p.111; G. H. Townes and A. L. Shawlow, Microwave Spectroscopy (McGraw-Hill, New York, 1955), P.344.

Table 1. Circuit parameters of the equivalent circuit.

Circuit parameter	Form (CGS-Gauss)	Numerical values (practical units)*	Typical vales for QP plasma**	Voltage across the elements (power or VA)	Notes
L	$\frac{k_{\parallel}^2 \ell}{\Pi_i^2}$	$2.05\times10^{-5}\frac{A_i \ell}{\lambda^2}(\frac{10^{12}}{n_i})$ (H)	8.19×10^{-5} (H) $\omega L=18.8\,(\Omega)$	505 (V) (13,600 VA)	$L=\frac{1}{\Omega_0^2}L_k$
C	$\frac{\Pi_i^2}{\Omega_i^2 k_{\parallel}^2 \ell}$	$1.24\times10^{-9}\frac{\lambda^2}{\ell f^2 A_i}(\frac{n_i}{10^{12}})$ (F)	1.93×10^{-9} (F) $(\omega C)^{-1}=22.6\,(\Omega)$	608 (V) (16,400 VA)	$(\omega C)^{-1}=\omega(L+L_k+L_z)$
L_k	$\frac{\ell}{c^2}$	$1\times10^{-9}\ell$ (H)	1×10^{-7} (H) $\omega L_k=2.29\,(\Omega)$	61.7 (V) (1,660 VA)	
L_z	$\frac{\nu^2 \ell}{\Pi_e^2}$	$4.15\times10^{-9}\frac{\ell}{p^2}(\frac{10^{12}}{n_i})$ (H)	6.63×10^{-8} (H) $\omega L_z=1.52\,(\Omega)$	40,9 (V) (1,100 VA)	$L_z=\mu L_1^2 L$
R	$\sim\frac{2}{\Omega^2}L\nu_{ij}$	$2.95\frac{\ell Z_i^4 \ln\Lambda}{\Omega^2\lambda^2 T^{3/2}(\ \ v)}(\frac{A_i+A}{A_i A_J})^{1/2}(\frac{n_j}{n_i})\,(\Omega)$	$1.39\times10^{-1}\,(\Omega)$ $n_j/n_i=0.5$, $T=3$eV	3.74 (V) (101 W)	
R_k	$\sim\frac{h\omega}{\Omega^2 c^2 k_{\parallel} d^2}$	$5.56\times10^{-3}\frac{\lambda f}{\Omega d^2}\,(\Omega)$	2.03 (Ω)	54,7 (V) (1,470 W)	$h\equiv2\times2.78$
Q_2	$\frac{c^2 k_{\parallel}^3 \ell d^2}{h\Pi_i^2}$	$2.31\times10^{4}\frac{\ell d^2}{\lambda^3}(\frac{10^{12}}{n_i})$	1.11×10^{1}		$Q_2=\frac{1}{R_p\omega C}$
Q	$\frac{4c_0^2 S_0^2}{\ell k_{\parallel}^3 p^4 d^2}$	$2.37\times10^{-1}\frac{\lambda^3 S_0^2}{\ell p^4 d^2}$	6.79×10^{2}		$Q=\frac{\omega L_e I_p^2}{R_p I_e^2}$, $S_0\equiv\frac{2I_1(ks)}{k_{\parallel}^2 K_1(ks)}$
M	$\frac{\sqrt{h}k_{\parallel}p^4\sqrt{\ell L_e}}{2C_0 S_0 \Omega c}$	$6.11\times10^{-8}\frac{p^2\sqrt{\ell L_e}}{\lambda\Omega S_0}$	$1.14\times10^{-8}\sqrt{L_e(\mu H)}$ (H)		$M\sim\sqrt{\frac{L_e R_p}{\omega Q}}$

Circuit parameter	Form (CGS-Gauss)	Numerical values (practical units) *	Typical Vales for QP plasma **	Voltage across the elements (power or VA)	Notes
K_c^2	$\frac{hp^4\Pi_i^2}{4C_0^2S_0^2c^2}$	$1.83\times10^{-4}\ \frac{p^4}{S_0^2}\ (\frac{n}{10^{12}})$	1.32×10^{-4}		$K_c^2=\frac{M^2}{L_eL_p}$
P_2^c	$\frac{\ell k_{\shortparallel}^3p^4d^2}{4C_0^2S_0^2}\ Q_1P_1$	$4.22\ \frac{\ell p^4d^2}{S_0^2\lambda^3}\ Q_1P_1$	$1.47\times10^{-1}P_1$		$P_2^c=R_pI_p^2$ $\sim K_c^2Q_1Q_2P_1$
I_p	$\frac{k_{\shortparallel}^2p^2d^2\Omega c\ \sqrt{\ell Q_1P_1}}{2C_0S_0\sqrt{\omega h}}$	$2.76\times10^1\ \frac{p^2d^2\sqrt{\Omega\ell Q_1P_1(w)}}{\lambda^2S_0\sqrt{f}}$ (A)	$2.69\times10^{-1}\sqrt{P_1(w)}$ (A)	26.9 (A)	$I_p=\frac{P_2^c}{R_p}$
R_z	$\sim\frac{\nu^2\ell}{4\pi}\ \eta$	$6.11\times10^{-3}\ \frac{\ell\ln\Lambda}{p^2T_e^{3/2}}$ (Ω)	1.88×10^{-1} (Ω)	5.06 (V) (136 w)	$R_k=\frac{1}{\omega CQ_2}$

* ℓ; cm
λ; cm
p; cm
f; MH_z

$$d^2 \equiv \frac{\Omega_0^2}{1-\Omega^2} \sim \frac{\Omega_0^2}{\mu L_1^2+\Omega_0^2}$$

** $n = 10^{12}/cc\ (H^+)$, $\lambda = 50$ cm, $\ell = 100$ cm, $P_1 = 10^4$ W

$T_e = T_i = 3$ eV, $f = 4\ MH_z$, $Q_1 = 100$

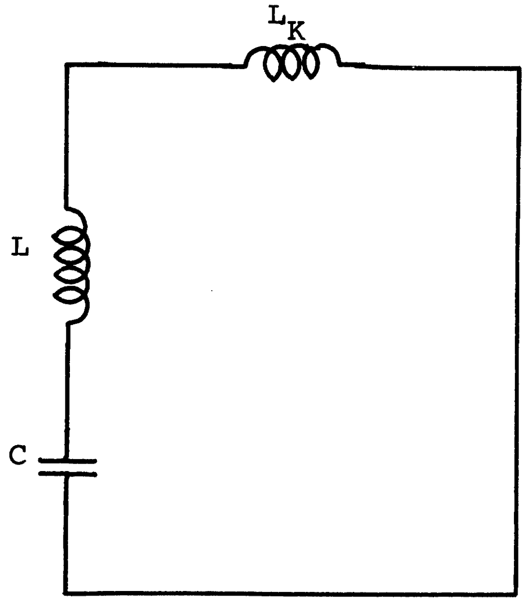

Fig. 1

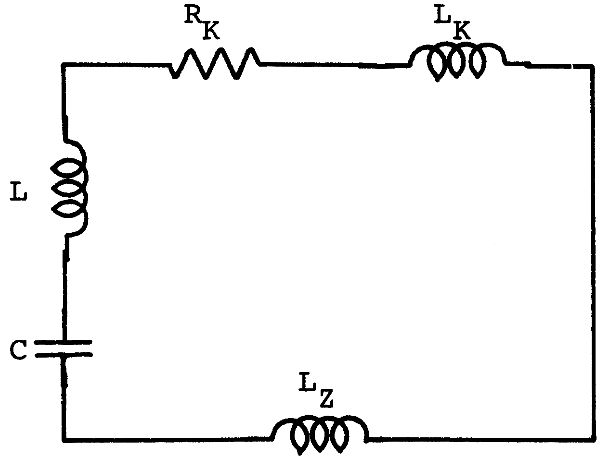

Fig. 2

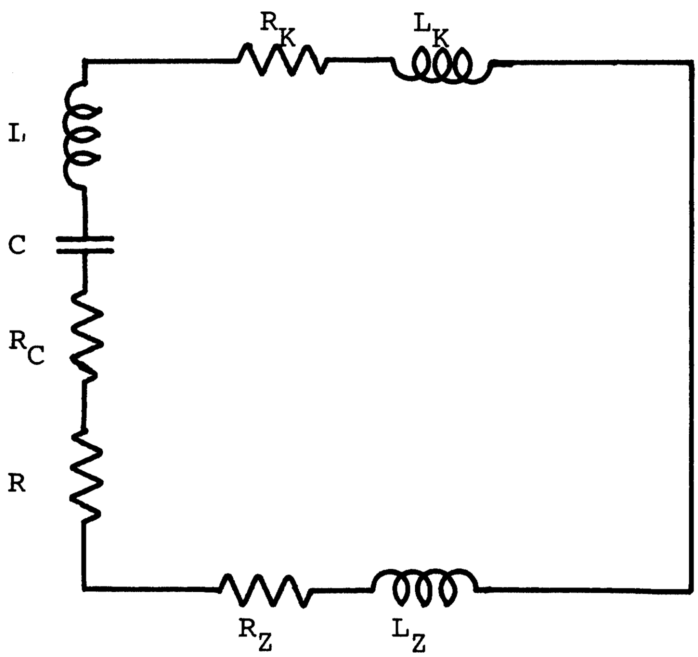

Fig. 3

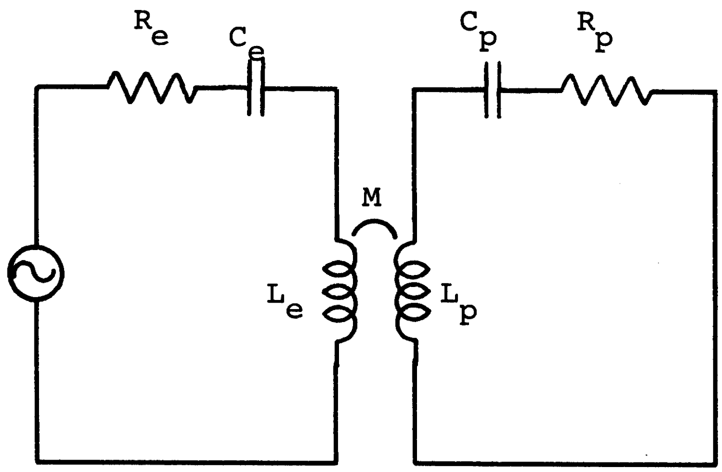

Fig. 4

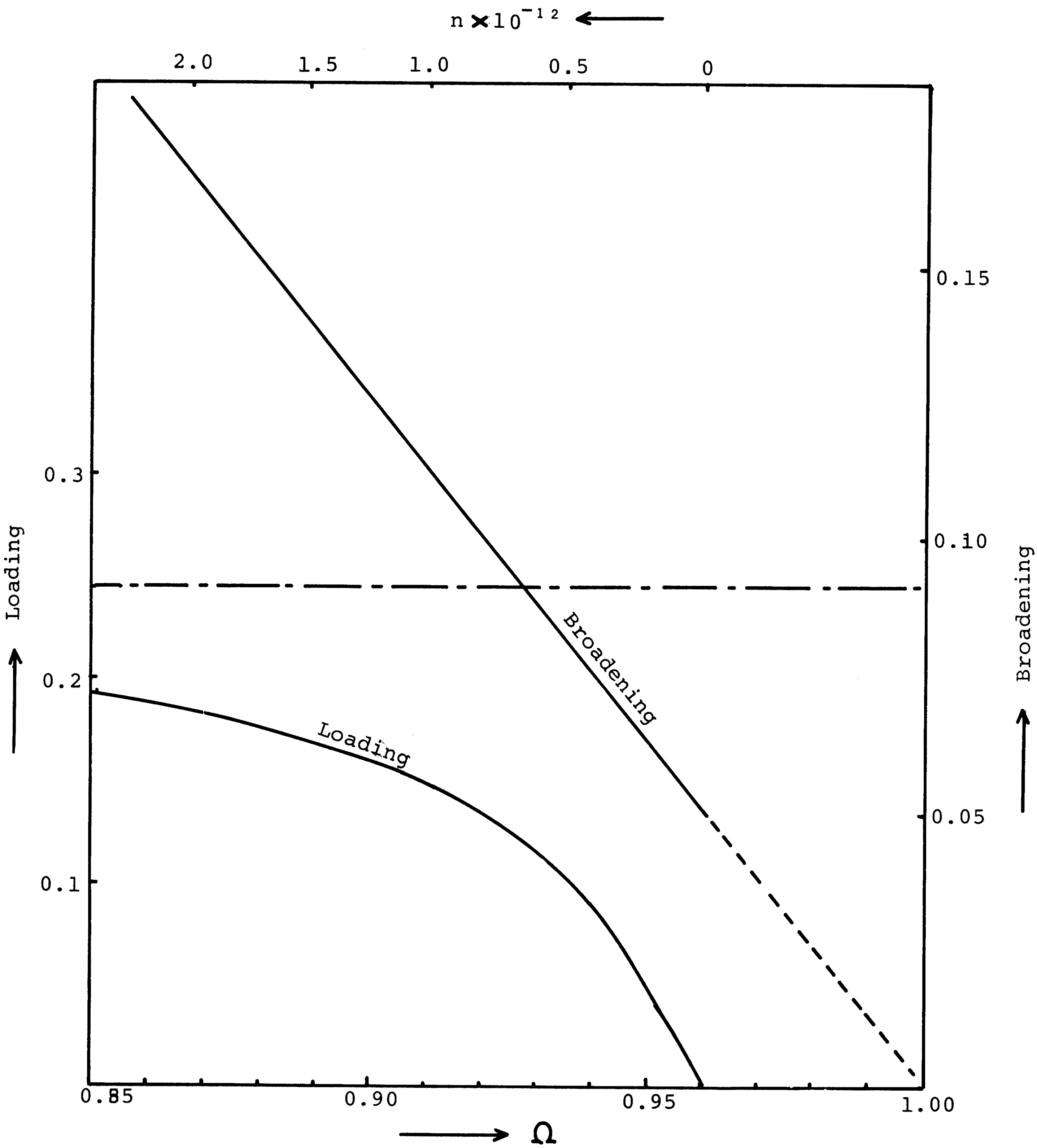
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Fig. 5

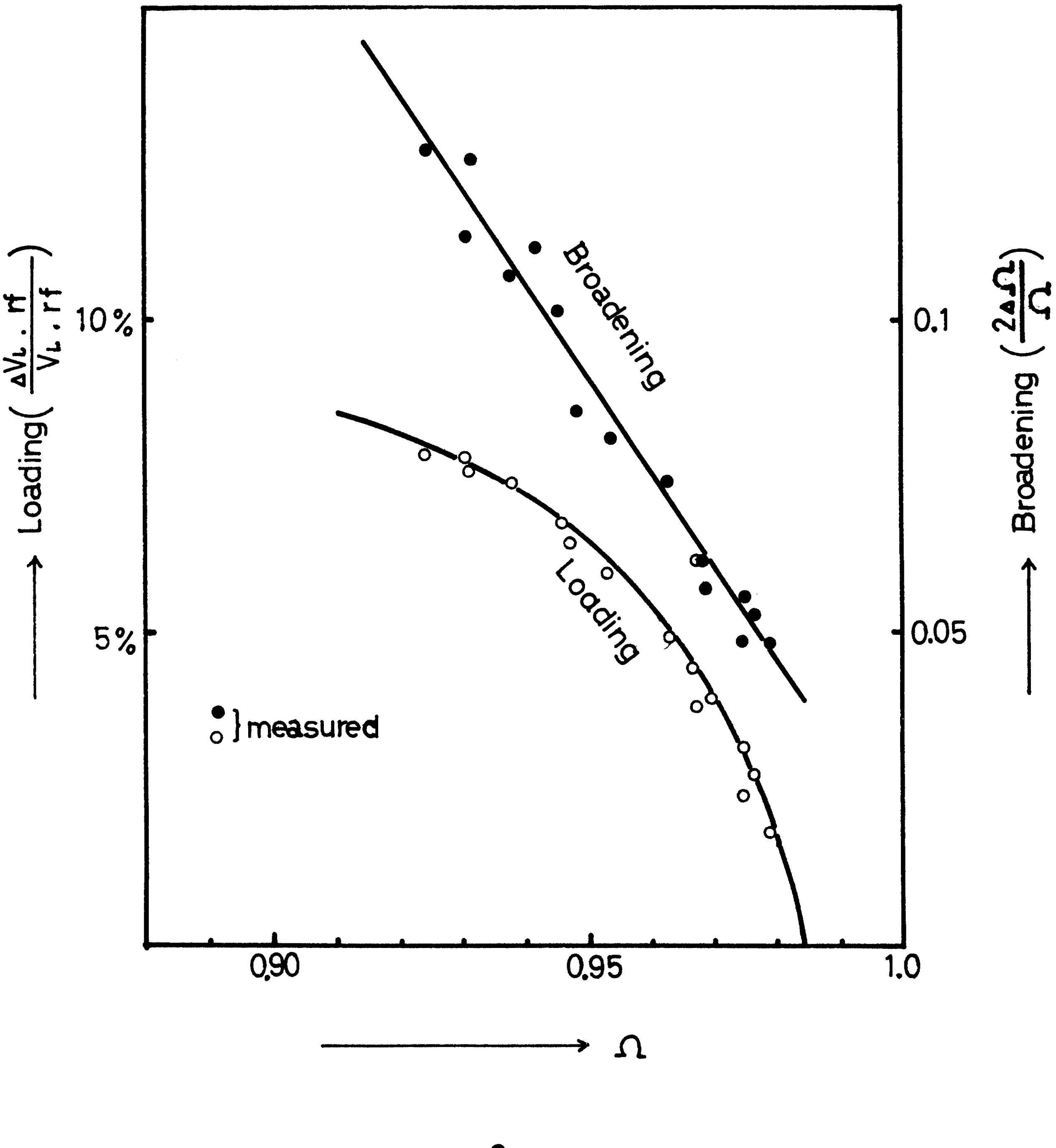
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Fig. 6