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Hydromagnetic Kink Mode of a Relativistic Electron Beam in a Current Carrying Plasma

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Synopsis

The stability of the kink mode of a relativistic electron beam propagating in a current carrying plasma is analysed. The treatments follows after Lee. It is found that the stability condition is more stringent than that of Kruskal and Shafranov. An available rotational transform ι must be $\frac{\iota}{2\pi} < [1 + 4\ln(d/a)]^{-1}$, where d and a are the radii of the metal wall and the beam. The relativistic effect appears in the characteristic oscillation.

§1. Introduction

In an approach to the problem of achieving controlled thermonuclear reactions by using toroidal magnetic confinement provided by the current carried by high energy electrons the low frequency hydromagnetic instability of relativistic beam in plasmas is one of the immediate problems to be clarified. Very recently, Lee[1)] has analysed this type of subject without taking into account of the plasma current. In general both plasma current and the beam one can flow in the equilibrium.

The purpose of the present paper is to investigate the effect of the equilibrium plasma current to the stability of the relativistic electron beam. Since the kink mode limits the total current we treat only the mode. The background cold, uniform plasma is assumed to be a perfect MHD fluid in which the intense monoenergetic electron beam propagates (See Figure). The basic equations are tabulated in §2. In §3 we treat the equilibrium configuration with an assumption that the electron beam energy is monochromatic. In §4 the linearized basic equations describing the cold background plasma, the field and the electron beam are given and these set of equations are applied to cylindrical equilibrium. In §5 we derive an ordinary differential equation from the linearized equations for a general equilibrium by an aid of an ordering scheme. Section 6 deals with the dispersion relation for a certain equilibrium.

§2. Basic Equations

The background cold plasma is assumed to be described by the following set of equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0 \ , \qquad (1)$$

$$\rho \frac{\partial \vec{u}}{\partial t} = \vec{J}_p \times \vec{B} \ , \qquad (2)$$

$$\vec{E} + \vec{u} \times \vec{B} = 0 \ , \qquad (3)$$

where ρ is the plasma mass density, $\vec{u}$ the plasma velocity and $\vec{J}_p$ the plasma current density. The quantities $\vec{E}$ and $\vec{B}$ are the electric field and the magnetic field respectively.

The intense relativistic electron beam is desribed by

$$m^* \gamma \frac{d\vec{V}}{dt} = -e(\vec{E} + \vec{v} \times \vec{B}) + \frac{e}{c^2} \vec{v}(\vec{v} \cdot \vec{E}) \qquad (4)$$

and

$$\frac{\partial n}{\partial t} + \nabla \cdot nv = 0 \ , \qquad (5)$$

where m^* and $-e$ are the rest mass and the charge of electron, respectively. The quantity $\vec{v}$ is the velocity of electron beam, $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$ and $\gamma = (1 - |\vec{v}|^2/c^2)^{-1/2}$. The Maxwell's equations are

$$\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \ , \tag{6}$$

$$\nabla \times \vec{B} = \mu_0 (\vec{J}_p + \vec{J}_b) \ , \tag{7}$$

and

$$\nabla \cdot \vec{B} = 0 \ , \tag{8}$$

where $\vec{J}_b = -\, ne\, \vec{v}$ and μ_0 is the magnetic permeability of the vacuum.

§3. Equilibrium State

In the equilibrium state the velocity of the background plasma is assumed to be zero. And this state is described by the following set of equations.

$$\vec{J}_{p_0} \times \vec{B}_0 = 0 \ , \tag{9}$$

$$\nabla \times \vec{B}_0 = \mu_0 (\vec{J}_{p_0} + \vec{J}_{b_0}) \ , \tag{10}$$

$$m^* \gamma_0 (\vec{v}_0 \cdot \nabla) \vec{v}_0 = -e \vec{v}_0 \times \vec{B}_0 \tag{11}$$

and

$$\nabla \cdot n_0 \vec{v}_0 = 0 \ , \tag{12}$$

where the suffix 0 represents the equilibrium quantities,

$\gamma_0 = (1 - |\vec{v}_0|^2/c^2)^{-1/2}$ and $\vec{J}_{b_0} = - n_0 e \, \vec{v}_0$.

From Eq.(9) we immediately have

$$\vec{J}_{p_0} = \alpha \, \vec{B}_0 \quad , \tag{13}$$

where α is a scalar function. If the beam energy is monochromatic in the whole region the following additional condition must be satisfied.

$$|\vec{v}_0/c|^2 = |\vec{J}_{b_0}/n_0 ec|^2 = \text{const.} \equiv \beta^2 \quad . \tag{14}$$

From Eqs.(10), (13), and (14)

$$n_0^2 e^2 \beta^2 c^2 = |\mu_0^{-1} \nabla \times \vec{B}_0 - \alpha \, \vec{B}_0|^2 \quad . \tag{15}$$

It is apparent from Eq.(15) that the equilibrium density profile of the relativistic electron beam is not uniform in space. In the cylindrical geometry (r, θ, z) with $\vec{B}_0 = (0, B_\theta(r), B_z(r))$ these set of equations are reduced to a single equation.

$$B_z \frac{dB_z}{dr} + \frac{B_\theta}{r} \frac{d}{dr} (rB_\theta) = \frac{m_0}{n_0 e^2 \mu_0} \frac{1}{r} [\frac{dB_z}{dr} + \delta B_\theta]^2 , \tag{16}$$

where $\delta = \alpha \, \mu_0$ and $m_0 = m^* \gamma_0$.

For ease of analysis we assume that the guide field B_z is exactly uniform. This assumption is possible only when the equilibrium plasma current is taken into account.

Then Eq.(16) is further reduced to

$$\frac{d}{dx}(xy) = \xi \frac{y}{f} \quad , \tag{17}$$

where $\xi = \delta c\beta/\omega_z$, $x = r/a$, $y = B_\theta/B_0$, $\omega_z = eB_0/m_0$, $n_0 = N_0\xi$, and a is the radius of the beam, and B_0 is the uniform guide field. The quantity N_0 is the density of the relativistic electron beam at x = 0, which is defined by

$$N_0 = \xi\, B_0{}^2/(m_0 c^2\beta^2\mu_0) \ . \tag{18}$$

The function f is defined by

$$f^2 = y^2 + (1 - \frac{c\beta}{\omega_z a}\frac{y}{xf})^2 \ . \tag{19}$$

In order to analytically proceed this investigation further we define a small quantity ε as follows. At x = 1

$$y = \varepsilon \quad << \quad 1. \tag{20}$$

Then from Eqs. (17) and (19) the function f is expanded as a power series for y.

$$f = 1 - \frac{c\beta}{\omega_z a}\frac{y}{x} + 0(\varepsilon^2). \tag{21}$$

And the Eq. (17) reduces to

$$\frac{dy}{dx} = (\xi - 1) \frac{y}{x} + 0(\varepsilon^2) \quad . \tag{22}$$

When δ is a constant the solution of Eq.(22) becomes

$$y = \varepsilon x^{\xi-1} + 0(\varepsilon^2). \tag{23}$$

The components of the equilibrium beam velocity v_{θ_0} and v_{z_0} can be written by

$$v_{\theta_0} = c\beta y/f$$

and

$$v_{z_0} = \frac{c\beta}{f}(1 - \frac{c\beta}{\omega_z a} \frac{y}{xf}) \quad . \tag{24}$$

The equilibrium angular velocity of the beam around the axis should be finite at $x = 0$. Hence, from Eqs.(23) and (24)

$$\xi \geqq 2 \quad . \tag{25}$$

Outside of the beam there is a cold dense plasma in which currents flow. Here the equilibrium is in the force free state, which is described by

$$B_z \frac{dB_z}{dx} + \frac{B_\theta}{x} \frac{d}{dx} (xB_\theta) = 0 \quad . \tag{26}$$

§4. Perturbed Equations

The quantities ρ , $\vec{u}$, $\vec{v}$, $\vec{B}$, $\vec{E}$, $\vec{J}_p$, $\vec{J}_b$ and n are assumed to have the form

$$\rho = \rho_0 + \rho_1 ,$$

$$\vec{u} = \vec{u}_1 ,$$

$$\vec{v} = \vec{v}_0 + \vec{v}_1 ,$$

$$\vec{B} = \vec{B}_0 + \vec{B}_1 ,$$

$$\vec{E} = \vec{E}_1 ,$$

$$\vec{J}_p = \vec{J}_{p_0} + \vec{J}_{p_1} ,$$

$$\vec{J}_b = \vec{J}_{b_0} + \vec{J}_{b_1} , \tag{27}$$

and

$$n = n_0 + n_1 ,$$

where the suffix 0 and 1 denote the equilibrium and the perturbed state respectively. Then the governing equations in the perturbed state are

$$\rho_0 \frac{\partial u_1}{\partial t} = \vec{J}_{p_0} \times \vec{B}_1 + \vec{J}_{p_1} \times \vec{B}_0 , \tag{28}$$

$$\vec{E}_1 + \vec{u}_1 \times \vec{B}_0 = 0 , \tag{29}$$

$$\frac{\partial \vec{B}_1}{\partial t} + \nabla \times \vec{E}_1 = 0 , \tag{30}$$

$$\nabla \times \vec{B}_1 = \mu_0 (\vec{J}_{p_1} + \vec{J}_{b_1}) \quad , \tag{31}$$

$$\frac{\partial \vec{V}_1}{\partial t} + (\vec{V}_0 \cdot \nabla)\vec{V}_1 + (\vec{V}_1 \cdot \nabla)\vec{V}_0 + \gamma_0^2 (\vec{V}_0 \cdot \nabla)\vec{V}_0 \frac{\vec{V}_0 \cdot \vec{V}_1}{c^2}$$

$$= - \frac{e}{m_0} (\vec{E}_1 + \vec{V}_1 \times \vec{B}_0 + \vec{V}_0 \times \vec{B}_1) + \frac{e}{m_0 c^2} \vec{V}_0 (\vec{V}_0 \cdot \vec{E}_1) \tag{32}$$

and

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (n_0 \vec{V}_1 + n_1 \vec{V}_0) = 0 \quad . \tag{33}$$

From Eqs.(13) and (28) we immediately have

$$\rho_0 \frac{\partial u_1}{\partial t} = \vec{B}_0 \times (\alpha \quad \vec{B}_1 - \vec{J}_{p_1}) . \tag{34}$$

And

$$\vec{B}_0 \cdot \frac{\partial \vec{u}_1}{\partial t} = 0 \quad . \tag{35}$$

Therefore the plasma velocity along the equilibrium line of force remains zero, provided it vanishes initially. We hereafter consider the case of $\vec{u}_1 \cdot \vec{B}_0 = 0$. Hence, from Eq.(29)

$$\vec{u}_1 = (\vec{E}_1 \times \vec{B}_0)/B_0^2 \quad . \tag{29'}$$

It is convenient to introduce the vector potential $\vec{A}$ with gauge chosen such that

$$\vec{B}_1 = \nabla \times \vec{A},$$

$$\vec{E}_1 = -\frac{\partial \vec{A}}{\partial t} \quad . \tag{36}$$

Then from Eqs.(29)', (31) and (34) we have

$$\vec{B}_0 \times \vec{I} = 0 \quad , \tag{37}$$

where

$$\vec{I} = \frac{\rho_0}{B_0{}^2} \frac{\partial^2 \vec{A}}{\partial t^2} + \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A} - \alpha \nabla \times \vec{A} - \vec{J}_{b_1} \quad . \tag{38}$$

Equation (37) is the governing equation of the perturbed field, and the perturbed state of the relativistic beam in the cold plasma is described by Eqs.(32), (33) and (37).

In a cylindrical configuration, equilibrium quantities are functions of the distance r from the axis of the cylinder. Taking advantage of the symmetry of the undisturbed system to Fourier-analyze the perturbed quantities we write

$$A_r = A_r(r,\ m,\ k)\sin S \ ,$$

$$A_\theta = A_\theta(r,\ m,\ k)\cos S \ ,$$

$$A_z = A_z(r,\ m,\ k)\cos S \ ,$$

$$v_{1_r} = v_r(r,\ m\ k)\sin S \ ,$$

$$v_{1_\theta} = v_\theta(r,\ m,\ k)\cos S \ ,$$

$$v_{1_z} = v_z(r,\ m,\ k)\cos S, \tag{39}$$

and

$$n_1 = n(r,\ m,\ k)\cos S,$$

where $S = \Omega t + m\theta + kz$ and $m = \pm 1$ (we treat only the kink mode). From Eqs.(37), (38) and (39) we have

$$I_r = \left(\frac{1}{r^2} + k^2 - \frac{c^2}{c_A{}^2}\right)A_r - \frac{d}{dr}\left(\frac{mA_\theta}{r} + kA_z\right) - \frac{2mA\theta}{r^2}$$

$$- \delta\left(kA\theta - \frac{mA_z}{r}\right) - \mu_0 J_{b_1 r} = 0 \tag{40}$$

and

$$B_\theta I_z - B_z I_\theta$$

$$= \frac{\Omega^2}{c_A{}^2}(B_z A_\theta - B_\theta A_z) + \left(kB_\theta - \frac{mB_z}{r}\right)\left\{\frac{1}{r}\frac{d}{dr}(rA_r) - \left(\frac{mA_\theta}{r} + kA_z\right)\right\}$$

$$+ B_z L A_\theta - B_\theta L A_z - \frac{B_z}{r} A_\theta$$

$$+ \left\{\frac{2mB_z}{r} + \delta\left(\frac{mB_\theta}{r} + kB_z\right)\right\}A_r - \delta\left\{\frac{B_\theta}{r}\frac{d}{dr}(rA_\theta) + B_z\frac{dA_z}{dr}\right\}$$

$$+ \mu_0(J_{b_1\theta}B_z - J_{b_1 z}B_\theta) = 0 \quad , \tag{41}$$

where $C_A{}^2 \equiv \dfrac{B_0{}^2}{\mu_0\rho_0}$ and $L \equiv \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - \left(\dfrac{1}{r^2} + k^2\right)$.

The r component I_r defines A_r in terms of A_θ and A_z. It is apparent from Eqs.(29) and (36) that

$$\vec{B}_0 \cdot \vec{A} = 0 \quad . \tag{42}$$

This relation is useful to eliminate the variable A_z from Eqs. (40) and (41). In the cylindrical system Eq.(42) is reduced to

$$A_z = -y\ A_\theta\ . \tag{43}$$

From Eq. (32) we have

$$\Gamma v_r = D_1 v_\theta + D_2 v_z + \frac{e}{m_0}\ \Gamma A_r - \frac{e}{m_0}\ \frac{c}{a}\ [D_3 A_\theta + D_4\ \frac{dA_\theta}{dx}]\ , \tag{44}$$

$$\Gamma v_\theta = D_5 v_r + \frac{e}{m_0}\ D_6 A_\theta\ , \tag{45}$$

$$\Gamma v_z = D_7 v_r - \frac{e}{m_0}\ D_8 A_\theta\ , \tag{46}$$

where

$$D_1 = \frac{c}{a}\ \frac{\beta_\theta}{x}\ (2 + \gamma_0{}^2\beta_\theta{}^2) - \omega_z\ ,$$

$$D_2 = \frac{c}{a}\ \frac{\beta_z}{x}\ \gamma_0{}^2\beta_\theta{}^2 + \omega_\theta\ ,$$

$$D_3 = \frac{\beta_\theta}{x} - \beta_z\ \frac{dy}{dx}\ ,$$

$$D_4 = \beta_\theta - y\ \beta_z\ ,$$

$$D_5 = \frac{c}{a}\ \frac{1}{x}\ \frac{d}{dx}\ (x\beta_\theta) - \omega_z\ ,$$

$$D_6 = (1 + y\beta_\theta\beta_z - \beta_\theta^2)\,\Omega - \frac{c}{a}\ \beta_z\ (\frac{my}{x} + \lambda)\ ,$$

$$D_7 = \frac{c}{a}\ \frac{d\beta_z}{dx} + \omega_\theta$$

and

$$D_8 = (y + \beta_\theta\beta_z - y\beta_z{}^2)\Omega \quad + \frac{c}{a}\beta_\theta(\frac{my}{x} + \lambda).$$

The quantities λ , ω_θ, β_θ and β_z are $\lambda = ka$, $\omega_\theta = \frac{eB_\theta}{m_0}$, $\beta_\theta = \frac{v_{0\theta}}{c}$ and $\beta_z = \frac{v_{0z}}{c}$.

The notation Γ means

$$\Gamma = \Omega \quad + \frac{c}{a}(\frac{m\beta_\theta}{x} + \lambda\beta_z). \tag{47}$$

The continuity equation is reduced to

$$n = (n_0 h + v_{1r}\frac{dn_0}{dx})/(a\Gamma) \quad , \tag{48}$$

where

$$h = \frac{1}{x}\frac{d}{dx}(xv_{1r}) - (\frac{mv_{1\theta}}{x} + \lambda v_{1z}) .$$

§5. Reduction to an Ordinary Differential Equation

To aid in the analysis we introduce an ordering in the quantity ε. It is found that the consistent scheme which is suitable for the problem of the present interest is

$$\beta \;,\; \beta_z \;,\; \frac{c}{\omega_{za}} \sim 0(1) \;,$$

$$\frac{\Omega a}{cA} \;,\; \frac{c_A}{c} \sim 0(\varepsilon^{1/2}) \;,$$

$$\lambda \;,\; \beta_\theta \sim 0(\varepsilon) \;. \tag{49}$$

Then dropping terms of order ε^2 Eqs. (40), (41), (44), (45), (46) and (48) are reduced into a single differential equation.

In the beam (i.e. $0 < x < 1$)

$$\left(\frac{a\Omega}{c_A}\right)^2 \frac{d}{dx}\left(x \frac{dY}{dx}\right) + \frac{2}{m}\xi\left(\frac{\omega_z a}{c\beta}\right)\left(\frac{my}{x} + \lambda\right)\frac{dY}{dx}$$

$$- \left[\left(\frac{a\Omega}{c_A}\right)^2 - \frac{x}{m}\left(\frac{my}{x} + \lambda\right)\frac{d}{dx}(\delta a)\right.$$

$$- \xi^2\left(\frac{\omega_z a}{c\beta}\right)\left\{\frac{x}{\xi}\frac{d}{dx}\left(\frac{Y}{x}\right) + \frac{Y}{x}\left(\left(1+\frac{2}{\xi}\right)\frac{\gamma^-}{\gamma^+} - \left(1-\frac{2}{\xi}\right)\right)\right]\frac{Y}{x} = 0 , \tag{50}$$

and out of the beam (i.e. $1 < x < d/a$)

$$\left(\frac{a\Omega}{c_A}\right)^2 \frac{d}{dx}\left(x \frac{dY}{dx}\right) + \frac{Y}{m}\left(\frac{my}{x} + \lambda\right)\frac{d}{dx}(\delta a) = 0 , \tag{51}$$

where $Y = xA_\theta$ and $\gamma^{\pm} = \frac{\Omega}{\omega_z} \pm \frac{c\beta}{\omega_z a}\left(m\frac{Y}{x} + \lambda\right)$. The quantity d is the radius of the wall surface.

It is very difficult to proceed the analysis analytically for the general ξ except for the case of $\xi = 2$ (critical equilibrium). We limit ourselves, hereafter, to discuss only the critical equilibrium. Then Eqs. (50) and (51) are reduced into simple differential equations. In the beam

$$\left(\frac{a\Omega}{c_A}\right)^2 \frac{d^2Y}{dx^2} + \frac{1}{x}\left[\left(\frac{a\Omega}{c_A}\right)^2 + \frac{4}{m}\left(\frac{\omega_z a}{c\beta}\right)(m\varepsilon + \lambda)\right]\frac{dY}{dx}$$

$$- \left[\left(\frac{a\Omega}{c_A}\right)^2 - 8\varepsilon\frac{\omega_z a}{c\beta}\frac{a\Omega - c\beta(m\varepsilon+\lambda)}{a\Omega + c\beta(m\varepsilon+\lambda)}\right]\frac{Y}{x^2} = 0 , \tag{52}$$

and out of the beam

$$\frac{d}{dx}\left(x\frac{dY}{dx}\right) \doteq 0 \quad , \tag{53}$$

where the assumption is made for simplicity that δa is chosen to be a step function so that the total current is continuous at $x = 1$.

The differential equation (52) has the following two solutions Y_+ and Y_-.

$$Y_\pm \propto \exp[(b \pm \sqrt{D})\,\ell n x] \quad , \tag{54}$$

where

$$b = -\frac{1+Z}{X^2}\,Q > 0 \quad ,$$

$$D = 1 - 8Q\,\frac{1}{X^2}\,\frac{X-Z-1}{X+Z+1} + 4Q^2\,\frac{(1+Z)^2}{X^4} \quad ,$$

$$X = \frac{ma\Omega}{c\beta\varepsilon} \ , \qquad Z = \frac{m\lambda}{\varepsilon} \quad \text{and} \quad Q = \frac{\omega_z a}{c}\,\frac{c_A{}^2}{\beta^3\varepsilon c^2} \ .$$

Since the plasma fills the metal cylinder to the wall Y should vanish at the wall surface $x = d^*$, where $d^* = d/a$. And by Eq.(53) the perturbed field is of the form

$$Y_0 = C\,\ell n(x/d^*) \ , \qquad 1 < x < d^* \ , \tag{55}$$

where C is a constant.

In the beam the solution becomes

$$Y_b = C_1\,x^b\,\sin(\sqrt{-D}\,\ell n x) + C_2\,x^b\,\cos(\sqrt{-D}\,\ell n x) , \tag{56}$$

where C_1 and C_2 are constants.

At $x = 1$ the following conditions should be satisfied.

$$\text{i)}\quad Y_0 = Y_b\ ,$$

$$\text{ii)}\quad \frac{dY_0}{dx} = \frac{dY_b}{dx}\ ,$$

$$\text{iii)}\quad \frac{d}{dx}\frac{1}{x}\frac{dY_0}{dx} = \frac{d}{dx}\frac{1}{x}\frac{dY_b}{dx}\ . \tag{57}$$

Then we obtain the dispersion relation

$$(D - b^2)\,\ell nd^* - 2b = 0\ . \tag{58}$$

§6. Analysis of Dispersion Relation and Conclusions

The expansion of the dispersion relation (58), valid near the zeros of X is

$$X^2 = -\frac{3}{2}\,Q(1 + Z)^2\ \ell nd^*/(4\ell nd^* + 1 + Z)\ . \tag{59}$$

This relation is sufficient to discuss the low frequency kink mode. The unstable wave numbers are associated with the value of Z for which X^2 is negative. Hence, the stability condition becomes

$$4\ \ell nd^* + 1 + Z < 0\ . \tag{60}$$

In a torus the wave number k is estimated to be

$$k = \frac{1}{R} \tag{61}$$

where R is the major radius of the torus. From Eqs. (60) and (61) we may conclude that the stability condition of relativistic beam in the current carrying plasma is more stringent than the one obtained by Kruskal and Shafranov.[2) 3)] An available rotational transform ι must be

$$\frac{\iota}{2\pi} \equiv \frac{R}{a}\varepsilon < \frac{1}{1 + 4\ \ell nd^{*}} \ . \tag{62}$$

The relativistic effect appears in the characteristic oscillations.

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References

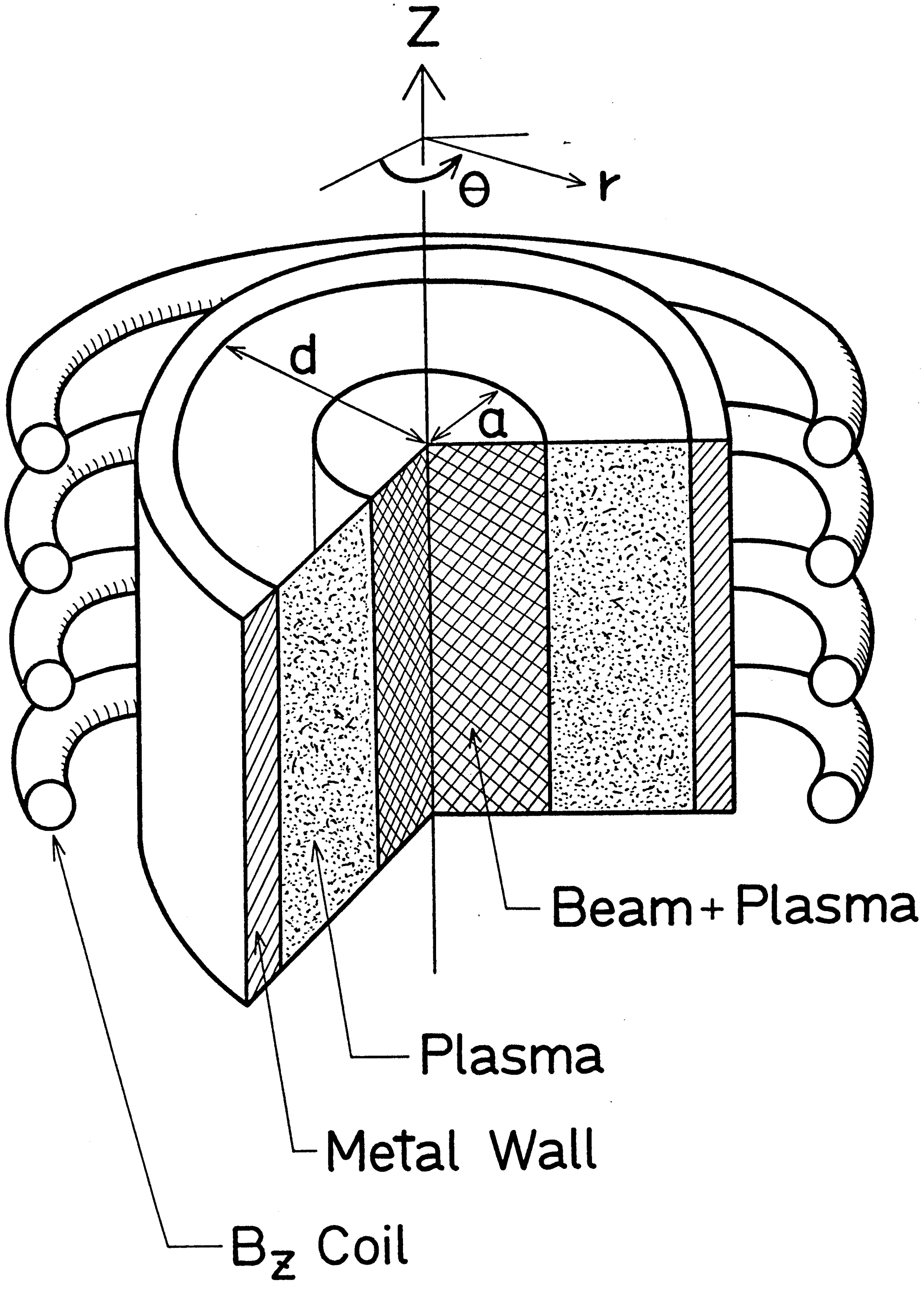
1) E.P. Lee; Phys. of Fluids 16 (1973) 1072.

2) M.D. Kruskal, et. al.; Phys. of Fluids 1 (1958) 421.

3) V.D. Shafranov, Plasma Physics and the Problem of Controlled Thermonuclear Reactions (Pergamon, London, 1959), Vol. II, P.197.

Figure Caption

Geometry.



Z
θ
r
d
a
Beam + Plasma
Plasma
Metal Wall
B_z Coil