INSTITUTE OF PLASMA PHYSICS

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Nonadiabatic Motion of Charged Particle in an Inhomogeneous Magnetic Field

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Abstract

A new analytic expression is obtained in a differential form for nonadiabatic change of magnetic moment μ of a charged particle in an inhomogeneous magnetic field. The expression gives a picture easier to be understood than the past. The results agree well with numerical calculations which were made previously.

§1. Introduction

Motion of a charged particle in an inhomogeneous magnetic field has been investigated by many authors[1,2]. Most of them focussed their interests on the behavior of the magnetic moment of charged particle. The magnetic moment of a charged particle is an adiabatic invariant, but it is not an absolute one and changes when the particle moves through a non-uniform magnetic field region. The change of magnetic moment is closely connected with the motion of plasma particles and the particle effusion process in a cusped magnetic field as in the Divertor plasma source[3]. Many authors have reported that the change of magnetic momet depends on the phase angle of gyration of the particle at the midplane of the nonadiabatic region. It seems, however, that the conclusion does not correspond to a picture easy to be understood. Previously, we have made a numerical calculation to trace the motion of a particle in a cusped magnetic field, and got the same conclusion as M. Rusbridge about the size of the scattering center of nonadiabatic zero field region.

Besides our numerical calculation on this problem, we made some analysis from another stand point, and have got a more clear picture than the past about the change of magnetic moment. Starting from the equation of motion of a charged particle and obtaining its solution in the vicinity of a point where the particle initially was, we have got an expression of magnetic moment change in a differential form. Averaging (or integrating) this result, we have got a final result and a picture of the process which is very easy to be understood. This result agrees well with our numerical calculations.

§2. Analysis

Suppose that a charged particle is travelling in an inhomogeneous magnetic field. To trace the orbit of the particle, we expand the magnetic field around a point P, where the particle was at time T = 0, as

$$\vec{B}(\vec{r}) = \vec{B}_0 + \|C_{ij}\| \cdot \vec{r}, \tag{1}$$

where $\vec{r}$ is the position vector and

$$C_{ij} = \frac{\partial B_i}{\partial x_j} \tag{2}$$

is the field gradient matrix element. Among nine elements of the C matrix, we have only one relation as

$$C_{ii} = 0, \tag{3}$$

which is equivalent to

$$\mathrm{div}\,\vec{B} = 0. \tag{4}$$

Choosing z-axis parallel with $\vec{B}_0$ and x-axis parallel with the principal normal of the magnetic line of force as in Fig.1, we follow the particle orbit with an initial velocity as

$$\vec{v}(0) = (v_\perp(0)\cos\theta,\ v_\perp(0)\sin\theta,\ v_\parallel(0)). \tag{5}$$

The equation of motion is

$$m\frac{d\vec{v}}{dT} = e\vec{v} \times \vec{B}, \tag{6}$$

where e and m are the electric charge and the mass of the particle. Setting $\vec{r}(T)$ in a form of power series in T as

$$\vec{r} = \vec{a}T + \vec{b}T^2 + \cdots\cdots \tag{7}$$

and

$$\dot{\vec{r}}(T) = \vec{v}(T) = \vec{a} + 2\vec{b}T + \cdots\cdots, \tag{8}$$

we can determine the coefficients $\vec{a}$ and $\vec{b}$ from Eq.(6), i.e.,

$$\vec{a} = \vec{v}(0),\ \vec{b} = \frac{e}{2m}\,\vec{v}(0) \times \vec{B}_0. \tag{9}$$

Now, we set a time τ, which represents a changing time scale of magnetic field as seen from the travelling particle, i.e.,

$$\tau = (|B_0|/|\,\|C_{ij}\|\vec{v}(0)|,\ \omega_c^{-1})_{max}, \tag{10}$$

where the notation $(\quad , \quad)_{max}$ means the larger value of the two values in the bracket. If T is infinitesimally smaller than τ, it will be good enough to take only the first degree term in T and neglect all the higher order terms in the expression of $\vec{r}$ and $\vec{v}$. Then

$$\vec{r} = \vec{v}(0)T,\ \vec{v} = \vec{v}(0) + \frac{e}{m}\vec{v}(0) \times \vec{B}_0 T. \tag{11}$$

The magnetic moment μ of the particle is defined as

$$\mu = \frac{E_\perp}{|B|}, \tag{12}$$

where $E_\perp$ is the kinetic energy of the particle corresponding to the perpendicular component of the particle velocity

$$E_\perp = \frac{v_\perp^2}{2m} \tag{13}$$

and B is the field strength at the position of the particle. It must be noticed that B is the field strength at the particle position and not at the point where the guiding center lies. But these two definitions of μ do not show any difference in a uniform magnetic field, and do not show any confusion in the final conclusion.

To express $v_\perp^2(T)$ in terms of the quantities at $T = 0$, we write $v_\perp^2(T)$ as

$$v_\perp^2(T) = v^2(T) - v_\parallel^2(T) = v^2(T) - \frac{[\vec{v}(T)\cdot\vec{B}(T)]^2}{B^2}. \quad (14)$$

Then, we substitute $v(\vec{T})$ and $\vec{B}(T)$ by the quantities given in Eqs.(1) and (11). Taking only the 0-th and 1-st order terms in T, we obtain for $v_\perp^2(T)$ as

$$v_\perp^2(T) = v_\perp^2(0)\{1 - 2\frac{v_\parallel^2(0)}{v_\perp^2(0)} \frac{v_i(0)C_{ij}v_j(0) - v_\parallel(0)C_{zj}v_j(0)}{B(0)v_\parallel(0)}T\}$$

(15)

Finally, from Eqs.(1), (10) and (11), we get $\mu(T)$ as a function of T, which is

$$\mu(T) = \frac{mv_\perp^2(T)}{2|B(T)|}$$

$$= \frac{2v_\perp^2(0)}{2|B(T)|}\{1 - 2\frac{v_\parallel^2(0)}{v_\perp^2(0)} \frac{v_i(0)C_{ij}v_j(0) - v_\parallel(0)C_{zj}(0)}{B(0)v_\parallel(0)}T\}$$

(16)

$$= \frac{mv_\perp^2(0)}{2|B(0)|}\{1 - [2\frac{v_\parallel^2}{v_\perp^2} \frac{v_iC_{ij}v_j - v_\parallel C_{zj}v_j}{Bv_\parallel} + \frac{v_\parallel C_{zj}v_j}{Bv_\parallel}]_{T=0}\cdot T\},$$

or

$$\delta\mu = -\mu(0)T\{ \frac{v_{\parallel}}{B}(C_{xx}\cdot 2\cos^2\theta + C_{yy}\cdot 2\sin^2\theta + C_{zz})$$

$$+ \frac{2v_{\parallel}}{B}(C_{xy} + C_{yx})\cos\theta\cdot\sin\theta$$

$$+ \frac{2v_{\parallel}^2}{Bv_{\perp}}[(C_{xz} + 2C_{zx})\cos\theta + (C_{yz} + 2C_{zy})\sin\theta \tag{17}$$

$$- \frac{v_{\perp}}{B}(C_{zx}\cos\theta + C_{zy}\sin\theta)\}_{T=0}$$

for the change of magnetic moment.

As mentioned before, Eqs.(16) and (17) dominate only if T is small enough to fulfil the condition

$$T << \tau. \tag{18}$$

After a time T_1 (T_1 << τ), when the particle is at a point

$$\vec{r} = \int_0^{T_1} \vec{v}(t)dt, \tag{19}$$

we shift our coordinate system to that point with the three axes set just as before, that is, z-axis parallel with $\vec{B}(T)$ and x-axis parallel with the principal normal of the magnetic line of force. In such a new coordinate system, we can expect that the change of the matrix elements C_{ij} will be the minimum (Fig.2).

Putting a new time scale as

$$T - T_1 \longrightarrow T, \tag{20}$$

we can get just the same expression as Eq.(17) for the magnetic moment change $\delta\mu$ of the particle, except the phase angle θ. The change of θ is due partly to the gyrating motion of the particle and partly to the rotation of the coordinate system. Therefore, θ' the new phase angle can be expressed as

$$\theta' = \theta + (\omega_c + \omega_1)T_1 \equiv \theta + \Omega(0)T_1, \tag{21}$$

where ω_c and ω_1 are angular velocities corresponding to the gyration motion of the particle and the rotation of the coordinate system respectively. $\omega_1 T_1$ is the angle between two x-z planes of two coordinate systems. In most cases, ω_c is much larger than ω_1, and Ω is nearly equal to ω_c,

$$\omega_c(0) + \omega_1(0) = \Omega(0) \fallingdotseq \omega_c(0). \tag{22}$$

Repeating the same procedure, we obtain the magnetic moment change $\delta\mu_i$ during the time interval of i-th computation stage as

$$\delta\mu_i = -\mu(T_i)\{\frac{v_\parallel}{B}[C_{xx}\cdot 2\cos^2\theta_i + C_{yy}\cdot 2\sin^2\theta_i + C_{zz}]$$

$$+ \frac{2v_{\parallel}}{B} [(C_{xy} + C_{yx}) \cos\theta_i \sin\theta_i]$$

$$+ \frac{2v_{\parallel}^2}{Bv_{\perp}} [(C_{xz} + 2C_{zx})\cos\theta_i \tag{23}$$

$$+ (C_{yz} + 2C_{zy})\sin\theta_i]$$

$$-\frac{v_{\perp}}{B}(C_{zx}\cos\theta_i + C_{zy}\sin\theta_i)\}_{T=T_i} \cdot t_i \equiv f(T_i, \theta_i)t_i,$$

where T_i is the time when the i-th computation stage starts, and t_i and θ_i are the time interval of each stage and the initial phase angle in each stage respectively, i.e.,

$$t_i = T_i - T_{i-1},$$

$$T_0 = 0 \tag{24}$$

$$\theta_i = \theta_0 + \sum_{j=0}^{i} \Omega(T_j)t_j = \theta_0 + \omega(T_i)T_i,$$

where $\omega(T_i)$ is the average value of Ω, as

$$\omega(T_i) = \frac{1}{T_i}\int_0^{T_i} \Omega(t)dt. \tag{25}$$

Therefore, the overall change of μ can be obtained with a satisfactory accuracy by summing up all $\delta\mu_i$'s, which is

$$\Delta\mu = \sum_i \delta\mu_i = \sum_i f(T_i,\theta_i)\Delta T_i. \tag{26}$$

In integral form it can be written as

$$\begin{aligned}
\Delta\mu &= \int_0^T f(t,\theta)d\theta \\
&= -\int_0^T \mu(t)\,[\frac{v_{\parallel}(t)}{B(t)}\{(C_{xx}(t) - C_{yy}(t))\cos 2\theta \\
&\qquad + (C_{xy}(t) + C_{yx}(t))\sin 2\theta\} \qquad (27) \\
&\quad + \frac{2v_{\parallel}^2(t)}{B(t)v(t)}\{(C_{xz}(t) + 2C_{zx}(t))\cos\theta + \\
&\qquad (C_{yz}(t) + 2C_{zy}(t))\sin\theta\} \\
&\quad - \frac{v_{\perp}(t)}{B(t)}\{C_{zx}(t)\cos\theta + C_{zy}\sin\theta\}]dt
\end{aligned}$$

where,

$$\theta(t) = \theta_0 + \Omega(t)t, \tag{28}$$

Now Eq.(27) can approximatedly be evaluated by keeping in view the following considerations:

(1) The rate of change of $\theta(t)$ with time is much larger than the change in other factors as μ, B, $v_{\perp}$ and $v_{\parallel}$.

(2) $C_{ij}(T)$'s are closely related to the field inhomogeniety and can be considered as leading factors in the nonadiabatic process of particle motion.

(3) The quantities μ, B, $v_{\perp}$ and $v_{\parallel}$ are subordinate factors,

are definitely constant when $C_{ij}(T) = 0$ as given by Eq.(1) and are less dependent on T as compared with $C_{ij}(T)$'s.

With these considerations, we are justified in assuming μ, B, $v_\perp$ and $v_\parallel$ as constants and finally get

$$\Delta\mu = -\mu\Big[\frac{v_\parallel}{B}\{(C_{xx} - C_{yy})_{2c} + (C_{xy} + C_{yx})_{2s}\} + \frac{2v_\parallel^2}{Bv_\perp}\{(C_{xz} + 2C_{zx})_{1c} + (C_{yz} + 2C_{zy})_{1s}\} - \frac{v_\perp}{B}\{(C_{zx})_{1c} + (C_{zy})_{1s}\}\Big], \tag{29}$$

where $(\quad)_{1s}$, $(\quad)_{1c}$, $(\quad)_{2s}$ and $(\quad)_{2c}$ are as follows:

$$\begin{aligned} (\quad)_{1s} &= \int_0^T (\quad)\sin\theta(t)dt, \\ (\quad)_{1c} &= \int_0^T (\quad)\cos\Theta(t)dt, \\ (\quad)_{2s} &= \int_0^T (\quad)\sin2\Theta(t)dt, \\ (\quad)_{2c} &= \int_0^T (\quad)\cos2\theta(t)dt. \end{aligned} \tag{30}$$

§3. Discussions and Summary

In general, the particle starts in a uniform magnetic field region and after passing through an inhomogeneous field region reaches another uniform region with some change of μ. In such a case, the C_{ij}'s of the magnetic field experienced by the particle in travelling along its way are zero both at initial and final stages but not zero at the intermediate stage.

In Fig.3, a typical curve of C_{ij} is shown. The region where C_{ij} is markedly different from zero corresponds to the inhomogeneous field region. At the same time, the integral

$$I = |\vec{v}| \cdot \int_0^T C_{ij} dt \tag{31}$$

corresponds roughly to the change in field intensity between initial and final positions.

Therefore, it can be said that the sharpness of C_{ij} and the total area in C_{ij}-t plane correspond to the inhomogeniety of the field and the overall change in field intensity respectively. The integral given in Eq.(30) can have a value other than zero only when C_{ij} is sharp enough to fulfil the condition

$$t_s \leqq \Omega^{-1}, \tag{32}$$

where t_s is a half value width of $C_{ij}(T)$. In other words, with a definition of inhomogeneous field region to be a region where C_{ij} takes a value more than the half of its peak value, the change in μ can occur only when the diameter of inhomogeneous field region is not larger than the helical pitch of the particle orbit.

The conclusion from Eqs.(17) and (29) are summarized in the following:

(1) In the case of both the initial and the final positions of the particle in uniform regions, the net change of μ is closely connected with the first and second component of the Fourier series of $C_{ij}(T)$ expanded with the fundamental frequency of Ω, where $C_{ij}(T)$ is the field gradient matrix element as seen from the travelling particle.

(2) The nonadiabatic process occurs only when the helical pitch of the particle orbit is comparable or larger than the diameter of the inhomogeneous field region.

(3) When a particle is travelling in an inhomogeneous field region, μ of the particle oscillates with a frequency nearly equal to Ω.

(4) The change of μ largely depends on the phase angle of the gyrating motion of the particle at the midpoint of the inhomogeneous field region.

We made some computations to investigate the behavior of charged particles in a line cusped magnetic field using FACOM-230-60 and 230-35 digital computers in Nagoya University

Computer Center. The magnetic field is given as

$$B_x = \pm Ax,$$

$$B_y = \mp Ay, \qquad (33)$$

$$B_z = 0,$$

where A is a constant. The zero field line coincides with z-axis, and separatrices are x-z and y-z planes. The details of these comutations have been reported elsewhere[4].

Among these computations, we picked up some data concerning the magnetic moment change of the particle. We calculated $v_\perp^2/B$ instead of μ, where $v_\perp$ is the perpendicular velocity component of the particle and B is the magnetic field strength at the particle position. In Figs.4 and 5, typical behaviors of $v_\perp^2/B$ are plotted as a function of t. At the same time, corresponding particle trajectories are shown in the same figures.

As expected from Eq.(29), $v_\perp^2/B$ or μ oscillates in time with a frequency of about Ω. At first, when the particle is approaching the midpoint of the inhomogeneous field region, the oscillation amplitude of μ goes up and takes its maximum value at the midpoint of the inhomogeneous field region keeping its average value constant. After the passage of the midpoint, the curve of μ follows after another oscillating curve being smoothly connected at the midpoint, but being different in its average. After that, as the particle moves away from the midpoint, the oscillation amplitude goes down and μ approaches a value different from the initial one. Notice, at the same time, that the curve sometimes shows some distortions, which means that the oscillation contains the second order higher harmonics. This will be easily deduced from Eq.(29). From these curve shown in Figs.4 and 5, it is clear that μ is strongly dependent on the gyration phase angle of the particle at the midpoint of the inhomogeneous field region.

Aknowledgements

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References

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Figure Captions

Fig.1. Coordinate system with the z-axis parallel with $\vec{B}$ and x-axis parallel with the principal normal of line of force.

Fig.2. Relation between two succesive coordinate systems.

Fig.3. Typical curve of field gradient matrix C_{ij}.

Fig.4. Magnetic moment μ as a function of t (a) and corresponding particle orbit (b). The magnetic field is as given in Eq.(33) with $A = 0.5$ Wb/m^3, and the initial conditions of the particle are $\vec{r}(0) = (10$ cm, 1 cm, 0), $\vec{P}/P = (0, 0.5, 0.87)$ and $P = 48$ keV/c which corresponds to an energy of 0.3 eV for He ion.

Fig.5. Magnetic moment μ as a function of t and corresponding particle orbit ((a) and (b)). The field parameter and the initial conditions of the particle are all the same except for $\vec{P}/P = (-0.7, -0.7, 0)$.

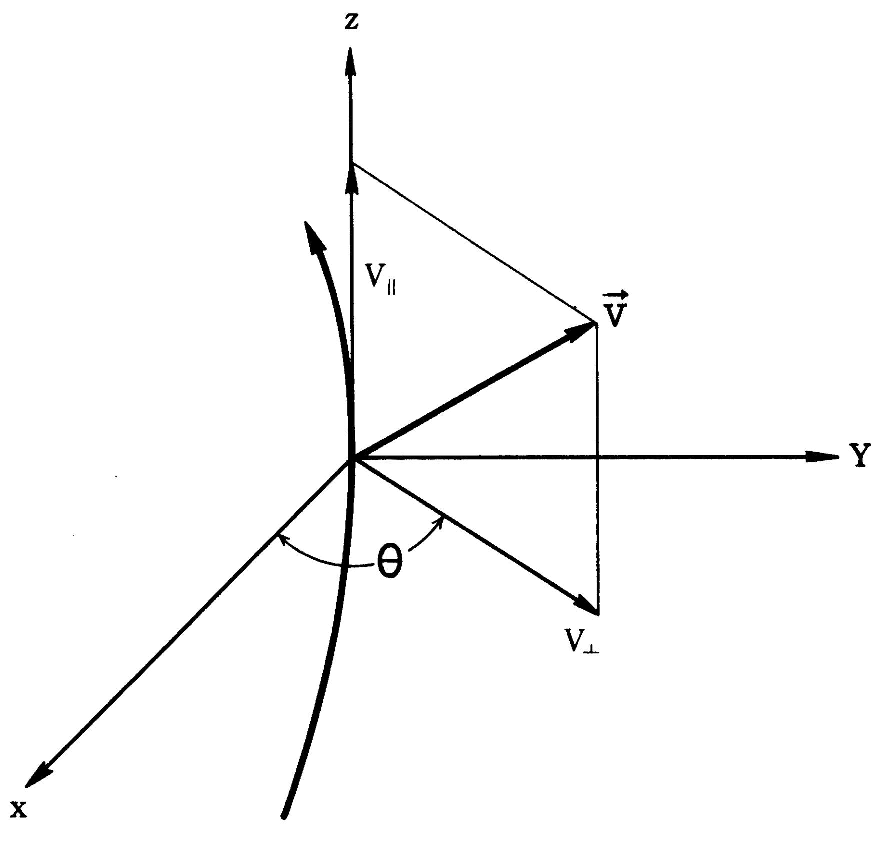
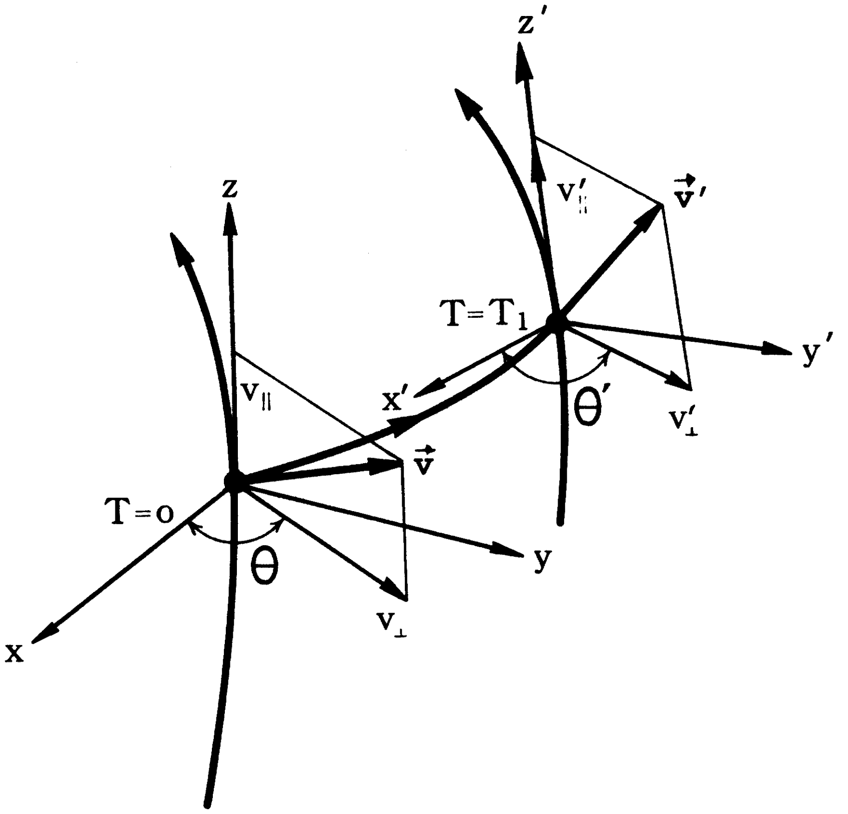


Fig. 1



z'
z
$v'_{\parallel}$
$\vec{v}'$
T=T$_1$
y'
x'
θ'
$v'_{\perp}$
$v_{\parallel}$
$\vec{v}$
T=o
y
θ
$v_{\perp}$
x

Fig. 2

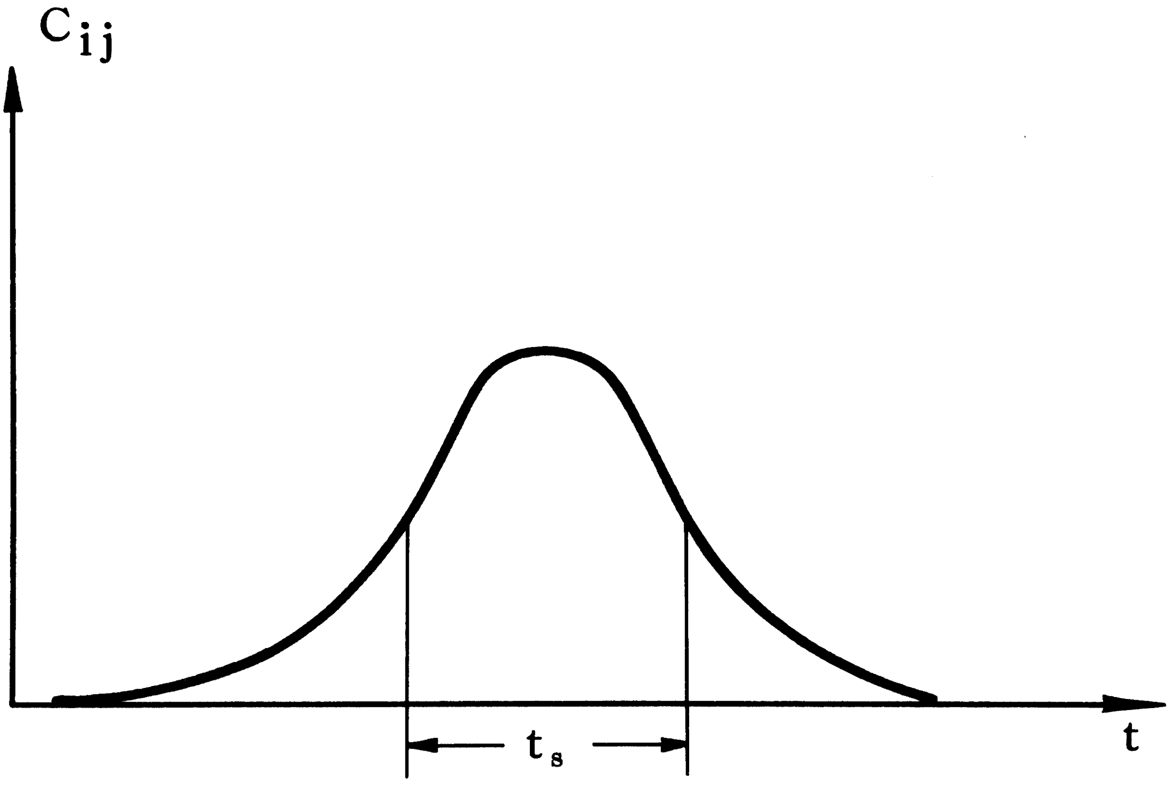
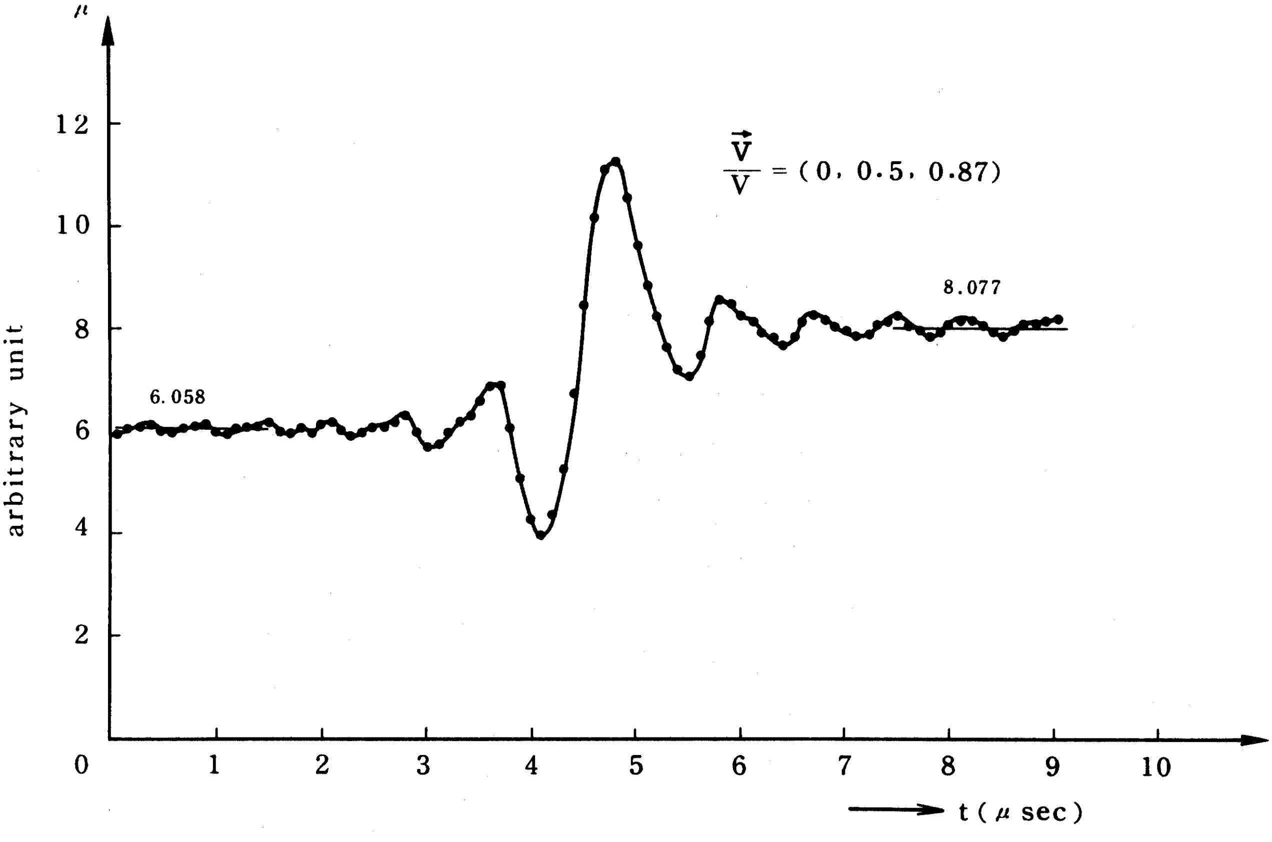


Fig. 3



μ
12
10
8
6
4
2
0
arbitrary unit
6.058
8.077
$\frac{\vec{V}}{V}$ = (0, 0.5, 0.87)
1
2
3
4
5
6
7
8
9
10
t (μ sec)

Fig. 4(a)

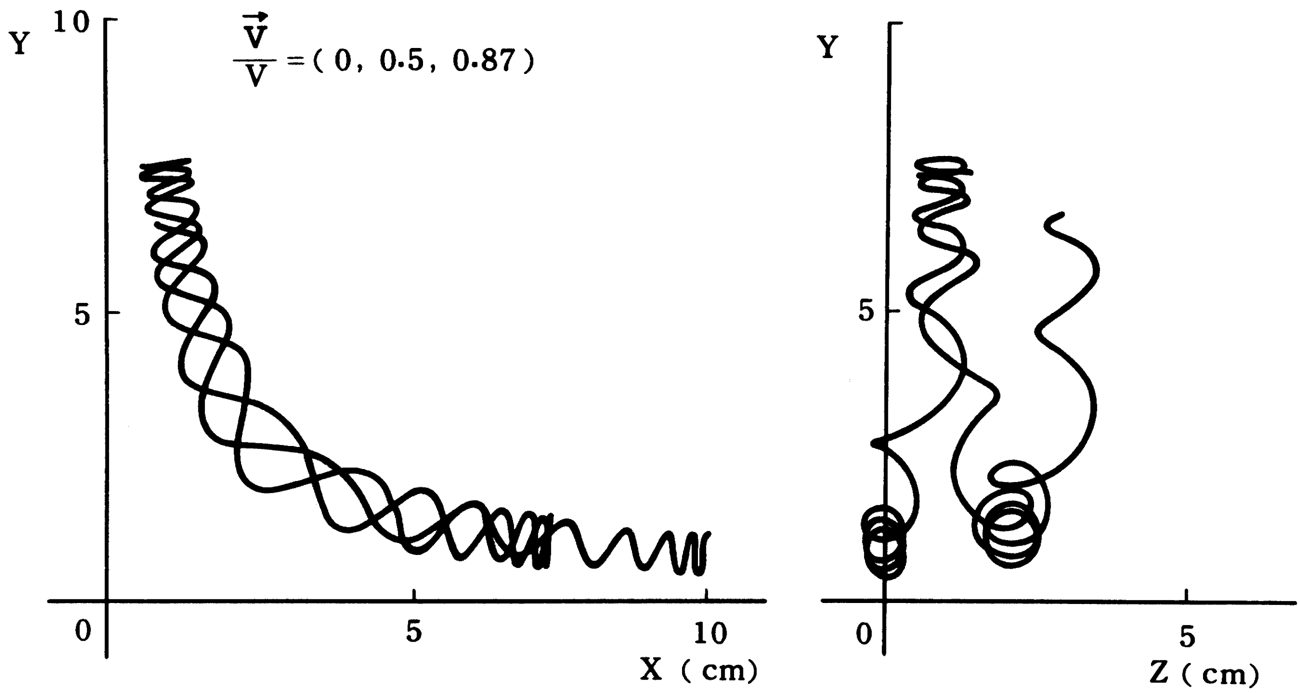
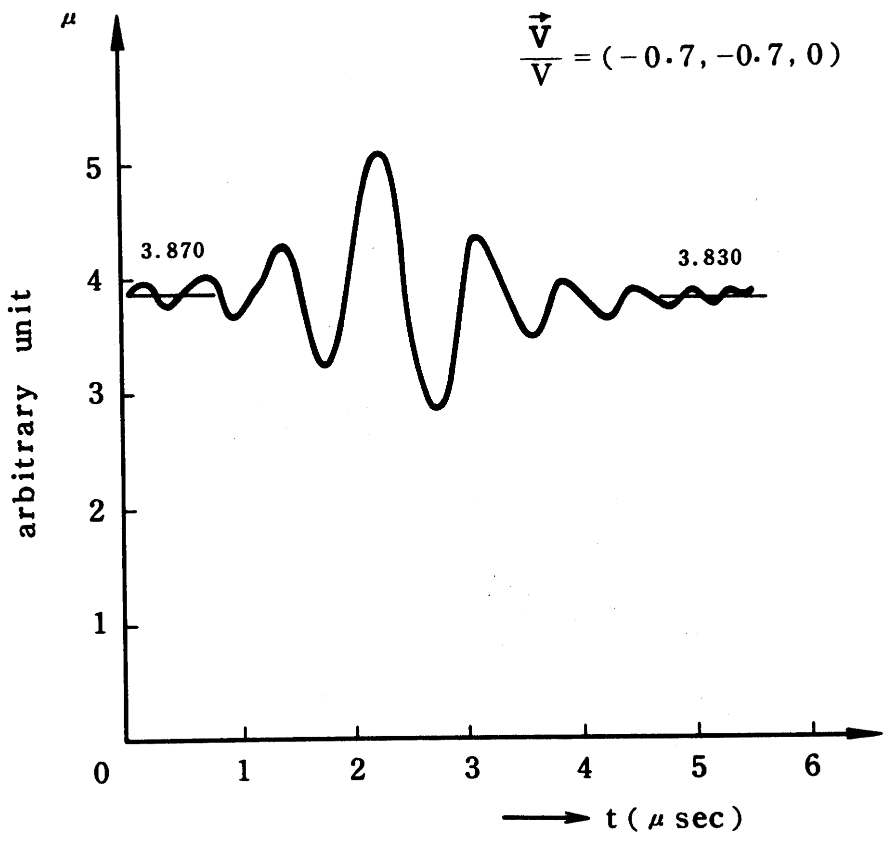


Fig. 4(b)



μ
$\frac{\vec{V}}{V} = (-0.7, -0.7, 0)$
5
3.870
3.830
4
arbitrary unit
3
2
1
0
1
2
3
4
5
6
t (μ sec)

Fig. 5(a)

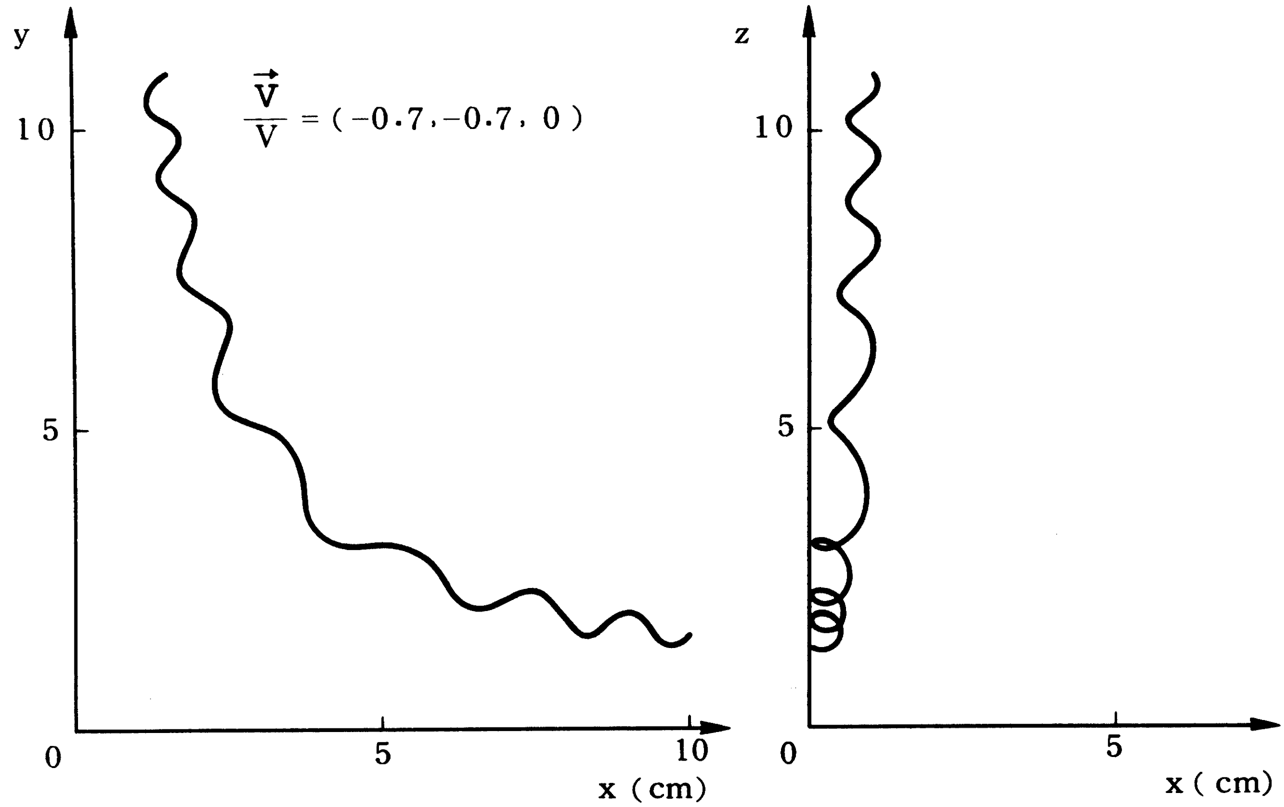


Fig. 5(b)