INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

RESEARCH REPORT

NAGOYA, JAPAN

ON TOKAMAK EQUILIBRIUM

Akihiko Yoshii*, Masahiro Wakatani**

and

Tsuneo Amano*

IPPJ-161 May 1973

Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan.

Permanent Address:

* Plasma Physics Laboratory, Faculty of Engineering, Osaka University, Suita, Osaka.

** Department of Electronic Engineering, Faculty of Engineering, Kyoto Univeristy, Kyoto.

Abstract

Two probelms on tokamak equilibrium are considered. One problem concerns with the maintenance of the toroidal equilibrium by means of the field due to external currents. The other problem concerns with the toroidal equilibrium of a high pressure tokamak plasma with β_θ (poloidal beta) >> 1. We adopt an expansion in fractional power of the inverse aspect ratio ε. We take the ordering $\beta_\theta \sim \varepsilon^{k-1}$ where $0 < k < 1$, and show that the equilibrium can be solved consistently to any order in ε by appropriately choosing the coordinates. We specifically treat the case $k = 1/3$ ($\beta_\theta \sim \varepsilon^{-2/3}$) and make the calculation up to the order $\varepsilon^{3k}(\varepsilon)$, thereby including the contributions from ellipticity and triangularity of the magnetic surfaces. The shape of the conducting shell is weakly deformed from a circle to study the effects of non-circularity of the shell.

§1. Introduction

The equilibrium of toroidal plasma in axi-symmetric tokamak type devices has been first studied by Shafranov in 1957.[1-3] This theory has been proved satisfactory in the analysis of the experiments when the plasma pressure is relatively low and $\beta_\theta \lesssim 1$. Here β_θ means the poloidal beta defined by $\beta_\theta = 8\pi p/B_a^2$ where p is the average plasma pressure and B_a is the value of poloidal field B_θ at the plasma boundary $r = a$.

The recent encouraging results obtained from the tokamak approach to the controlled thermonuclear program, stimulated new interest in the tokamak equilibrium problem.

One important problem concerns with the role of conducting shell in the tokamak equilibrium.[4] At present, the confinement time of the plasma in tokamak experiments is about several times 10 msec. For this time scale, the conducting shell acts as a perfect conductor, thereby assuring the equilibrium by means of the image current flowing in the shell. As the confinement time is increased, however, the shell cannot act as the perfect conductor and it is necessary to support the plasma equilibrium by use of the field due to external currents.

The other problem concerns with the toroidal equilibrium of a high pressure plasma with $\beta_\theta >> 1$.[5-7,13] In the present day tokamak's experiments, the plasma pressure is still low $\beta_\theta \lesssim 1$, and the original Shafranov's theory is fully applicable. Now it is expected that by the application of new heating techniques other than Joule heating, much higher plasma pressure with $\beta_\theta \gg 1$ can be achieved. From the point view of fusion reactors, it is desirable that beta should increase as high as possible, needless to say.

It is the purpose of this report to investigate these two problems systematically.

Our method is to find the curvilinear (r, θ, ϕ) coordinates in which the surfaces r = const. coincide with the magnetic surfaces. This point of view in solving the equilibrium has been pushed forward by Green, Johnson and Weimer.[8] The M.H.D. equilibrium equation

$$\left.\begin{array}{l} \vec{J}\times\vec{B} = \vec{\nabla}p \\ \vec{\nabla}\times\vec{B} = \vec{J} \end{array}\right\} \tag{1-1}$$

is solved by means of an asymptotic expansion in the inverse aspect ratio $\varepsilon \simeq a/R$.

The boundary conditions imposed to Eq.(1-1) on plasma boundary r = a are

$$\chi_I\big|_{r=a} = \chi_{II}\big|_{r=a} ,$$

$$\left.\frac{\partial\chi_I}{\partial r}\right|_{r=a} = \left.\frac{\partial\chi_{II}}{\partial r}\right|_{r=a} ,$$

where χ_I is the poloidal flux in the plasma region and χ_{II} is the flux outside the plasma. The first represents the continuity of the flux and the second represents the continuity of the contravariant θ component of the magnetic field. It should be noted that in our coordinates the plasma surface is simply represented by r = a = const., even when the shape of the plasma cross-section is much distoreted from circle. One of the important advantages of our method is that the fitting of the boundary conditions is much simplified. The other advantage is that the higher order calculations in the expansion of ε can be done in a straightforward way.

In Sec.2, we describe the general expressions for the equilibrium and formulation that is applied to the tokamak equilibrium problem with $\beta_\theta \lesssim 1$ and the results of Shafranov are recovered.

In §3-1, we study the maintenance of the toroidal equilibrium by means of the field due to external currents. Special attention is paid to the distortion of the magnetic surfaces from co-centric circles, due to the inhomogeneity of the maintaining external field. The shape of the plasma column cross-section is expressed in terms of the Fourier analyzed coefficients of surface density of the external current.

When the conducting shell is absent, it is required that maintaining field provides a stabilizing (focusing) mechanism.

In §3-2, we obtain the stability condition against axisymmetric displacement by calculating the change in the potnetial energy of the system.

In §4, we treat the toroidal equilibrium of a high pressure plasma with $\beta_\theta >> 1$. As has been pointed out by Shafranov and Yurchenko,[13] in the usual calculation of the toroidal equilibrium, the actual expansion parameter is not

ε (inverse of the aspect ratio), but $\varepsilon\beta_\theta$. When we consider the plasma with $\beta_\theta \lesssim 1$, this expansion scheme is quite satisfactory, since $\varepsilon\beta_\theta << 1$. However, we get into a difficulty when dealing with a plasma with $\beta_\theta >> 1$.

In this paper, we adopt an expansion in fractional power of ε to circumvent the above difficulty. We take the ordering $\beta_\theta \sim \varepsilon^{k-1}$, $\beta_\theta \sim \varepsilon^k$, where $0 < k < 1$, and show that the equilibrium can be solved consistently to any order in ε by appropriately choosing the coordinates. It is shown that our coordinates characterize the geometrical property of the problem more clearly than the coordinates used by Shafranov and Yurchenko. Then we specifically treat the case $k = 1/3$ ($\beta_\theta \sim \varepsilon^{-2/3}$) and make the calculation up to the order $\varepsilon^{3k}(\varepsilon)$, thereby including the contributions from ellipticity and triangularity of the magnetic surfaces. The shape of the conducting shell is weakly deformed from a circle to study the effects of non-circularity of the shell.[9-12] Several examples of the equilibrium are illustrated in Figs. 4-8.

§2. Equilibrium of a Plasma Column in a Perfectly Conducting Shell

Before getting into an argument with the equilibrium equation, we first describe a general formalism to the toroidal equilibria, which is convenient for application to tokamak systems.

Following to J. M. Green et al.[8] we use an axially symmetric curvilinear (r, θ, ϕ) coordinate system in which the magnetic surfaces coincide with the coordinate r=const. The transformation between the (r, θ, ϕ) and the cylindrical (X, ϕ, Z) coordinate systems can be written formally as (See Fig.1)

$$X = X(r,\theta) \;, \quad Z = Z(r,\theta) \quad .$$

In the (r, θ, ϕ) coordinates, the magnetic axis is at $r = 0$, or $X = R$ and $Z = 0$.

In this coordinate system, the contravariant components of the magnetic field are given by the formulas,

$$B^r = 0 \;, \quad B^\theta = \frac{RB_0}{X\mathcal{J}} f(r) \;, \quad B^\phi = \frac{RB_0}{X^2} g(r) \;,$$

where

$$\mathcal{J} = \frac{\partial X}{\partial \theta}\cdot\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\cdot\frac{\partial Z}{\partial \theta}$$

is the Jacobian. Then the magnetic field is given by

$$\vec{B} = B^{\theta}\vec{e}_{\theta} + B^{\phi}\vec{e}_{\phi} \qquad (2\text{-}1)$$

RB_0 was introduced to make f(r) and g(r) dimensionless and the factor $1/xg$ make (2-1) divergence free magnetic field. From (2-1), $\vec{J} = \mathrm{rot}\,\vec{B}$ can be written as

$$\vec{J} = J^{\theta}\vec{e}_{\theta} + J^{\phi}\vec{e}_{\phi} \quad ,$$

$$J^{\theta} \equiv -\frac{RB_0}{xg} g'(r) \; ,$$

$$J^{\phi} \equiv \frac{RB_0}{xg}\left[\frac{\partial}{\partial r}\frac{f}{xg}\left\{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2\right\} - \frac{\partial}{\partial \theta}\frac{f}{xg}\left(\frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial r}\right)\right] , \qquad (2\text{-}2)$$

where the prime denotes the derivative with respect to r.

Substituting (2-1) and (2-2) into Eq.(1-1), we have

$$\frac{p'}{R^2 B_0^2} + \frac{g g'}{x^2} + \frac{f}{xg}\left[\frac{\partial}{\partial r}\frac{f}{xg}\left\{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2\right\} - \frac{\partial}{\partial \theta}\frac{f}{xg}\left(\frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}\right)\right] = 0 \; . \qquad (2\text{-}3)$$

To proceed further we expand in powers of the inverse aspect ratio, $\varepsilon = a/R$.

In this section we adopt the ordering,

$$g = 1 + \varepsilon^2 g(r) \; , \quad f \sim \varepsilon \; , \quad p \sim \varepsilon^2 \quad ,$$

which corresponds to $q = rB_{\phi}/RB_{\theta} \sim 1$ and the poloidal $\beta(\beta_{\theta}) \sim 1$. To carry the calculation to the second order expnasion, we put

$$\left.\begin{aligned} X &= R - \varepsilon r\cos\theta - \varepsilon^2\Delta(r) + \varepsilon^3 E(r)\cos\theta \\ Z &= \varepsilon r\sin\theta + \varepsilon^3 E(r)\sin\theta \end{aligned}\right\} \quad (2\text{-}4)$$

Here $\Delta(r)$ shows the shift of the centers of the magnetic surfaces from the magnetic axis; $E(r)$ determines the ellipticity of the surfaces.

Inserting (2-4) into Eq.(2-3) and setting each Fourier coefficient equal to zero, we obtain the following equations;

$$\frac{P'}{B_0^2} + gg' + \frac{f}{r}(rf)' = 0 \quad , \qquad (2\text{-}5)$$

$$\Delta'' + \left\{2\frac{(rf)'}{rf} - \frac{1}{r}\right\}\Delta' - \frac{1}{R} - \frac{2(rf)'}{Rf} - \frac{2rgg'}{Rf^2} = 0, \qquad (2\text{-}6)$$

$$E'' + \left\{2\frac{(rf)'}{rf} - \frac{1}{r}\right\}E' - \frac{3}{r^2}E + \frac{(rf)'}{2rf}\left(\Delta'^2 + \frac{r^2}{R^2}\right)$$
$$+ \frac{r}{R^2} + \frac{\Delta'}{2R} - \frac{r\Delta''}{2R} - \frac{\Delta'^2}{r^2} + \Delta''\Delta' + \left(\frac{r}{2R} + \Delta'\right)\frac{rgg'}{R^2f^2} = 0 . \qquad (2\text{-}7)$$

In the fourth order there are terms independent of θ, but this terms do not contribute to the parameters Δ and E. If we expand to higher order, this contribution appears. This will be seen in Sec.5.

From Eq.(2-5) and Eq.(2-6), we obtain

$$\Delta'(r) = \frac{1}{Rrf^2}\int_0^r (f^2 - \frac{2rP'}{B_0^2}) r\,dr , \qquad (2-8)$$

$$\Delta(r) = \int_0^r \left\{ \frac{1}{Rrf^2}\int_0^r (f^2 - \frac{2rP'}{B_0^2}) r\,dr \right\} dr. \qquad (2-9)$$

If f(r) and p(r) are known in the plasma region $\Delta(r)$ and $\Delta'(r)$ are obtained directly. In the vacuum region p(r) and g(r) vanish and from Eq.(2-5)

$$f(r) = f_a \frac{a}{r} , \qquad (2-10)$$

where a denotes the plasma surface. Substitution of this into Eq.(2-9) gives

$$\Delta(r) = \Delta_a + \frac{r^2 - a^2}{4}\left(\frac{2\Delta_a'}{a} - \frac{1}{R}\right) + \frac{r^2}{2R^2}\ln\frac{r}{a} , \qquad (2-11)$$

where

$$\Delta_a' = \frac{1}{Raf_a^2}\int_0^a (f^2 - \frac{2rP'}{B_0^2}) r\,dr . \qquad (2-12)$$

Eq.(2-7) determines E;

$$\begin{aligned} E(r) = {} & \frac{E_a}{4}\left(\frac{r^3}{a^3} + \frac{3a}{r}\right) + \frac{aE_a'}{4}\left(\frac{r^3}{a^3} - \frac{a}{r}\right) \\ & + \frac{a^3}{64R^2}\left(1 + \frac{4R\Delta_a'}{a}\right)\left(\frac{r^3}{a^3} - \frac{a}{r}\right) \\ & - \frac{r^3}{16R^2}\left(1 + \frac{4R\Delta_a'}{a}\right)\ln\frac{r}{a} - \frac{r^3}{8R^2}\left(\ln\frac{r}{a}\right)^2 . \end{aligned} \qquad (2-13)$$

Let the equation of the surface of the conducting shell be

$$\left.\begin{aligned} X &= R_0 - b_0 \cos\theta' + E_0 \cos\theta' \\ Z &= b_0 \sin\theta' + E_0 \sin\theta' \end{aligned}\right\} \qquad (2\text{-}14)$$

which are written in the usual toroidal coordinates. This means that the shell has a form of weakly elliptic surface with radius b_0. If the shell is perfectly conducting, magnetic lines lie on the surface of the shell, hence

$$\left.\begin{aligned} R - R_0 &\simeq \Delta(b) \\ E_0 &\simeq E(b) \end{aligned}\right\} \qquad (2\text{-}15)$$

Let δ be the displacement of the plasma cross section from the center of the shell. δ can be written as follows

$$\delta = R - R_0 - \Delta_a \simeq \Delta_b - \Delta_a \quad ,$$

and from Eq.(2-11), we have

$$\delta = \frac{b^2}{2R}\left[\ln\frac{b}{a} + \left(1 - \frac{a^2}{b^2}\right)\left(\frac{R\Delta_a'}{a} - \frac{1}{2}\right)\right] \quad . \qquad (2\text{-}16)$$

Let us now consider a special plasma model; constant current density, and parabolic pressure distribution, i.e.,

$$
\left.\begin{aligned}
f(r) &= \frac{r}{a} f_a \\
p(r) &= B_0^2 f_a^2 \beta_\theta \left(1 - \frac{r^2}{a^2}\right)
\end{aligned}\right\} , \qquad (2\text{-}17)
$$

where

$$
\beta_\theta \equiv \frac{4}{B_0^2 f_a^2 a^2} \int_0^a r p(r) dr \quad . \qquad (2\text{-}18)
$$

β_θ is often called "poloidal β". From (2-17) and (2-12), it becomes

$$
\Delta_a' = \frac{a}{R} \left(\beta_\theta + \frac{1}{4}\right) \quad . \qquad (2\text{-}19)
$$

In this case

$$
\delta = \frac{b^2}{2R} \left[\ln \frac{b}{a} + \left(1 - \frac{a^2}{b^2}\right)\left(\beta_\theta - \frac{1}{4}\right) \right] . \qquad (2\text{-}20)
$$

Substituting (2-17) into Eq.(2-7) we can find the relation between E_a and E_a' as follows

$$
E_a' = \frac{E_a}{a} - \frac{a^2(3 + 16\beta_\theta^2)}{32R^2} ,
$$

then from (2-13) and (2-15) E_a takes the form

$$
\begin{aligned}
E_a = 2\left(\frac{b^3}{a^3} + \frac{a}{b}\right)^{-1} & \left\{ E_a + \frac{a^3}{128R^2}(8\beta_\theta^2 - 1)\left(\frac{b^3}{a^3} - \frac{a}{b}\right) \right\} \\
& + \frac{b^3}{16R^2}(2 + 4\beta_\theta) \ln \frac{b}{a} + \frac{b^3}{8R^2}\left(\ln \frac{b}{a}\right)^2 . \qquad (2\text{-}21)
\end{aligned}
$$

If $E_b \sim b^3/R^2$ and is larger than other terms, (2-21) is approximately

$$E_a \simeq 2\left(\frac{b^3}{a^3} + \frac{a}{b}\right)^{-1} \cdot E_b \quad . \qquad (2\text{-}22)$$

§3. Maintaining of a Plasma Column by the Magnetic Field of External Conductors

3-1. Equilibrium without conducting shell.

In the previous section, we have discussed the equilibrium of a plasma column in a nearly circular conducting shell. There, it has been sufficient to take the coordinates (2-4) and the magnetic surfaces have been described by ellipces with small eccentricity.

In this section we consider a tokamak equilibrium when the shell is absent. We maintain the equilibrium of a plasma by use of the field of external coils. This field is assumed to contain many Fourier components which distort the magnetic surfaces.

To consider this effect, the coordinates (2-4) is not sufficient and we shall take

$$\left.\begin{aligned} X &= R - \varepsilon r\cos\theta + \varepsilon^2 \xi(r,\theta) \\ Z &= \varepsilon r\sin\theta + \varepsilon^2 \eta(r,\theta) \end{aligned}\right\} \qquad (3-1)$$

where ξ and η are to be determined. Substituting it into Eq.(2-3), we note that terms in the lowest order in ε is the same as Eq.(2-5).

In the next order we obtain

$$F(r,\theta) + \left\{ \frac{1}{R} + 2\frac{(rf)'}{Rf} + 2\frac{rgg'}{Rf^2} \right\} \cos\theta = 0 , \qquad (3-2)$$

where

$$F(r,\theta) = \left\{ \frac{\partial^2 \xi}{\partial r^2} + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right)\frac{\partial \xi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \xi}{\partial \theta^2} \right\} \cos\theta$$

$$- \left\{ \frac{\partial^2 \eta}{\partial r^2} + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right)\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2} \right\} \sin\theta$$

$$- \frac{2}{r^2}\frac{\partial \xi}{\partial \theta}\sin\theta - \frac{2}{r^2}\frac{\partial \eta}{\partial \theta}\cos\theta$$

From the consideration of the symmetry with respect to the Z = 0 plane, it is evident that ξ and $\sqrt{\xi^2 + \eta^2}$ are even functions and η is an odd function. Therefore one is suggested to take

$$\left.\begin{aligned} \xi_c &= -\left\{ \sum_m S_m(r)\cos m\theta \right\}\cos\theta \\ \eta_c &= \left\{ \sum_m S_m(r)\cos m\theta \right\}\sin\theta \end{aligned}\right\} \quad (3-3)$$

Similar choice of coordinate is adopted by V. D. Shafranov et al.[13] In our notations, his choice is

$$\left.\begin{aligned} \xi &= -S_1(r) - \sum_{m=2} \left\{ S_m(r)\cos m\theta \right\}\cos\theta \\ \eta &= \sum_{m=2} \left\{ S_m(r)\cos m\theta \right\}\sin\theta \end{aligned}\right\}$$

On the other hand ξ_s and η_s defined by

$$\left.\begin{aligned} \xi_s &= \left\{ \sum_m S_m(r)\sin m\theta \right\}\sin\theta \\ \eta_s &= \left\{ \sum_m S_m(r)\sin m\theta \right\}\cos\theta \end{aligned}\right\} \quad (3-4)$$

satisfy the homogeneous equation $F(r, \theta) = 0$ identically. Now we choose ξ and η as follows;

$$\left.\begin{aligned} \xi &= \xi_c - \xi_s = -\sum_m S_m \cos(m-1)\theta \\ \eta &= \eta_c - \eta_s = -\sum_m S_m \sin(m-1)\theta \end{aligned}\right\} ,$$

then our coordinate becomes

$$\left.\begin{aligned} X &= R - \varepsilon r\cos\theta - \sum_m S_m \cos(m-1)\theta \\ Z &= \varepsilon r.\sin\theta - \sum_m S_m \sin(m-1)\theta \end{aligned}\right\} \quad (3\text{-}5) .$$

The coordinate in this form has been used by J. M. Green et al.[8] It can easily be seen that the coordinate (2-4) in Sec2 is the special case of (3-5).

The difference between (3-3) and (3-5) does not appear as far as the first order approximation is concerned. The difference will be discussed in the later section, where we shall be interested in tokamak equilibrium with $\beta_\theta >> 1$.

Substituting (3-5), we find the θ-dependent terms of Eq.(3-2)

$$\sum_{m=1} \left\{ S_m'' + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right) S_m' + \frac{1-m^2}{r^2} S_m \right\} \cos m\theta$$

$$= \left(\frac{1}{R} + 2\frac{(rf)'}{Rf} + 2\frac{r g g'}{Rf^2}\right)\cos\theta \quad . \quad (3\text{-}6)$$

From this equation it can be seen that $S_1(r)$ is the only essential factor to maintain the toroidal equilibrium, since the right hand side of Eq.(3-6) has only the m = 1 component. As will be shown later, the other terms $S_m(r)$ with $m \geqq 2$ is related to distortions of magnetic surfaces.

As g', p' = 0 and $f = f_a \frac{a}{r}$ in the vacuum region, the solution to Eq.(3-6) is

$$S_1(r) = S_{1a} + \frac{r^2 - a^2}{4}\left(\frac{2S_{1a}'}{a} - \frac{1}{R}\right) + \frac{r^2}{2R}\ln\frac{r}{a} \qquad (3-7)$$

for m = 1, and

$$S_m(r) = \frac{S_{ma}}{2m}\left\{(m-1)\frac{r^{m+1}}{a^{m+1}} + (m+1)\frac{a^{m-1}}{r^{m-1}}\right\}$$

$$+ \frac{aS_{ma}'}{2m}\left(\frac{r^{m+1}}{a^{m+1}} - \frac{a^{m-1}}{r^{m-1}}\right) \qquad (3-8)$$

for $m \geqq 2$.

Here we introduce a poloidal magnetic flux χ defined by

$$\chi \equiv \int \vec{B} \cdot d\vec{S}_p \quad , \qquad (3-9)$$

where $d\vec{S}_p$ is the poloidal surface element. In the (r, θ, ϕ) coordinates, χ is written by

$$\chi(r) = 2\pi R B_0 \int_0^r f(r) dr \quad . \qquad (3-10)$$

The poloidal magnetic field is determined by the relation

$$\vec{B_p} = \frac{1}{2\pi} \vec{\nabla}\phi \times \vec{\nabla}\chi \quad . \tag{3-11}$$

In the following consideration, it is convenient to introduce the toroidal coordinates (ξ, ψ, ϕ), defined by

$$\left.\begin{aligned} X &= \frac{R \sinh \xi}{\cosh \xi - \cos \varphi} \\ Z &= \frac{R \sin \varphi}{\cosh \xi - \cos \varphi} \end{aligned}\right\} \tag{3-12}$$

which we call "(R, ξ, ϕ), coordinates" hereafter.

In the vacuum region, χ must satisfy the equation

$$\frac{\partial}{\partial \xi} \frac{(\cosh \xi - \cos \varphi)}{\sinh \xi} \frac{\partial \chi}{\partial \xi} + \frac{\partial}{\partial \varphi} \frac{(\cosh \xi - \cos \varphi)}{\sinh \xi} \frac{\partial \chi}{\partial \varphi} = 0 \, . \tag{3-13}$$

The general solution of Eq.(3-13) is

$$\chi = \chi^{o} + \chi^{i} \quad , \tag{3-14}$$

where

$$\chi^{o} = C + \frac{\sinh \xi}{(\cosh \xi - \cos \varphi)^{1/2}} \sum_{n=0} C_n P^{1}_{n-\frac{1}{2}} (\cosh \xi) \cos n\varphi \, , \tag{3-15}$$

and

$$\chi^{i} = \frac{\sinh \xi}{(\cosh \xi - \cos \varphi)^{1/2}} \sum_{n=0} B_n Q^{1}_{n-\frac{1}{2}} (\cosh \xi) \cos n\varphi \, . \tag{3-16}$$

Here $P^1_{n-1/2}$ and $Q^1_{n-1/2}$ are the associated Legendre functions of first and second kinds, respectively. χ° and χ^i are the outer and inner solutions, respectively, and χ° diverges as ξ approaches infinite and χ^i diverges as ξ tends to zero. It should be noted that $\xi \to \infty$ as $X \to R$ and $Z \to 0$.

Now let the external maintaining coils lie on a toroidal surface

$$\left.\begin{aligned} X &= \frac{R_0 \sinh \xi_0}{\cosh \xi_0 - \cos \varphi} \\ Z &= \frac{R_0 \sin \varphi}{\cosh \xi_0 - \cos \varphi} \end{aligned}\right\} ,$$

where R_o and ξ_o are constants.

Let us now consider the three regions as bellow.

Region I: inside the plasma column .

Region II: the region between the plasma column and the coils.

Region III: outside the coils.

In the region I, χ_I has the form

$$\chi_I = 2\pi R B_0 \int_0^r f(r)\,dr \quad . \tag{3-17}$$

The poloidal flux χ_p due to the current in the plasma is obtained as the outer solution in the (R, ξ', ϕ') coordinates, and using Fock toroidal functions[14] we can expand it in the form

$$\frac{\chi_p}{2\pi R B_0} = C^{(2)} + C_0^{(2)}(\xi' + 2\ln 2 - 2)$$
$$+ C_0^{(2)}(\xi' + 2\ln 2 - 2)e^{-\xi'}\cos\varphi' - \sum_{m=1} C_m^{(3+m)} e^{m\xi'} \cos m\varphi', \tag{3-18}$$

where $C^{(n)}$'s are the constants whose magnitude are of the order ε^n.

In the same manner, the poloidal flux χ_e due to the current in the external coils is given by the inner solusion in the (R_o, ξ, ϕ) coordinates in the form

$$\frac{\chi_e}{2\pi R B_o} = \sum_{m=1} B_m^{(3-m)} e^{-m\xi} \cos m\varphi \quad . \qquad (3\text{-}19)$$

The new coordinates (R_o, ξ, ϕ) introduced here is related to the coordinates (R, ξ', ϕ') as, (See Fig.2)

$$\left.\begin{aligned} \frac{R \sinh \xi'}{\cosh \xi' - \cos \varphi'} &= \frac{R_o \sinh \xi}{\cosh \xi - \cos \varphi} \\ \frac{R \sin \varphi'}{\cosh \xi' - \cos \varphi'} &= \frac{R_o \sin \varphi}{\cosh \xi - \cos \varphi} \end{aligned}\right\} , \qquad (3\text{-}20)$$

or approximately related as

$$\left.\begin{aligned} e^{2\xi'} &\simeq e^{2\xi}\left(1 + \frac{\delta}{R} e^{\xi} \cos \varphi\right) \\ \xi' &\simeq \xi + \frac{\delta}{2R} e^{\xi} \cos \varphi \end{aligned}\right\} . \qquad (3\text{-}21)$$

Then χ_p can be rewritten in the new coordinate (R_o, ξ, ϕ) as

$$\begin{aligned} \frac{\chi_p}{2\pi R B_o} = {} & C^{(2)} + C_o^{(2)} (\xi + 2\ln 2 - 2) \\ & + C_o^{(2)} \left\{ (\xi + 2\ln 2 - 2) e^{-\xi} + \frac{\delta}{2R} e^{\xi} \right\} \cos \varphi \\ & - C_m^{(3+m)} e^{m\xi} \cos m\varphi \quad . \end{aligned} \qquad (3\text{-}22)$$

The solution in the region II is given by a sum of χ_p and χ_e :

$$\chi_{II} = \chi_p + \chi_e \quad . \tag{3-23}$$

This can be transformed into the (r, θ) coordinates, and becomes

$$\begin{aligned} \frac{\chi_{II}}{2\pi R B_0} &= C^{(2)} + C_0^{(2)} \left(\ln \frac{8R}{r} - 2 \right) \\ &- C_0^{(2)} \left(\frac{1}{r} \sum_{m=1} S_m \cos m\theta + \frac{r}{2R} \cos\theta \right) \\ &- C_0^{(2)} \left(\ln \frac{8R}{r} - 2 \right) \frac{r}{2R} \cos\theta \\ &- (-1)^m \sum_m C_m^{(m+3)} \left(\frac{2R}{r} \right)^m \cos m\theta \\ &+ (-1)^m \sum_m B_m^{(3-m)} \left(\frac{r}{2R} \right)^m \cos m\theta \quad . \end{aligned} \tag{3-24}$$

The unknown parameters C_n's and B_n's are determined from the boundary conditions

$$\chi_I \Big|_{r=a} = \chi_{II} \Big|_{r=a} \quad , \tag{3-25}$$

and

$$\frac{\partial \chi_I}{\partial r} \Big|_{r=a} = \frac{\partial \chi_{II}}{\partial r} \Big|_{r=a} \quad . \tag{3-26}$$

which represent the continuity of the poloidal flux and the continuity of the contravariant θ component of the magnetic field at the plasma surface, respectively. Then we get

$$C_0^{(2)} = -a f_a \tag{3-27}$$

$$C^{(2)} = \int_0^a f(r)\,dr + a f_a \left(\ln\frac{8R}{a} - 2\right) \tag{3-28}$$

$$C_1^{(4)} = -\frac{a f_a}{2R}\left(S_{1a} - \frac{a S_{1a}'}{2} + \frac{a^2}{4R}\right) \tag{3-29}$$

$$B_1^{(2)} = a f_a \left(\ln\frac{8R}{a} - \frac{3}{2} + \frac{R S_{1a}'}{a}\right) \tag{3-30}$$

and for m $\geqq$ 2

$$C_m^{(m+3)} = (-1)^m a f_a \left(\frac{a}{2R}\right)^m \frac{1}{2m}\left\{(m+1)\frac{S_{ma}}{a} - S_{ma}'\right\} \tag{3-31}$$

$$B_m^{(3-m)} = -(-1)^m a f_a \left(\frac{2R}{a}\right)^m \frac{1}{2m}\left\{(m-1)\frac{S_{ma}}{a} + S_{ma}'\right\} \tag{3-32}$$

In the region III, not far from the external coils, χ_{III} is obtained as the outer solution in the (R_0, ξ, ϕ) coordinates, then

$$\begin{aligned}\frac{\chi_{III}}{2\pi R B_0} &= A^{(2)} + A_0^{(2)}(\xi + 2\ln 2 - 2) \\ &\quad + A_0^{(2)}(\xi + 2\ln 2 - 2)e^{-\xi}\cos\varphi \\ &\quad - \sum_{m=1} A_m^{(3+m)} e^{m\xi}\cos m\varphi \end{aligned} \tag{3-33}$$

At the boundary ($\xi = \xi_o$) where the external conductors are located, we expand the surface density of the current in Fourier series as

$$I_\phi(\varphi) = \sum_m I_m \cos m\varphi \quad .$$

Then, the boundary condition at $\xi = \xi_o$ is

$$\chi_{II}\Big|_{\xi=\xi_o} = \chi_{III}\Big|_{\xi=\xi_o} \qquad (3-34)$$

$$I_\phi(\varphi) = \frac{\cosh\xi - \cos\varphi}{2\pi R}\left(\frac{\partial\chi_{II}}{\partial\xi} - \frac{\partial\chi_{III}}{\partial\xi}\right)\Big|_{\xi=\xi_o} \quad . \qquad (3-35)$$

From this condition, $I_\phi(\phi)$ becomes

$$I_\phi(\varphi) = -B_o \sum_{m=1} m B_m^{(3-m)} e^{(1-m)\xi_o} \cos m\varphi$$

$$\simeq \sum_{m=1} I_m \cos m\varphi \ , \qquad (3-36)$$

then

$$I_m = -mB_o B_m^{(3-m)} \left(\frac{b}{2R}\right)^{m-1} \quad . \qquad (3-37)$$

To obtain this, we assumed that $I_\phi(\phi)$ does not contain uniform current component I_o.

From Eq.(3-30) and Eq.(3-37) the dipole component of the external current, I_1, is uniquely determined and takes the form

$$I_1 = -B_0 a f_a \left(\ln \frac{8R}{a} - \frac{3}{2} + \frac{R}{a} S'_{1a} \right) . \qquad (3-38)$$

The other constants $B_m^{(3-m)}$ ($m \geqq 2$) can not be determined uniquely, unless we specify the shape of the magnetic surface at the plasma boundary by giving S_{ma}. Conversely, the shape of the magnetic surface adjust themselves so as to satisfy Eq.(3-37), if we give I_m arbitrarily.

Let us again consider the specific plasma model discussed in the §2, constant toroidal current density inside the plasma and parabolic pressure distribution, then inside the plasma column,

$$S_m(r) = S_{ma} \left(\frac{r}{a} \right)^{m-1} \qquad (3-39)$$

and

$$S'_{ma} = (m-1) S_{ma} / a \qquad (3-40)$$

for $m \geqq 2$.

From (3-32), (3-37), and (3-40), S_{ma} is obtained in the form

$$S_{ma} = (-1)^m \frac{a}{2R} \frac{I_m}{B_0 f_a} \left(\frac{a}{b} \right)^{m-1} . \qquad (3-41)$$

This shows that the shaping parameter S_{ma} become sufficiently small and can be neglected for large m and b. That is to say, the contribution of the higher Fourier components of external current is negligible, if the external conductors are located sufficiently far from the plasma surface.

3-2. Stability against Axi-symmetric Displacements.

In §3-1 we have considered the maintaining of equilibrium of a plasma column by means of a external current. Since it is difficult to control the external current rapidly in time, it is desirable that the external field provides a stabilizing mechanism, at least against axi-symmetric displacements.

Let us consider the displacement in the form

$$\vec{\xi} = (\alpha \cos\phi,\ \alpha \sin\phi,\ \beta) \tag{3-42}$$

where α and β are constant.

We introduce new coordinates (r', θ', ϕ) which are related to (r, θ, ϕ) as

$$\left.\begin{aligned} R' &= R + \alpha \\ r' \cos\theta' &= r \cos\theta + \alpha \\ r' \sin\theta' &= r \sin\theta - \beta \end{aligned}\right\}, \tag{3-43}$$

neglecting the contributions of the plasma distortions, (See Fig. 3).

In this system, the displacement (3-42) is rewritten as

$$\left.\begin{aligned} \vec{\xi} &= \xi^r \vec{e_r} + \xi^\theta \vec{e_\theta} \\ \xi^r &\simeq -\alpha \cos\theta' + \beta \sin\theta' \\ \xi^\theta &\simeq \frac{1}{r'}(\alpha \sin\theta' + \beta \cos\theta') \end{aligned}\right\}. \tag{3-44}$$

To maintain the equilibrium the applied vertical field must satisfy

$$B_z^{eq} = -\frac{B_0 a}{2R'} f_a' \left(\ln\frac{8R'}{a} - \frac{3}{2} + \frac{RS_{1a}'}{a} \right) \quad . \tag{3-45}$$

But actually, the applied vertical field is

$$B_z^{ext} = \frac{1}{2\pi J X}\left(\frac{\partial z}{\partial \theta}\frac{\partial \chi_e}{\partial r} - \frac{\partial z}{\partial r}\frac{\partial \chi_e}{\partial \theta} \right)$$

$$\simeq -\sum_m (-1)^m \frac{I_m}{2R}\left(\frac{r}{b}\right)^{m-1} \cos(m-1)\theta \quad , \tag{3-46}$$

where χ_e is given by (4-19). Then the following excessive vertical field acts on the plasma;

$$\delta B_z = B_z^{ext} - B_z^{eq}$$

$$\simeq \frac{I_1}{2R'}\frac{\alpha}{R'} - \frac{I_1}{2R'}\frac{\delta f_a}{f_a'} + \frac{I_2}{2R'}\frac{\alpha}{b} - \frac{I_2}{2R'}\frac{r'}{b}\cos\theta'$$

$$+ \frac{I_3}{2R'}\left(\frac{r'}{b}\right)^2\cos 2\theta' + \frac{I_3}{2R'}\frac{\alpha}{R'}\frac{r'^2}{b^2}\cos 2\theta' - \frac{I_3}{R'}\frac{r'\alpha}{b^2}\cos\theta'$$

$$- \frac{I_3}{R'}\frac{r'\beta}{b^2} + \cdots \tag{3-47}$$

where $\delta f_a = f_a' - f_a$, and we have neglected the higher order terms with respect to α and β. In the same manner, the excessive horizontal field acting on the plasma is

$$\delta B_x = B_x = \frac{1}{2\pi J X}\left(\frac{\partial X}{\partial \theta}\frac{\partial \chi_e}{\partial r} - \frac{\partial X}{\partial r}\frac{\partial \chi_e}{\partial \theta} \right)$$

$$= \sum_m (-1)^m \frac{I_m}{2R}\left(\frac{r}{b}\right)^{m-1} \sin(m-1)\theta$$

$$\simeq \frac{I_2}{2R'}\frac{\alpha}{R'}\frac{r'}{b}\sin\theta' + \frac{I_2}{2R'}\frac{\beta}{b} - \frac{I_3}{2R'}\frac{r'^2\alpha}{b^2}\sin 2\theta' + \frac{I_2}{2R}\frac{r'}{b}\sin\theta'$$

$$- \frac{I_3}{2R'}\left(\frac{r'}{b}\right)^2\sin 2\theta' + \frac{I_3}{R'}\frac{r'\alpha}{b}\sin\theta' - \frac{I_3}{R} r'\beta\sin\theta' + \cdots \quad . \tag{3-48}$$

From now on we omitt the prime, since all the calculations will be done in the (r', θ', ϕ) coordinates.

The Lorentz force acting on the volume element of the plasma column, is given by

$$\vec{F} = \vec{J} \times \delta\vec{B} \tag{3-49}$$

where

$$\vec{J} \simeq -\frac{B_0}{r} f' \vec{e_\theta} + \frac{(rf)'}{Rr} B_0 \vec{e_\phi} \tag{3-50}$$

and

$$\delta\vec{B} \simeq (-\cos\theta \cdot \delta B_x + \sin\theta \cdot \delta B_z)\vec{e_r}$$

$$+ \left(\frac{\sin\theta}{r}\delta B_x + \frac{\cos\theta}{r}\delta B_z\right)\vec{e_\theta} \quad . \tag{3-51}$$

The change in potential energy of the system under the displacement $\vec{\xi}$ is

$$W = -\frac{1}{2}\int \vec{\xi}\cdot\vec{F} \times \mathcal{J}\, dr\, d\theta\, d\phi \quad . \tag{3-52}$$

Then putting (3-44) and (3-49) into (3-52), we have

$$W = \frac{\pi^2 B_0 a f_a}{2}\left\{ \frac{\beta^2}{b} I_2 - \left(\frac{I_1}{R} + \frac{I_2}{b}\right)\alpha^2 + I_1 \frac{\delta f_a}{f_a}\alpha \right\} \quad . \tag{3-53}$$

Now δf_a is determined from

$$L_e I_p + \pi R^2 B_z^{ext} = const \tag{3-54}$$

which means the conservation of magnetic flux enclosed in the plasma column. Here L_e is the external inductance of the plasma column and has the form

$$L_e \simeq 4\pi R \left(\ln\frac{8R}{a} - 2\right)$$

and $I_p \simeq B_o a f_a$ is the total toroidal plasma current. From (3-54) we approximately obtain

$$\frac{\delta f_a}{f_a} \simeq -\frac{\alpha}{2R} \tag{3-55}$$

where we neglected $(\ln\frac{8R}{a})^{-1}$, in comparison with unity. Substitution (3-55) into (3-53), leads

$$W = \frac{\pi^2}{2} B_o a f_a \left\{ \frac{I_2}{b}\beta^2 - \left(\frac{3I_1}{2R} + \frac{I_2}{b}\right)\alpha^2 \right\} \tag{3-56}$$

The stability condition is $W > 0$, so we can find the stability criterion

$$-\frac{3b}{2R} I_1 > I_2 > 0 \tag{3-57}$$

This condition is in accordance with the stability criterion

$$n < \frac{3}{2} \tag{3-58}$$

where

$$n = -\frac{R}{B_z^{ext}} \frac{\partial B_z^{ext}}{\partial R} ,$$

which was derived by S. Yoshikawa[15] and S. M. Osovets.[16] The stability criterion (3-57) is derived only for the axi-symmetric displacement. This displacement corresponds to m=1, n=0 mode of the displacement $\xi \sim \xi_{m,n} e^{i(m\theta+n\phi)}$. We can also calculate the change in potential energy against the other, m=1, n=1 mode. In this mode, the major radius of the plasma column is not changed. One typical case is $\xi_x=\alpha$, $\xi_y=\xi_z=0$. This displacement means the shift of the plasma column in the same direction as a whole.

In this case W=0, therefore the stability is neutral. Against a twist displacement $\xi_x=\xi_y=0$, $\xi_z=\beta\cos\phi$, it can be shown that $W = \frac{\pi^2 B_o a f_a}{4} \frac{\beta}{b} I_2$, then the system is stable against this mode. In general we must find out the displacement which makes W minimum. Only when W minimum is positive, the equilibrium is stable.

§4. Equilibrium of a High β Plasma in Tokamak

Until now our discussions on toroidal equilibria have been limited to the ordering, $f \sim \varepsilon$ and $p \sim \varepsilon^2$, which have meant $q \sim aB\phi/RB_\theta \sim 1$ and $\beta_\theta \sim 1$.

Now in this section we treat an equilibrium of high β plasma column in the ordering,

$$p \sim \varepsilon^{1+k} \quad (g = 1 + \varepsilon^{1+k}\tilde{g}), \quad f \sim \varepsilon \quad , \beta_\theta \sim \varepsilon^{k-1} \quad (1 \geq k > 0).$$

We have retained the ordering $f \sim \varepsilon$, since we should satisfy $q > 1$ to assure the stability against the M.H.D. instability.

Let us now take the coordinates,

$$\left.\begin{aligned} X &= R - \varepsilon r\cos\theta - \varepsilon^{1+2k} T_2 \cos\theta - \sum_{m=1} \varepsilon^{1+mk} S_m \cos(m-1)\theta \\ Z &= \varepsilon r\sin\theta + \varepsilon^{1+2k} T_2 \sin\theta - \sum_{m=1} \varepsilon^{1+mk} S_m \sin(m-1)\theta \end{aligned}\right\} \quad (4\text{-}1)$$

which are different from (3-5) in the ordering and the addition of the term proportional to T_2. The parameter T_2 is the labeling parameter of the magnetic surfaces and can be chosen at our liberty.[8] Then the metric tensors $g_{\theta\theta}$, $g_{r\theta}$ and Jacobian J are written in the form,

$$\begin{aligned} g_{\theta\theta} &= \left(\frac{\partial X}{\partial \theta}\right)^2 + \left(\frac{\partial Z}{\partial \theta}\right)^2 \\ &= \varepsilon^2 r^2 \left\{ 1 + 2\varepsilon^{2k}\frac{T_2}{r} + \varepsilon^{4k}\frac{T_2^2}{r^2} + \sum_j \varepsilon^{2jk}(j-1)\frac{S_j^2}{r^2} \right\} \\ &\quad - \varepsilon^2 r^2 \sum_{m=1} \left\{ 2\varepsilon^{mk}(m-1)\frac{S_m}{r} - \sum_j \varepsilon^{(m+2j)k}(m+j-1)(j-1)\frac{S_{m+j}S_j}{r^2} \right. \\ &\quad \left. + 2\varepsilon^{(2+m)k}(m-1)\frac{T_2 S_m}{r^2} \right\} \cos m\theta \end{aligned} \quad (4\text{-}2)$$

$$g_{r\theta} = \frac{\partial X}{\partial \theta}\frac{\partial X}{\partial r} + \frac{\partial Z}{\partial \theta}\frac{\partial Z}{\partial r}$$

$$= -\varepsilon r \sum_{m=1} \Big\{ \varepsilon^{mk} \Big(S_m' + (m-1)\frac{S_m}{r} \Big)$$

$$+ \sum_{j=1} \varepsilon^{(m+2j)k} (m+j-1) \frac{S_{m+j} S_j'}{r}$$

$$+ \varepsilon^{(2+m)k} \Big(\frac{T_2 S_m'}{r} + (m-1) \frac{T_2' S_m}{r} \Big) \Big\} \sin m\theta ,$$

(4-3)

and

$$J = \frac{\partial X}{\partial \theta}\frac{\partial Z}{\partial r} - \frac{\partial X}{\partial r}\frac{\partial Z}{\partial \theta}$$

$$= \varepsilon r \Big\{ 1 + \varepsilon^{2k} \Big(T_2' + \frac{T_2}{r} \Big) + \varepsilon^{4k} \frac{T_2 T_2'}{r} - \sum_{j=1} \varepsilon^{2jk} (j-1) \frac{S_j S_j'}{r} \Big\}$$

$$+ \varepsilon r \sum_{m=1} \Big\{ \varepsilon^{mk} \Big(S_m' - (m-1)\frac{S_m}{r} \Big)$$

$$- \sum_{j=1} \varepsilon^{(m+2j)k} (m+j-1) \frac{S_{m+j} S_j'}{r}$$

$$+ \varepsilon^{(2+m)k} \Big(\frac{T_2 S_m'}{r} - (m-1) \frac{S_m T_2'}{r} \Big) \Big\} \cos m\theta. \quad (4-4)$$

Taking notice of the coefficient of $\cos m\theta$ (or $\sin m\theta$) in (4-2) and (4-3), we find that S_m itself contributes to the order ε^{mk}, while the other terms due to the coupling $\Big(\frac{S_{m+j} S_j}{r^2}$ and $\frac{S_{m+j} S_j'}{r}\Big)$ give ε^{2k} times higher order contributions. Then, if we desire to calculate up to the order ε^{mk}, we need not consider S_{m+j} $(j \geqq 1)$.

On the other hand, if we take the coordinates (3-3),

$$\left. \begin{aligned} X &= R - \varepsilon r \cos\theta - \sum_m \varepsilon^{1+mk} (\sigma_m \cos m\theta) \cos\theta \\ Z &= \varepsilon r \sin\theta + \sum_m \varepsilon^{1+mk} (\sigma_m \cos m\theta) \sin\theta \end{aligned} \right\} , \quad (4-5)$$

$g_{\theta\theta}$, $g_{r\theta}$ and Jacobian J take the form,

$$
\begin{aligned}
g_{\theta\theta} &= \varepsilon^2 r^2 \Big[1 + \sum_m \varepsilon^{mk} \Big\{ 2\frac{\sigma_m}{r} - \sum_j (mj - j^2 - 1)\frac{\sigma_{m-j}\sigma_j}{2r^2} \\
&\quad + \sum_j \varepsilon^{2jk} (mj + j^2 + 1)\frac{\sigma_{m+j}\sigma_j}{2r^2} \Big\} \cos m\theta \Big] , \\
g_{r\theta} &= -\varepsilon r \sum_m \Big\{ \varepsilon^{mk} \Big(m\frac{\sigma_m}{r} + \sum_j (m-j)\frac{\sigma_{m-j}\sigma_j'}{2r} \Big) \\
&\quad + \sum_j \varepsilon^{(m+2j)k} (m+j)\frac{\sigma_{m+j}\sigma_j'}{2r} \Big\} \sin m\theta , \\
J &= \epsilon r \Big[1 + \sum_m \Big\{ \varepsilon^{mk} \Big(\sigma_m' + \frac{\sigma_m}{r} + \sum_j \frac{\sigma_{m-j}\sigma_j'}{2r} \Big) + \sum_j \varepsilon^{(m+2j)k} \\
&\quad \times \frac{\sigma_{m+j}\sigma_j}{2r} \Big\} \cos m\theta \Big] .
\end{aligned}
\tag{4-6}
$$

In this case the coefficient of cosmθ or sinmθ and of the order ε^{mk} consists of not only σ_m itself but also coupling terms. That is to say, S_m characterize the geometrical properties of metric tensors, more clearly than σ_m. For example, S_2 determines the ellipticity exactly but σ_2 does not.

In this section we specifically consider the case, $k = \frac{1}{3}(\beta_\theta \sim \varepsilon^{-2/3})$, and the calculation can be done to any ε^{mk} ($m \geqq 4$), if desired.

Putting (4-2), (4-3) and (4-4) into Eq. (2-3) and setting each Fourier coefficient equal to zero, we have the following equations;

$$
\begin{aligned}
&\frac{P'}{f^2 B_0^2} + \frac{g g'}{f^2} + \frac{(rf)'}{rf} \\
&= \varepsilon^{2k} \Big[T_2'' + \Big(2\frac{(rf)'}{rf} - \frac{1}{r} \Big) T_2' + \frac{T_2}{r^2} + \Big(\frac{3}{2}\frac{(rf)'}{rf} - \frac{1}{r} \Big) S_1'^2 + \frac{rP'}{B_0^2 f^2 R}\Big(3S_1' + 2\frac{S_1}{r} \Big) \Big] ,
\end{aligned}
\tag{4-9}
$$

$$S_1'' + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right)S_1' + 2\frac{rp'}{B_0^2 f^2 R}$$

$$= \varepsilon^{2k}\Big[\frac{1}{R} - 2\frac{rp'}{B_0^2 f^2 R}\left(\frac{3}{2}S_2' + \frac{3S_2}{2r} + \frac{9}{4}S_1'^2 + 3\frac{S_2' S_1}{r} + 3T_2' + \frac{T_2}{r}\right)$$

$$- S_1'\left\{3\left(\frac{(rf)'}{rf} - \frac{1}{r}\right)S_2' - 3\frac{S_2}{r^2} + 6\frac{(rf)'}{rf}T_2' - 2\left(\frac{T_2'}{r} - \frac{T_2}{r^2}\right) + \frac{3(rf)'}{4rf}S_1'^2\right\}\Big], \tag{4-10}$$

$$S_2'' + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right)S_2' - \frac{3S_2}{r^2} = -\left(\frac{3rp'}{B_0^2 f^2 R} + \frac{3}{2}\frac{(rf)'}{rf}S_1'\right)S_1', \tag{4-11}$$

$$S_3'' + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right)S_3' - \frac{8S_3}{r^2}$$

$$= \frac{3S_1' S_2}{r^2} - \frac{S_1' S_2'}{r} - \frac{(rf)}{rf}\left(3S_1' S_2' + \frac{1}{4}S_1'^3\right) - \frac{2rp'}{B_0^2 f^2 R}\left(\frac{3}{4}S_1'^2 - \frac{S_2}{2r} + \frac{3S_2'}{2}\right). \tag{4-12}$$

Eq. (4-9) arises from the terms independent of θ. There are several alternative possibilities[8] for solving this equation. If the only purpose of the calculation is to obtain an equilibrium solution, it is convenient to adjust the labeling parameter T_2 so as to satisfy Eq. (4-13).

$$T_2'' + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right)T_2' + \frac{T_2}{r^2} + \left(\frac{3}{2}\frac{(rf)'}{rf} - \frac{1}{r}\right)S_1'^2 + \frac{rp'}{B_0^2 f^2 R}\left(3S_1' + 2\frac{S_1}{r}\right) = 0. \tag{4-13}$$

Then, we have,

$$\frac{p'}{B_0^2} + gg' + \frac{f(rf)'}{r} = 0. \tag{4-14}$$

We solve Eq. (4-10) by the successive approximation. Let $S_1^{(0)}(r)$ be the solution of equation,

$$S_1^{(0)\prime\prime} + \left(2\frac{(rf)'}{rf} - \frac{1}{r}\right) S_1^{(0)\prime} + 2\frac{rp'}{B_0^2 f^2 R} = 0, \tag{4-15}$$

and let us put

$$S_1(r) = S_1^{(0)}(r) + \varepsilon^{2k} S_1^{(2)}(r). \tag{4-16}$$

In the vacuum region, outside the plasma surface, $p'(r)=0$, $g'(r)=0$ and $f(r)=f_a a/r$, then $S_1^{(0)}(r)$ is obtained as

$$S_1^{(0)}(r) = S_{1a}^{(0)} + \frac{S_{1a}^{(0)\prime}}{2a}(r^2 - a^2) \tag{4-17}$$

and from Eq. (4-11) we have

$$S_2(r) = \frac{S_{2a}}{4}\left(\frac{r^3}{a^3} + \frac{3a}{r}\right) + \frac{aS_{2a}'}{4}\left(\frac{r^3}{a^3} - \frac{a}{r}\right). \tag{4-18}$$

Inserting (4-17) and (4-18) into Eq. (4-12), we find

$$S_3(r) = \frac{S_{3a}}{6}\left(\frac{2r^4}{a^4} + \frac{4a^2}{r^2}\right) + \frac{aS_{3a}'}{6}\left(\frac{r^4}{a^4} - \frac{a^2}{r^2}\right)$$

$$+ \left\{\frac{1}{24}\left(\frac{r^4}{a^4} - 1\right) + \frac{1}{12}\left(\frac{a^2}{r^2} - 1\right)\right\} S_{1a}^{(0)\prime}(3S_{2a} - aS_{2a}'). \tag{4-19}$$

In the same way Eq. (4-13) leads to

$$T_2(r) = \frac{r}{a}\left(1 - \ln\frac{r}{a}\right) T_a + r\ln\frac{r}{a} T_a'$$

$$+ r S_{1a}^{(0)\prime}\left\{\frac{1}{4}\left(\frac{r^2}{a^2} - 1\right) - \frac{1}{2}\ln\frac{r}{a}\right\}. \tag{4-20}$$

Substituting (4-16), (4-17), (4-18) and (4-20) into Eq. (4-10) we obtain $S_1^{(2)}$ as follows;

$$S_1^{(2)} = S_{1a}^{(2)} + \frac{S_{1a}^{(2)\prime}}{2a}(r^2-a^2) + \left\{\frac{1}{2R} + \frac{S_{1a}^{(0)\prime}}{a}\left(T_a' - \frac{T_a}{a} - \frac{S_{1a}^{(0)\prime 2}}{2}\right)\right\}$$

$$\times r^2 \ln\frac{r}{a} - \left\{\frac{1}{4R} + \frac{S_{1a}^{(0)\prime}}{a}\left(T_a' - \frac{T_a}{a} - \frac{S_{1a}^{(0)\prime 2}}{2}\right)\right\}(r^2-a^2)$$

$$+ \frac{S_{1a}^{(0)\prime}}{8a}\left\{\frac{3}{a^3}(S_{2a} + aS_{2a}') + \frac{1}{a^2}S_{1a}^{(0)\prime 2}\right\}(r^2-a^2)^2. \qquad (4\text{-}21)$$

To obtain explicit values for the quantities T_a' and S_{ma} it is necessary to specify the plasma model. We again take the special model (2-17). Then, inside the plasma the quantities S_m, T_2, S_{ma}' and T_a' become

$$\left.\begin{aligned} S_1^{(0)}(r) &= \frac{r^2}{2R}\beta_p \\ S_{1a}^{(0)} &= \frac{a^2}{2R}\beta_\theta \end{aligned}\right\}, \qquad (4\text{-}22)$$

$$S_{1a}^{(0)\prime} = \frac{a}{R}\beta_\theta \quad , \qquad (4\text{-}23)$$

$$S_2(r) = \frac{r}{a}S_{2a} + \frac{\beta_\theta^2}{4}\frac{a^3}{R^2}\left(\frac{r^3}{a^3} - \frac{r}{a}\right), \qquad (4\text{-}24)$$

$$S_{2a}' = \frac{S_{2a}}{a} + \frac{a^2}{2R}\beta_\theta^2, \qquad (4\text{-}25)$$

$$\left.\begin{aligned} T_2(r) &= \frac{3}{8}\frac{\beta_\theta^2}{R^2}r^3 \\ T_{2a} &= \frac{3}{8}\frac{\beta_\theta^2}{R^2}a^3 \end{aligned}\right\}, \qquad (4\text{-}26)$$

$$T_{2a}' = \frac{9}{8}\frac{\beta_\theta^2}{R^2}a^2, \qquad (4\text{-}27)$$

$$S_{1a}^{(2)} = \frac{a^2}{8R}\left(1 + 12\frac{S_{2a}}{a}\beta_\theta + 4\frac{a^2}{R^2}\beta_\theta^3\right), \qquad (4\text{-}28)$$

$$S_{1a}^{(2)\prime} = \frac{a}{4R}\left(1 + 12\frac{S_{2a}}{a}\beta_\theta + 11\frac{a^2}{R^2}\beta_\theta^3\right), \qquad (4\text{-}29)$$

$$S_3(r) = \frac{r^2}{a^2} S_{3a} - \frac{1}{8}\frac{\beta_\theta^3}{R^3} a^2 r^2 \left(1 - \frac{r^2}{a^2}\right), \tag{4-30}$$

$$S_{3a}' = 2\frac{S_{3a}}{a} + \frac{1}{4}\frac{a^3}{R^3}\beta_\theta^3 \quad . \tag{4-31}$$

The remaining parameters S_{2a} and S_{3a} are determined from the shape of the shell. Let the shape of the shell be

$$\left.\begin{aligned} X &= R - S_{1b} - b\cos\theta - T_{2b}\cos\theta - S_{2b}\cos\theta - S_{3b}\cos 2\theta \\ Z &= b\sin\theta + T_{2b}\sin\theta - S_{2b}\sin\theta - S_{3b}\sin 2\theta \end{aligned}\right\} \tag{4-32}$$

Then, from equations (4-18), (4-19) if follows

$$S_{2b} = \frac{S_{2a}}{4}\left(\frac{b^3}{a^3} + \frac{3a}{b}\right) + \frac{aS_{2a}'}{4}\left(\frac{b^3}{a^3} - \frac{a}{b}\right), \tag{4-33}$$

$$S_{3b} = \frac{S_{3a}}{6}\left(\frac{2b^4}{a^4} + \frac{4a^2}{b^2}\right) + \frac{aS_{3a}'}{6}\left(\frac{b^4}{a^4} - \frac{a^2}{b^2}\right)$$

$$+ \left\{\frac{1}{24}\left(\frac{b^4}{a^4} - 1\right) + \frac{1}{12}\left(\frac{a_2}{b^2} - 1\right)\right\} S_{1a}^{(0)\prime}\left(3S_{2a} - aS_{2a}'\right). \tag{4-34}$$

The displacement of the plasma cross section from the center of the shell, δ, is determined by use of Eq. (4-17) and (4-21) as

$$\delta = S_{1b} - S_{1a}^{(0)} - S_{1a}^{(2)}$$

$$= \frac{S_{1a}^{(2)\prime}}{2a_0}\left(b_0^2 - a_0^2\right) + \left\{\frac{1}{2R} + \frac{S_{1a}^{(0)\prime}}{a}\left(T_a' - \frac{T_a}{a} - \frac{S_{1a}^{(0)\prime 2}}{2}\right)\right\} b^2 \ln\frac{b}{a}$$

$$- \left\{\frac{1}{4R} + \frac{S_{1a}^{(0)\prime}}{a}\left(T_a' - \frac{T_a}{a} - \frac{S_{1a}^{(0)\prime 2}}{2}\right)\right\}\left(b^2 - a^2\right)$$

$$+ \frac{S_{1a}^{(0)\prime}}{8a}\left\{\frac{3}{a^3}\left(S_{2a} + aS_{2a}'\right) + \frac{1}{a^2} S_{1a}^{(0)\prime 2}\right\}\left(b^2 - a^2\right)^2, \tag{4-35}$$

where a_o and b_o are

$$\left.\begin{aligned} a_0 &= a - T_2(a) \\ b_0 &= b - T_2(b) \end{aligned}\right\}$$

Substituting Eq. (4-22) ∿ (4-31) into Eq. (4-33), (4-34) and (4-35) and neglecting a^4/b^4, we have

$$\delta_{2a} = \frac{2a^3}{b^3}\delta_{2b} - \frac{a^3}{4R^2}\beta_\theta^2 \quad , \qquad (4\text{-}36)$$

$$\delta_{3a} = \frac{3}{2}\frac{a^4}{b^4}\delta_{3b} - \frac{1}{4}\frac{a^4}{Rb^3}\beta_\theta\delta_{2b} \quad , \qquad (4\text{-}37)$$

and

$$\delta = \frac{b^2}{2R}\left\{\left(\beta_\theta - \frac{1}{4} + 6\frac{a^2}{b^3}\delta_{2b}\right)\left(1 - \frac{a_2}{b^2}\right) + \ln\frac{b}{a}\right\}$$
$$+ \left(\frac{3}{2}\frac{b}{R}\delta_{2b} - \frac{3b^4}{8R^3}\beta_\theta^3\right)\left(1 - \frac{a^2}{b^2}\right)^2 \quad . \qquad (4\text{-}38)$$

Several examples of high β tokamak equilibrium are shown in Fig. 4 ∿ Fig. 8. These examples are calculated for the uniform current distribution (2-17).

Fig. 4 shows magnetic surfaces of tokamak equilibrium surrounded by a circular conducting shell and, Fig. 5 and Fig. 6 show those surrounded by an elliptical conducting shell. Magnetic surfaces of the equilibrium surrounded by a triangular shell are shown in Fig. 7 and Fig. 8. In these figures the plasma columns occupy the shaded regions.

Acknowledgement

The authors wish to express their deepest thanks to Prof. H. Itô and R. Itatani for their helpfull discussions. And this work was carried out under the collaborating Research Program at Institute of Plasma Physics, Nagoya University, Nagoya.

References

(1) V. D. Shafranov, Soviet Phys. JETP 6 (1958) 545.

(2) V. D. Shafranov, J. Nucl. Energy Pt. C 5 (1963) 251.

(3) A. A. Ware and F. A. Haas, Phys. Fluids 9 (1966) 956.

(4) V. S. Mukhovatov and V. D. Shafranov, Nucl. Fusion 11 (1971) 605.

(5) H. R. Strauss, Phys. Rev. Letters 26 (1971) 616.

(6) J. D. Callen and R. A. Dory, Phys. Fluids 15 (1972) 1523.

(7) F. A. Haas, Phys. Fluids 15 (1972) 141.

(8) J. M. Green, J. L. Johnson and K. E. Weimer, Phys. Fluids 14 (1971) 671.

(9) R. Gajewski, Phys. Fluids 15 (1972) 70.

(10) H. Grad and J. Hogan, Phys. Rev. Letters 24 (1970) 1337.

(11) G. Laval, E. K. Maschke, R. Pellat and M. N. Rosenbluth, International Center for Theoretical Physcis, Report IC/70/35, 1970.

(12) T. Ohkawa and H. G. Voorhies, Phys. Rev. Letters 22 (1969) 1275.

(13) V. D. Shafranov and E. I. Yurchenko, Fourth conference on Plasma Physics and Controlled Nuclear Fusion Research, Madison, Wisconsin (1971) CN-28/F-13.

(14) V. A. Fock, Physik Z. Sovjetunion 1 (1932) 215.

(15) S. Yoshikawa, Phys. Fluids 7 (1963) 278.

(16) S. M. Osovets, Izv. Akad. Nauk SSR, Ser. Fiz. (1958) 238.

Figure Captions

Fig. 1 Coördinate systems.

Fig. 2 The relations between (R_o, ξ, φ) coordinate and (R, ξ', φ') coordinate systems. The solid lines are (R_o, ξ, φ) coordinates and the dotted lines are (R, ξ', φ') coordinates.

Fig. 3 The relations between (r, θ, ϕ) coordinates and (r', θ', ϕ) coordinates.

Fig. 4 Equiliblium surrounded by a circular conducting shell;

$\beta_p = 3.0, \quad S_{2b} = 0.0, \quad S_{3b} = 0.0.$

Fig. 5 Equiliblium surrounded by an elliptical conducting shell;

$\beta_p = 3.0, \quad S_{2b} = -0.4, \quad S_{3b} = 0.0.$

Fig. 6 Equiliblium surrounded by an elliptical conducting shell;

$\beta_p = 3.0, \quad S_{2b} = 0.2, \quad S_{3b} = 0.0.$

Fig. 7 Equiliblium surrounded by a triangular conducting shell;

$\beta_p = 3.0, \quad S_{2b} = 0.0, \quad S_{3b} = 0.4.$

Fig. 8 Equiliblium surrounded by a triangular conducting shell;

$\beta_p = 3.0, \quad S_{2b} = 0.0, \quad S_{3b} = -0.4.$

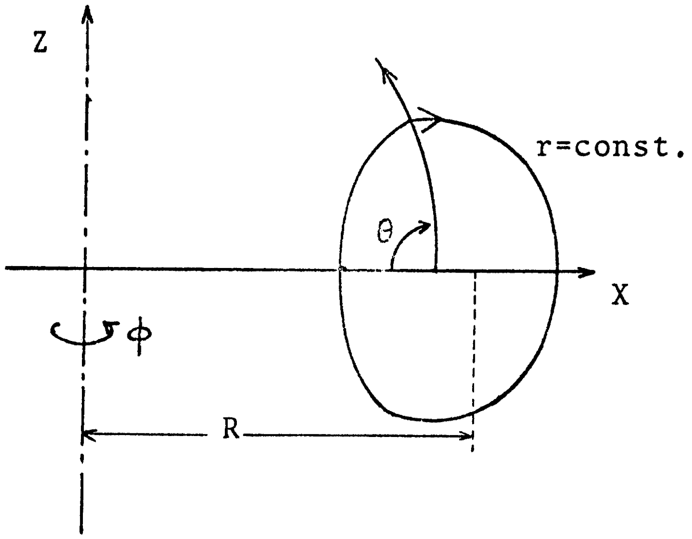

Fig. 1

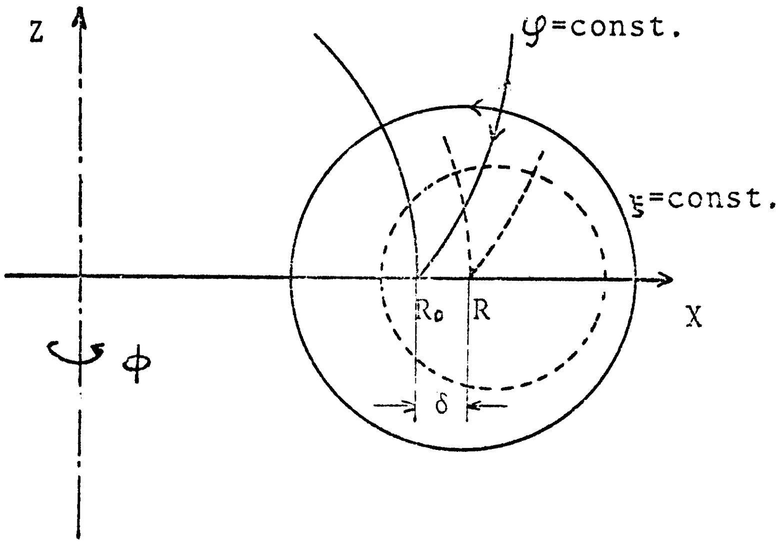

Fig. 2

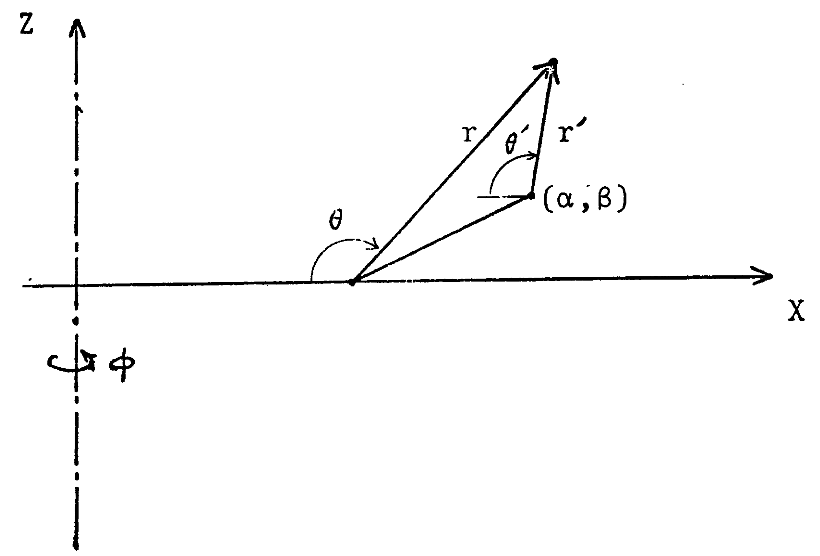

Fig. 3

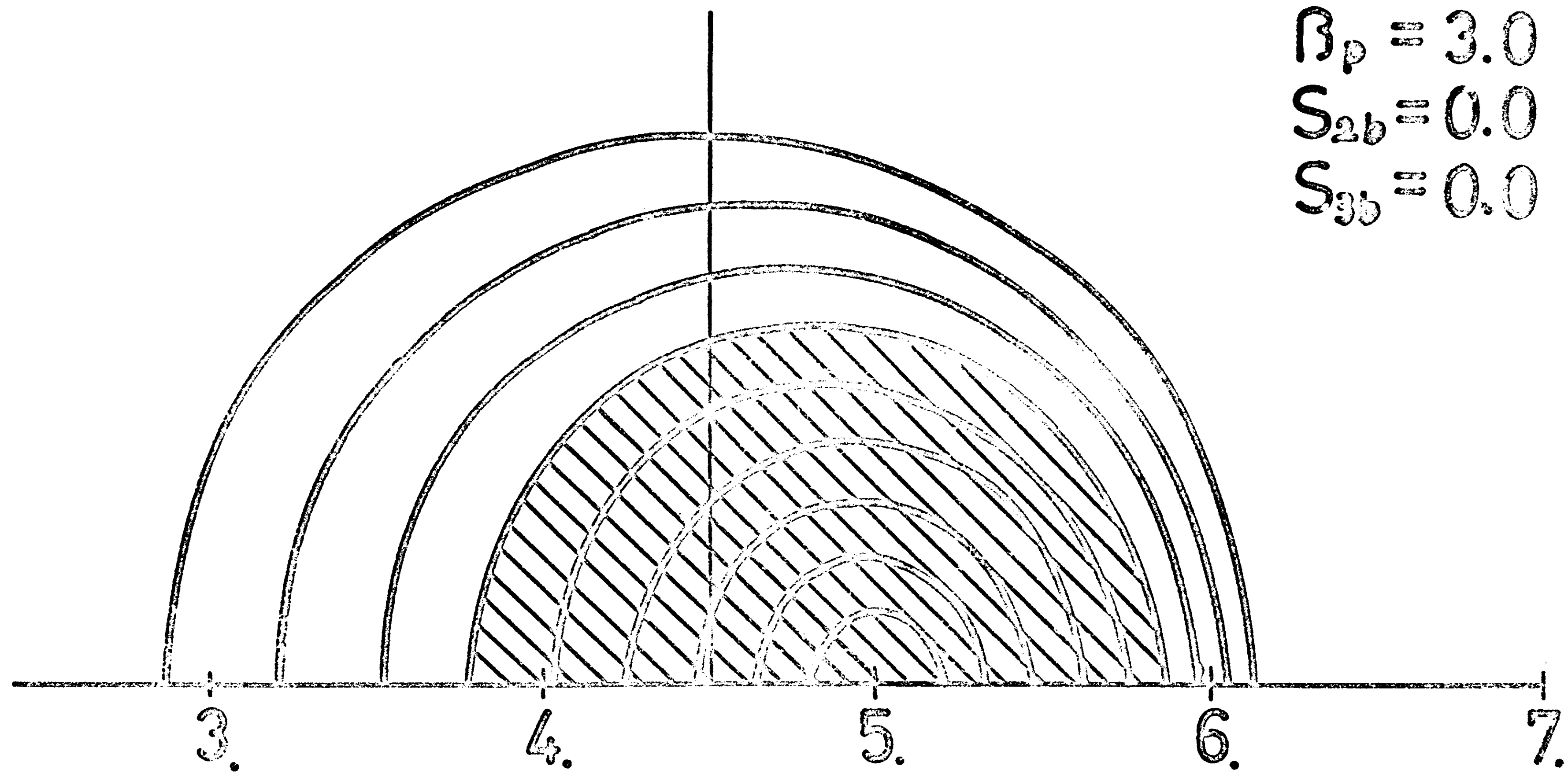

Fig. 4

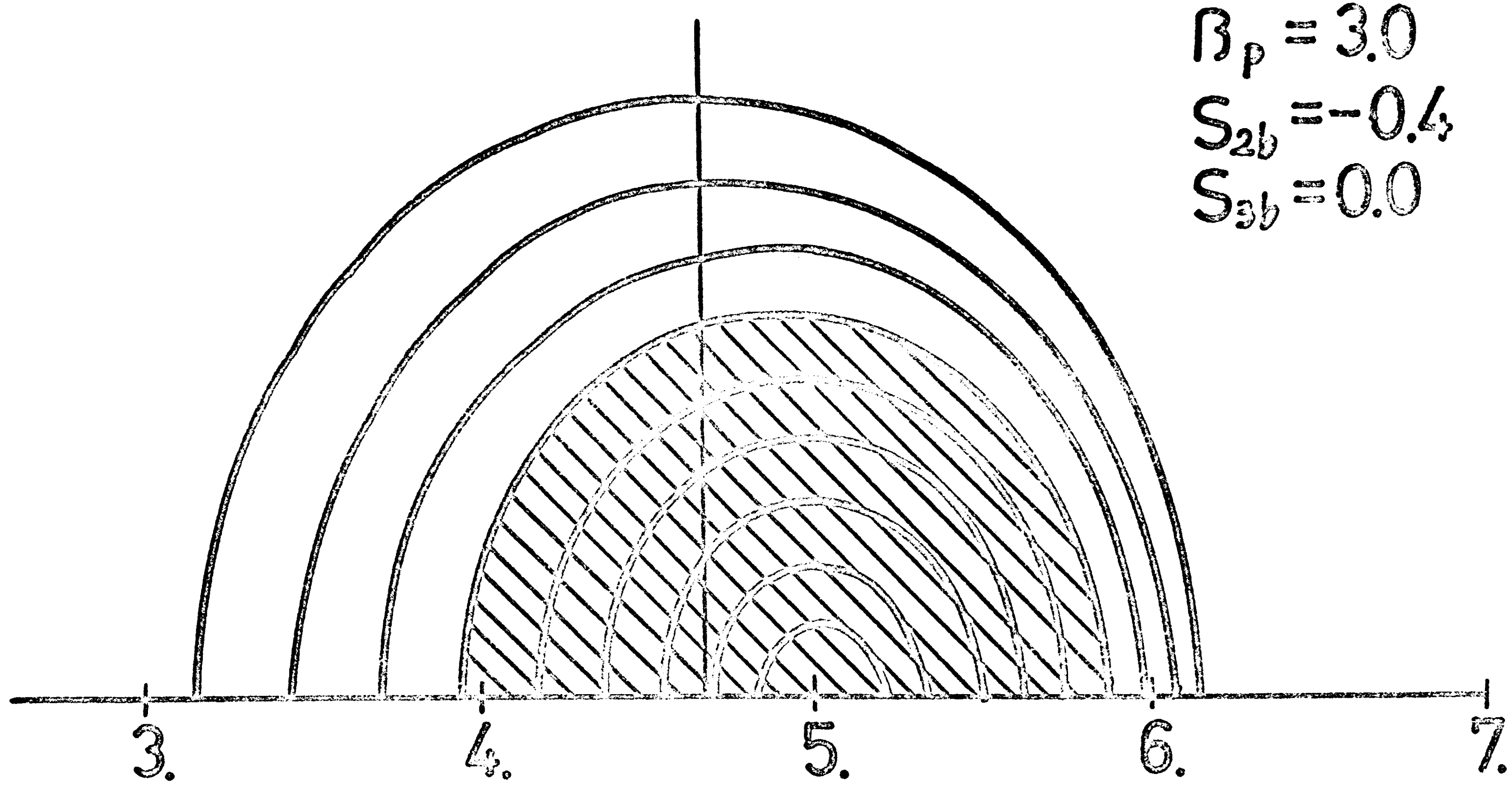

Fig. 5

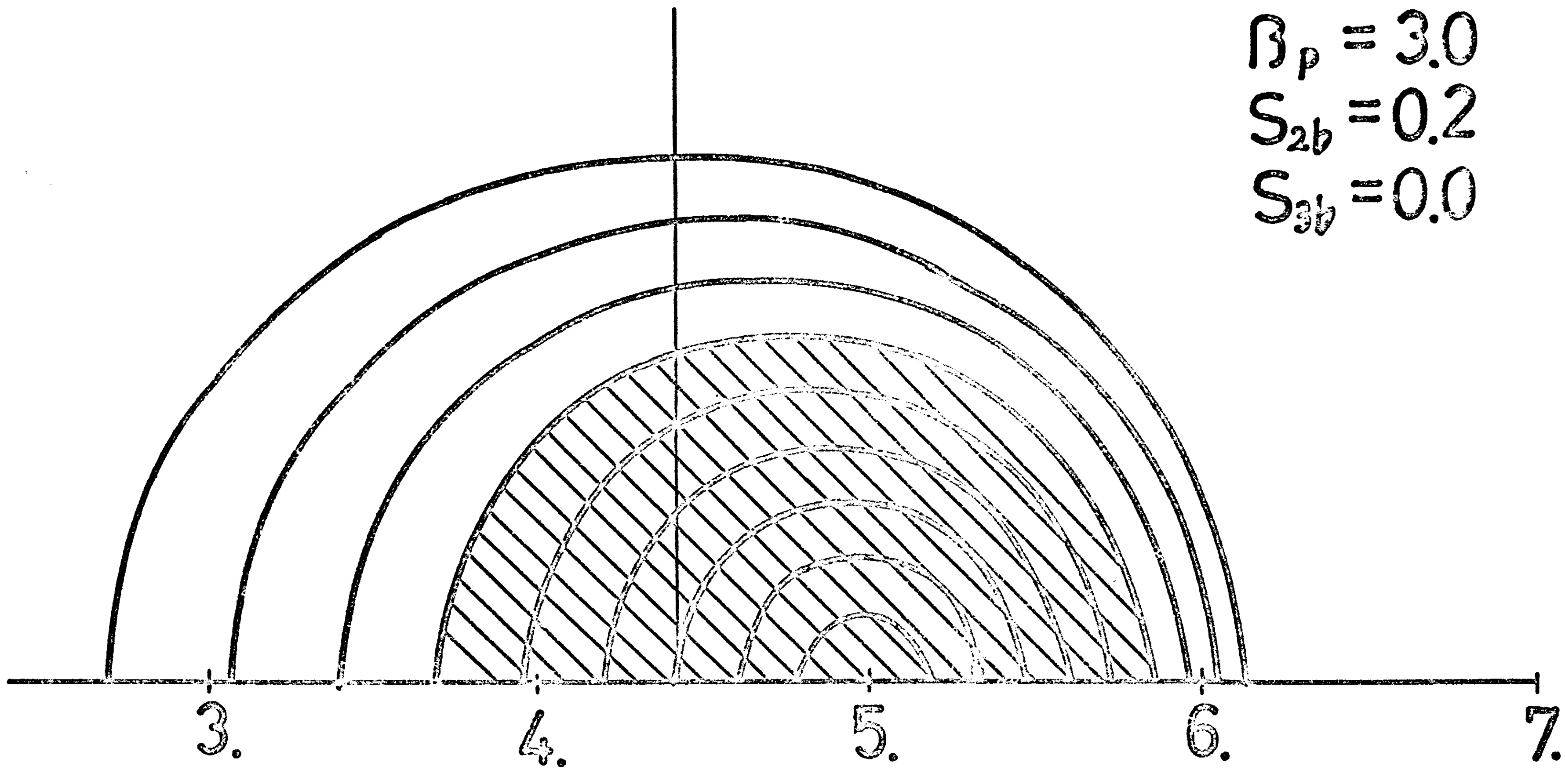

Fig. 6

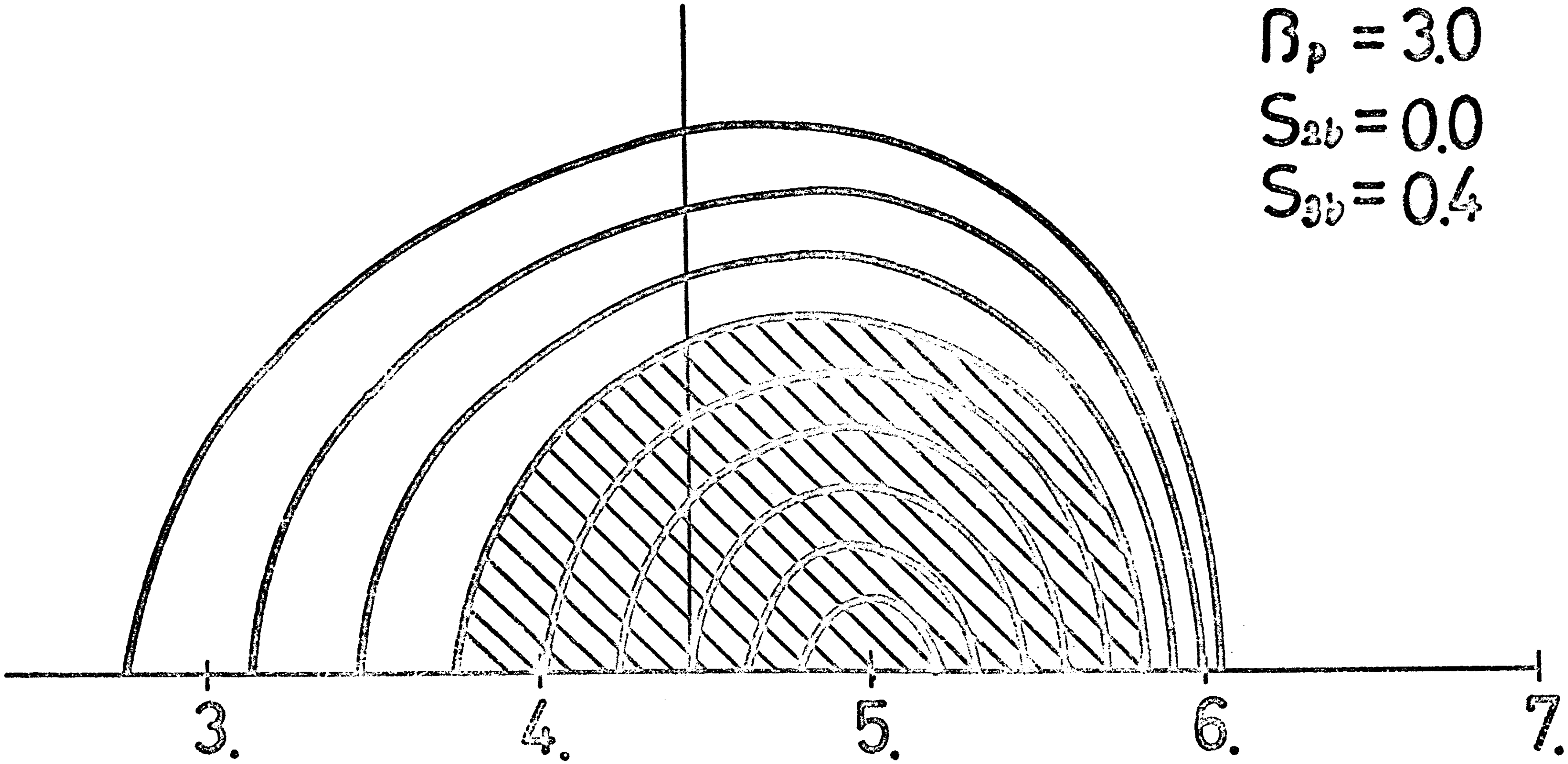

Fig. 7

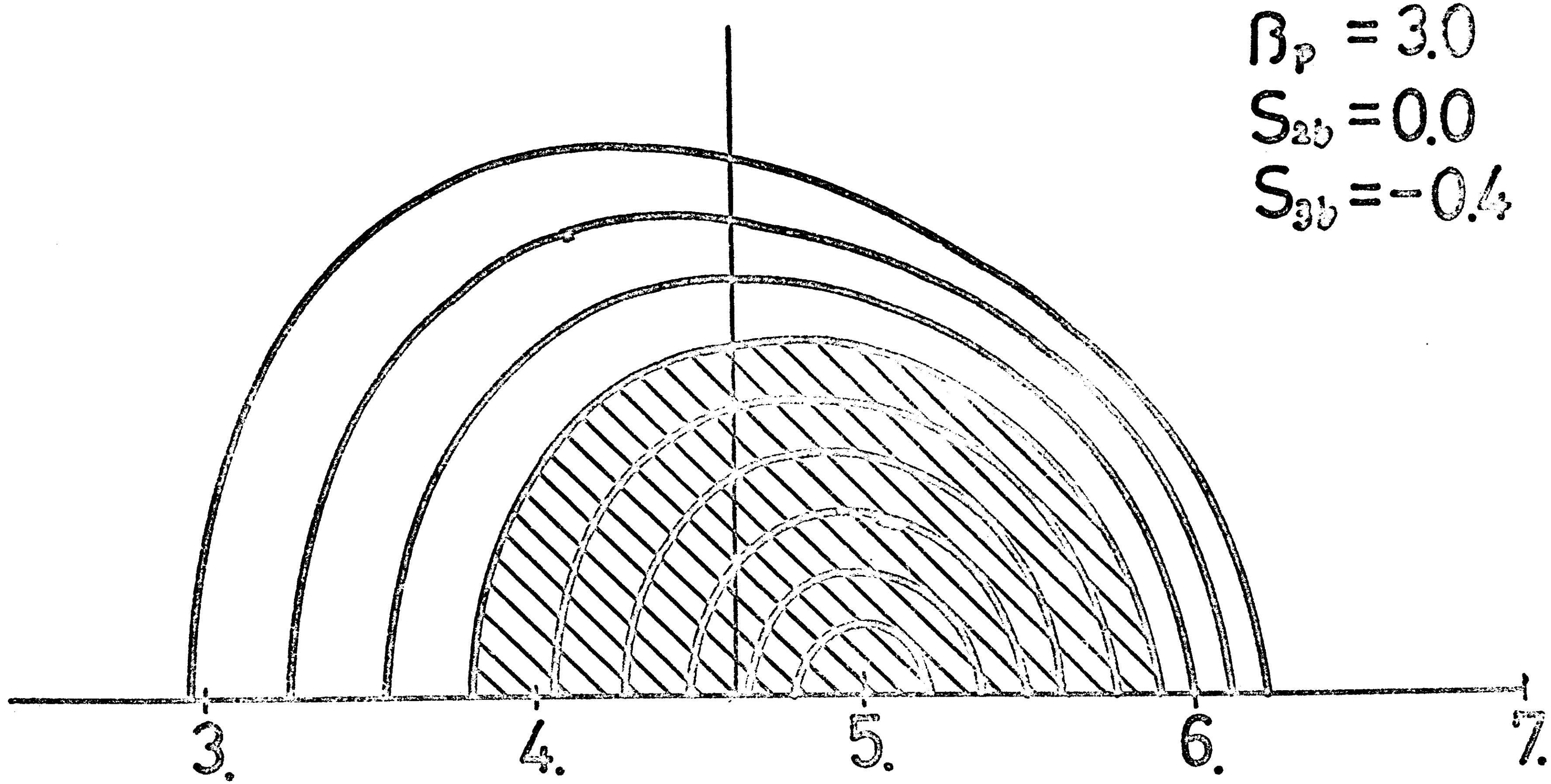

Fig. 8