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COUPLED NONLINEAR ELECTRON-PLASMA AND ION-ACOUSTIC WAVES

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Abstract

Solutions describing stationary one-dimensional propagation of coupled nonlinear electron plasma wave and nonlinear ion acoustic wave are obtained. They have the amplitudes linearly proportional to each other, and propagate with approximately the ion-acoustic velocity in the form of periodic wave-trains, including solitary waves as special case.

Nonlinear stationary propagation of plasma waves has been investigated extensively in recent years[1-4]. One=dimensional propagation of small but finite amplitude ion-acoustic wave in a collisionless cold-ion plasma is described by a Korteweg-deVries equation[5], and the theoretical prediction of steepening and soliton formation has been confirmed by experiments[6]. A long wavelength electron plasma wave obeys a nonlinear Schrödinger equation[7]. Its stationary solutions in one-dimensional case include envelope-soliton, periodic wave-train and finite-amplitude plane wave. The latter is subject to a modulational instability under certain conditions.

In this paper, we present some special solutions which describe coupled stationary propagation of one-dimensional, nonlinear electron-wave and nonlinear ion-wave. The basic equations are the Schrödinger equation for the electron-wave with a potential proportional to the ion-density perturbation and the cold-ion fluid equations for the ion-wave supplemented by the electron pressure balance equation. Our solutions have the form of periodic wave-trains, including solitary waves as special cases, and have the following properties; i) both electron and ion waves move with a group velocity very close to the ion-acoustic velocity C_s, and ii) the amplitudes of the two waves are proportional to each other.

Our solution appears to be of particular importance in the nonlinear stage of parametric instabilities by an electron plasma wave acting as the pump. In general, a finite amplitude electron plasma wave always induces an ion density perturbation by the ponderomotive force, but if the group velocity of the pump wave is not very close to C_s, the induced ion perturbation is very small, being of second order in the pump amplitude. On the other hand, if it is close to C_s and ions are sufficiently cold, the induced ion perturbation moves resonantly with the pump modulation and is thereby strongly enhanced, becoming of first order in the pump amplitude and hence nonlinear. Our solution can be regarded as describing such a situation. It corresponds to the one-dimensional scattering process modified by the nonlinear ion response. As a consequence of this nonlinear effect, the process can occur even in the overdense region, in contrast to the linear decay process which is restricted to the underdense region.

Basic equations The following equation gives the adequate description of one-dimensional propagation of a small but finite amplitude, long wavelength electron plasma wave[7]:

$$\frac{\partial^2}{\partial t^2} u_e - 3v_e^2 \frac{\partial^2}{\partial x^2} u_e + \omega_{pe}^2 [1 + \frac{\delta n_e}{n_0}] u_e = 0, \qquad (1)$$

where u_e, v_e, ω_{pe}, n_0 and δn_e are respectively the fluid velocity, thermal velocity, plasma frequency, average density and low-frequency density perturbation of the electron. We write

$$u_e(x,t) = \tilde{u}_e(x,t)\, e^{-i\omega_0 t} + \tilde{u}_e^*(x,t)\, e^{i\omega_0 t}$$

and assume that $\omega_0 \simeq \omega_{pe}$ and $\tilde{u}_e$ is slowly-varying in time. We then neglect $\partial^2\tilde{u}_e/\partial t^2$ and approximate $(\omega_0^2 - \omega_p^2)$ by $2\omega_{pe}\Delta$, where $\Delta = \omega_0 - \omega_{pe}$. From now on, we use ω_{pe}^{-1} and $\lambda_D = v_e/\omega_{pe}$ as units of time and length and denote non-dimensional variables, $\tilde{u}_e/v_e$, Δ/ω_{pe} and $\delta n_e/n_0$, simply by $\tilde{u}_e$, Δ and δn_e. Equation (1) then becomes

$$i\frac{\partial}{\partial t}\tilde{u}_e + \frac{3}{2}\frac{\partial^2}{\partial x^2}\tilde{u}_e + (\Delta - \frac{\delta n_e}{2})\tilde{u}_e = 0. \qquad (2)$$

As will be shown later, δn_e depends only on the amplitude of $\tilde{u}_e$ and not on its phase. Then, if $\tilde{u}_e = w(x,t)$ is a solution of (2), any function produced by the following transformation is also a solution:

$$w(x-x_0-Vt,\ t)\ \exp\{i\frac{V}{3}x - i\frac{V^2}{6}t + i\theta\}, \qquad (3)$$

where V, x_0 and θ are arbitrary parameters. Keeping this in mind, we look for a stationary solution, $w(\xi)$, which satisfies

$$\frac{3}{2}\frac{\partial^2}{\partial\xi^2}w(\xi) + [\Delta - \frac{v(\xi)}{2}]w(\xi) = 0, \qquad (4)$$

where $\xi = x-x_0-Vt$ and we put $\delta n_e = v(\xi)$ which is also assumed to be stationary.

For low-frequency perturbation, we can neglect the electron inertia, obtaining from the electron equation of motion

$$\frac{\partial}{\partial x}\,|\tilde{u}_e|^2 = \frac{\partial}{\partial x}\,\{\phi - \log[1 + \delta n_e]\}, \tag{5}$$

where ϕ is the low-frequency potential measured in the unit of T/e, T being the electron temperature and -e the electron charge. The left-hand side describes the ponderomotive force. We combine this equation with the ion equations of continuity and motion:

$$\frac{1}{\varepsilon}\frac{\partial}{\partial t}\,\delta n_i + \frac{\partial}{\partial x}\,(1 + \delta n_i)u_i = 0 \tag{6}$$

$$\frac{1}{\varepsilon}\frac{\partial}{\partial t}\,u_i + u_i\frac{\partial}{\partial x}\,u_i + \frac{\partial \phi}{\partial x} = 0 \tag{7}$$

and the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = (\delta n_e - \delta n_i), \tag{8}$$

where ε^2 is the electron-to-ion mass ratio, δn_i the ion density perturbation normalized by n_0 and u_i the ion fluid velocity normalized by $C_s = \varepsilon v_e$. Ion temperature is neglected in (7). Since we are interested in the stationary solution moving with velocity V, we can replace $\partial/\partial t$ by $-V\partial/\partial\xi$ and $\partial/\partial x$ by $\partial/\partial\xi$.

If we make the linear approximation, we get from (6) and (7),

$$\delta n_i = \varepsilon u_i/V = \varepsilon^2\phi/V^2. \tag{9}$$

If in addition we assume the local charge neutrality, $\delta n_e = \delta n_i$, we get from (5) and (9)

$$\delta n_e = |\tilde{u}_e|^2 \cdot (V^2/\varepsilon^2 - 1)^{-1}. \tag{10}$$

Substitution of (10) into (2) yields the usual nonlinear Schrödinger equation for the long wavelength electron wave. It is modulationally unstable when the group velocity is subsonic (i.e. $V < \varepsilon$).

The linear approximation breaks down if V is very close to ε. In order to derive an appropriate nonlinear equation, we differentiate (8) with respect to ξ and add the result to the sum of (7) and ε/V times (6). Using (5) and keeping the terms up to the second order in δn_e, one obtains

$$\frac{\partial}{\partial\xi}\{\frac{\varepsilon}{V}\,[(1 - \frac{V^2}{\varepsilon^2})u_i + \delta n_i u_i] + \frac{u_i^2}{2} + |\tilde{u}_e|^2 - \frac{\delta n_e^2}{2} + \frac{\partial^2\phi}{\partial\xi^2} = 0. \tag{11}$$

This equation contains only small terms, either nonlinear or linear with higher derivative or small coefficient $(1 - V^2/\varepsilon^2)$. One can therefore use the linear relations (9) as well as the local charge neutrality, $\delta n_e = \delta n_i = v(\xi)$. Also, V/ε may be replaced by unity except for the term $(V - \varepsilon)$. Equation (11) can then be reduced to the form

$$\frac{\partial^2}{\partial \xi^2} v - 2\lambda v + v^2 + |w|^2 + W = 0, \tag{12}$$

where we replaced $|\tilde{u}_e|^2$ by $|w|^2$, λ is the excess Mach number, $\lambda = (V - \varepsilon)/\varepsilon$, and W the integration constant. Equations (4) and (12) are our basic equations.

Solutions We expect the solution for which $|w|$ and $|v|$ are of comparable order. It is then natural to assume the form

$$|w|^2 = a + bv + cv^2. \tag{13}$$

Equation (12) with (13) has a general solution expressible in terms of Jacobi's elliptic function $\mathrm{cn}(\alpha\xi;k)$:[8]

$$v = v_0 + A\,\mathrm{cn}^2(\alpha\xi;k), \tag{14}$$

where v_0 is determined by the condition that the spatial average of v should vanish:

$$v_0 = -\frac{A}{2\alpha K(k)} \int_0^{2K} dx\,\mathrm{cn}^2(x;k), \tag{15}$$

K(k) being the complete elliptic integral of first kind. The other constants are to be determined such that the coefficient in each power of $\mathrm{cn}^2(\alpha\xi;k)$ vanishes in (12). There are three such relations.

Our next procedure is to separate w into the amplitude and phase by writing $w = R^{1/2}e^{i\psi}$; since v, as a solution of (13), is a function of R only, one can easily find two integrals of (4) as

$$R\, d\psi/d\xi = M = \text{Const.} \tag{16}$$

$$\frac{1}{R}\left\{\left(\frac{dR}{d\xi}\right)^2 + \frac{8}{3}\left(\Delta + \frac{b}{4c}\right)R^2 - \frac{R}{9c^2}(b + 2cv)^3 + 4M^2\right\} = E = \text{Const.} \tag{17}$$

Substituting (14) into (13) and then into (17) gives an algebraic equation for $cn^2(\alpha\xi;k)$ in fifth power. Setting the coefficient in each power equal to zero, we get six relations, of which only five are found to be independent. For given value of Δ, there are twelve independent parameters, a, b, c, A, α, k, λ, W, M, E, x_0 and θ or $\psi(\xi = 0)$. Of these, we can determine only eight parameters, four being left free to choose.

Particularly simple solutions are obtained in the case M = 0, i.e. $d\psi/d\xi = 0$. In this case, we find a solution in the form

$$w = B\, cn(\alpha\xi;\, k)\; sn(\alpha\xi;\, k) \tag{18}$$

with

$$A = -18k^2\alpha^2,\quad B = (432)^{1/2}k^2\alpha^2,$$

$$\alpha^2 = \frac{2}{3}\,\frac{1}{5k^2 - 4}\left[\frac{v_0}{2} - \Delta\right],$$

$$k^2 - 1 = [(v_0 - 2\lambda)v_\theta + W]/2A\alpha^2,$$

$$-\lambda + v_0 = 2\alpha^2(1 + 4k^2).$$

Clearly, $|A|$ and $|B|$ are of the same order, so are $|w|$ and $|v|$. The general form of (14) and (18) describes a periodic wave-train with three parameters, W, x_0 and θ, being left free to choose. In the special case in which $k^2 = 1$, the period of the wave-train becomes infinite and the solution is reduced to a solitary wave. In this case, $K \to \infty$ and hence $v_0 \to 0$, so that W must be zero. The explicit form of the solution is

$$v(\xi) = 12\Delta \operatorname{sech}^2[(-2\Delta/3)^{1/2}\xi] \tag{19}$$

$$w(\xi) = (192)^{1/2}\Delta \operatorname{sech}[(-2\Delta/3)^{1/2}\xi] \tanh[(-2\Delta/3)^{1/2}\xi]. \tag{20}$$

Since Δ has to be negative, this solution can exist only in the overdense region ($\omega_0 < \omega_{pe}$). The density perturbation $v(\xi)$ is negative, implying a density depletion, but is of the same order as $|w(\xi)|$. The excess Mach number λ is negative, i.e. subsonic, and is given by $20\Delta/3$. Whereas the ion-density perturbation is symmetric around $\xi = 0$, the electron wave is antisymmetric and shows a phase jump at $\xi = 0$. For given Δ, the only free parameters are the initial position x_0 and the initial phase θ, all the other parameters being uniquely determined by Δ.

Let us finally discuss the effect of Landau damping. First, the ion Landau damping due to a finite ion temperature prevents a sharp resonance at $V = C_s/v_e = \varepsilon$, $|\lambda|$ ($= |V - \varepsilon|$) becoming at least of order ν_i/ω_s, where ν_i and ω_s are the damping rate and the frequency of the ion-acoustic wave. On the other hand, a large ion-density perturbation (of order $|v| \sim |w|$) predicted by the present theory assumes $|\lambda|$ to be of order $|w|$ or less; otherwise, $|v|$ becomes much smaller being of order $|w|^2$. This implies that in the case $|w|$ acts as a pump the ion Landau damping brings in a threshold ($|w| > \nu_i/\omega_s$) for the occurrence of a large ion-density perturbation. Secondly, the Landau damping of the electron-wave will cut down the large wavenumber components and thereby tends to smooth the perturbation. Finally, a large ion-density perturbation will benefit the ion heating as compared with the usual parametric instabilities where only electrons are selectively heated[9]. However, this ion heating will eventually destroy the present solution by increasing the threshold.

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