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Two-dimensional stability of ion-acoustic solitons

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Abstract

Nonlinear ion-acoustic waves are analyzed in two-dimensional space. Korteweg - de Vries equation modified with a subsidiary equation is derived, from which it is shown that solitons are stable against disturbances at a right angle to their propagation direction.

§1. Introduction

Washimi and Taniuti (1968) first showed that nonlinear ion-acoustic waves could be described by the Korteweg - de Vries equation. Subsequently it was shown that this equation describes a wide class of weakly nonlinear dispersive systems (Taniuti & Wei 1968, Jeffrey & Kakutani 1972). It has a solution corresponding to a uniformly propagating pulse, the so-called soliton. The remarkable stability of these solitons has given rise to much interest in the last few years. However, all previous stability analyses, whether numerical (Zabusky 1967) or analy tic (Jeffrey & Kakutani 1972), have been restricted to one-dimensional perturbations.

To study the stability of a soliton to a more general perturbation it is first necessary to obtain the two-dimensional generalization of the Korteweg - de Vries equation. In this paper, the case of ion-acoustic waves is considered, the relevant equations obtained and then solved to show that the soliton is marginally stable to more general two-dimensional perturbations.

§2. Two-Dimensional Propagation of Ion-Acoustic Waves

Following Washimi and Taniuti (1968), the relevant set of basic equations is taken to be

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) + \frac{\partial}{\partial y}(nv) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial \phi}{\partial y}$$

$$-\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = n - e^{\phi} \qquad (1)$$

where quantities are dimensionless after normalization, that is ion density is normalized by n_o, the x- and y- components of ion velocity, u and v, by the ion acoustic velocity $(T_e/m_i)^{1/2}$, and the electrostatic potential ϕ by T_e/e.

We assume the ion-acoustic wave to propagate along the x-axis with a velocity V, and hence following Washimi and Taniuti, we introduce the stretched co-ordinates

$$\xi = \varepsilon^{1/2}(x - Vt)$$

$$\tau = \varepsilon^{3/2}t \qquad (2)$$

where ε is a small parameter essentially related to the amplitude of the wave. To describe any variation in the y- direction we introduce a new stretched co-ordinate

$$\eta = \varepsilon^{3/2} y \qquad (3)$$

That is, the variation in the y- direction is taken to be the same order as the time variation observed in a co-ordinate system moving with the velocity V along the x- axis. This is a reasonable assumption under which we consider stability of ion-acoustic wave solitons against a small variation in the transverse direction, because this effect must compete other terms in the Korteweg-de Vries equation, in which there is a first derivative of wave function with respect to τ.

We can introduce another stretched co-ordinate

$$\eta = \varepsilon^{1/2} y \qquad (4)$$

This stretching must be used when we treat a genuine two-dimensional propagation, in other words, when we consider a travelling wave involving variations in the transverse direction which can not be assumed as higher order perturbations. We analyze this case in the last section.

We apply, to the set of equations (1), the perturbation method developed by Taniuti and his collaborators. With expansions

$$\begin{aligned} n &= 1 + \varepsilon n^{(1)}(\tau,\xi,\eta) + \varepsilon^2 n^{(2)}(\tau,\xi,\eta) + \cdots\cdots \\ u &= \varepsilon u^{(1)} + \cdots\cdots \\ v &= \varepsilon v^{(1)} + \cdots\cdots \\ \phi &= \varepsilon \phi^{(1)} + \cdots\cdots \end{aligned} \qquad (5)$$

we find to lowest order

$$-V \frac{\partial}{\partial \xi} n^{(1)} + \frac{\partial}{\partial \xi} u^{(1)} = 0$$

$$-V \frac{\partial}{\partial \xi} u^{(1)} = -\frac{\partial \phi^{(1)}}{\partial \xi}$$

$$- V \frac{\partial}{\partial \xi} v^{(1)} = 0$$

$$n^{(1)} = \phi^{(1)} \qquad (6)$$

Setting $n^{(1)} = \Psi$ one gets $V^2 = 1$ and

$$u^{(1)} = V \Psi$$

$$\phi^{(1)} = \Psi \qquad (7)$$

To this order, the transverse variations are decoupled from a one-dimensional propagating wave described by the first two and the last of equations (6). The wave travels with the ion-acoustic speed since from these equations one obtains $V^2 = 1$. To second order, we have

$$- V\frac{\partial}{\partial \xi}n^{(2)} + \frac{\partial}{\partial \xi}u^{(2)} + \frac{\partial}{\partial \tau}n^{(1)} + \frac{\partial}{\partial \xi}n^{(1)}u^{(1)} + \frac{\partial}{\partial \eta}v^{(1)} = 0$$

$$- V\frac{\partial}{\partial \xi}u^{(2)} + \frac{\partial}{\partial \tau}u^{(1)} + u^{(1)}\frac{\partial}{\partial \xi}u^{(1)} = - \frac{\partial}{\partial \xi}\phi^{(2)}$$

$$(8)$$

$$- V\frac{\partial}{\partial \xi}v^{(2)} + \frac{\partial}{\partial \tau}v^{(1)} + u^{(1)}\frac{\partial}{\partial \xi}v^{(1)} = - \frac{\partial}{\partial \eta}\phi^{(1)}$$

$$- \frac{\partial^2}{\partial \xi^2}\phi^{(1)} = n^{(2)} - \left(\phi^{(2)} + \frac{1}{2}\phi^{(1)}\phi^{(1)}\right)$$

From the first two equations, we have

$$\frac{\partial}{\partial \xi} n^{(2)} = V^{-1}[V^{-1}\frac{\partial}{\partial \xi} \phi^{(2)} + 2\frac{\partial \psi}{\partial \tau} + 3V \psi\frac{\partial \psi}{\partial \xi} + \frac{\partial}{\partial \eta} v^{(1)}]$$

Substituting this into an equation obtained after differenciating the last equation of (8) with respect of ξ, one gets

$$\frac{\partial \psi}{\partial \tau} + V\psi \frac{\partial \psi}{\partial \xi} + \frac{V}{2} \frac{\partial^3 \psi}{\partial \xi^3} + \frac{1}{2} \frac{\partial}{\partial \eta} v^{(1)} = 0$$

where terms including $\phi^{(2)}$ have cancelled since $V^2 = 1$. For simplicity we restrict our attention to the case $V = 1$. Hence we have a Korteweg - de Vries equation modified by a transverse variation, which is expressed through the term involving $v^{(1)}$,

$$\frac{\partial \psi}{\partial \tau} + \psi \frac{\partial \psi}{\partial \xi} + \frac{1}{2} \frac{\partial^3 \psi}{\partial \xi^3} + \frac{1}{2} \frac{\partial}{\partial \eta} v^{(1)} = 0 \qquad (9)$$

From the third of equations (8), we find

$$\frac{\partial}{\partial \xi} v^{(2)} = \frac{\partial}{\partial \tau} v^{(1)} + \frac{\partial \psi}{\partial \eta} \qquad (10)$$

Since $v^{(1)}$ is independent of ξ as is seen from (6). Equations (9) and (10) govern the two-dimensional propagation of nonlinear ion-acoustic waves. As these two equations contain three quantities ψ , $v^{(1)}$ and $v^{(2)}$, one needs a supplimentary condition before one can obtain a unique solution. For this it is sufficient to demand that $v^{(2)}$ be finite. Integrating (10) with respect to ξ and dividing it by ξ, one gets

$$v^{(2)}/\xi = \frac{\partial}{\partial\tau} v^{(1)} + \frac{\partial}{\partial\eta}\frac{1}{\xi}\int \psi \, d\xi$$

In the limit of ξ tending to infinity the left hand side vanishes. Then since $v^{(1)}$ is independent of ξ, one may express $v^{(1)}$ in terms of ψ in the form

$$\frac{\partial}{\partial\tau} v^{(1)} = - \lim_{\xi\to\infty} \frac{1}{\xi}\int \frac{\partial\psi}{\partial\eta} d\xi \tag{11}$$

Equations (9) and (11) constitute the complete set of equations describing the two-dimensional propagation of nonlinear ion-acoustic waves.

§3. Stability Analyses

We now study, using equations (9) and (11), the stability of a soliton, propagating in the x-direction, to perturbations which depend on the transverse direction. To this end, we write

$$\psi(\xi,\eta,\tau) = \psi_0(\xi) + \delta\psi(\xi)e^{-i\omega\tau+ik\eta}$$

$$v^{(1)} = \delta v \, e^{-i\omega\tau+ik\eta} \tag{12}$$

where ψ_0 is a steady soliton solution of the Korteweg-de Vries equation (in other words, a steady solution of eqs.(9) and (11) with $v^{(1)}=0$). Substituting of the above forms into (9) and (11) and subsequent linearization gives

$$- i\omega\delta\psi + \frac{d}{d\xi}(\psi_0\delta\psi) + \frac{1}{2}\frac{d^3}{d\xi^3}\delta\psi + \frac{ik^2}{2\omega}\delta A = 0 \tag{13}$$

where

$$\delta A = \lim_{\xi\to\infty} \frac{1}{\xi} \int \delta\psi \; d\xi \quad . \tag{14}$$

The solution ψ_0 is unstable if there exists a solution for $\delta\psi$ which is a bounded function of ξ and for which the imaginary part of ω is greater than zero. Now if one writes

$$\delta\psi = \delta\tilde{\psi} + \frac{k^2}{2\omega^2} \delta A \tag{15}$$

and substitutes into eq.(13) one obtains

$$- i\omega\delta\tilde{\psi} + \frac{d}{d\xi}(\psi_0 \delta\tilde{\psi}) + \frac{1}{2} \frac{d^3}{d\xi^3} \delta\tilde{\psi} = - \frac{k^2}{2\omega^2} \frac{d\psi_0}{d\xi} \delta A \tag{16}$$

In the limit of $\xi\to\infty$, $\psi_o\to 0$ and $\frac{d\psi_0}{d\xi} \to 0$ so that for large ξ, $\delta\tilde{\psi}$ satisfies

$$\frac{1}{2} \frac{d^3}{d\xi^3} \delta\tilde{\psi} - i\omega\delta\tilde{\psi} = 0 \tag{17}$$

If ω is imaginary, $\delta\tilde{\psi}$ is found to be infinite from the equation (17). While $\delta\psi$ must be finite because it is a small perturbation imposed on the soliton solution (see eq.(12)), eqs.(14) and (15) are not consistent with each other. Hence ω is real. This leads a consistent discussion as follows.

With real ω , one sees from (17) that $\delta\tilde{\psi}$ is oscillating in the infinity of ξ , i.e., $\delta\tilde{\psi} = \exp[i(-2\omega)^{1/3}\xi]$. Therefore $\lim_{\xi\to\infty} \frac{1}{\xi}\int\delta\tilde{\psi}d\xi=0$, while at the lower limit of integration $\delta\tilde{\psi}$ is finite. Inserting Eq.(15), one finds a relation between k and ω

$$1 = k^2/2\ \omega^2 \tag{18}$$

where it must be noted that δA is independent of ξ. Hence one has got the same result that ω is real,

$$\omega = \pm k\ /\sqrt{2} \tag{19}$$

From this one concludes that any perturbations involving spatial variation in a direction at right angles to the direction of propagation will give rise to wave propagation but not to instabilities.

We have discussed stability of steady solitons represented by $\psi_0(\xi)$ for simplicity. The same analysis is valid when solitons are travelling with a speed λ, i.e., $\psi_0 = \psi_0(\xi-\lambda\tau)$. We just shift ψ to $\psi-\lambda$.

§4. Genuine Two Dimensional Propagation

Substituting the expansion (5) with the stretched co-ordinates (2) and (4) one finds in lowest order

$$\begin{aligned}
&- V \frac{\partial}{\partial \xi} n^{(1)} + \frac{\partial}{\partial \xi} u^{(1)} + \frac{\partial}{\partial \eta} v^{(1)} = 0 \\
&- V \frac{\partial}{\partial \xi} u^{(1)} = - \frac{\partial}{\partial \xi} \phi^{(1)} \\
&- V \frac{\partial}{\partial \xi} v^{(1)} = - \frac{\partial}{\partial \eta} \phi^{(1)} \\
&n^{(1)} = \phi^{(1)}
\end{aligned} \tag{20}$$

Writing

$$n^{(1)} = \psi$$

one gets

$$\phi^{(1)} = \psi$$

$$u^{(1)} = \psi/V$$

$$\frac{\partial}{\partial\xi} v^{(1)} = \frac{\partial\psi}{\partial\eta}/V$$

Differenciating the first equation of (20) with respect to ξ and substituting above expressions gives

$$(V^{-1} - V) \frac{\partial^2}{\partial\xi^2} \psi + V^{-1} \frac{\partial^2}{\partial\eta^2} \psi = 0 \qquad (21)$$

Unless $V^2 = 1$, one can transform co-ordinates according to

$$\alpha = \xi + i\gamma\eta$$

$$\beta = \xi - i\gamma\eta \qquad (22)$$

where $\gamma^2 = 1 - V^2$. Then eq.(21) will become simpler

$$\frac{\partial^2}{\partial\alpha\partial\beta} \psi = 0 \qquad (23)$$

Hence ψ is represented by two independent functions, each of which is arbitrary,

$$\psi = f(\alpha) + g(\beta) \tag{24}$$

If $V^2 = 1$, ψ must be independent of η, that is, two dimensional propagation is not proper. We must note here that ψ should be finite. The case $V^2 = 1$ is excluded henceforth.

To second order, one gets

$$-V \frac{\partial}{\partial \xi} n^{(2)} + \frac{\partial}{\partial \xi} u^{(2)} + \frac{\partial}{\partial \tau} n^{(1)} + \frac{\partial}{\partial \xi} n^{(1)} u^{(1)} + \frac{\partial}{\partial \eta} n^{(1)} u^{(1)} + \frac{\partial}{\partial \eta} v^{(2)} = 0$$

$$-V\frac{\partial}{\partial \xi} u^{(2)} + \frac{\partial}{\partial \tau} u^{(1)} + u^{(1)} \frac{\partial}{\partial \xi} u^{(1)} + v^{(1)} \frac{\partial}{\partial \eta} u^{(1)} = - \frac{\partial \phi^{(2)}}{\partial \xi}$$

(25)

$$-V\frac{\partial}{\partial \xi} v^{(2)} + \frac{\partial}{\partial \tau} v^{(1)} + u^{(1)} \frac{\partial}{\partial \xi} v^{(1)} + v^{(1)} \frac{\partial}{\partial \eta} v^{(1)} = - \frac{\partial \phi^{(2)}}{\partial \xi}$$

$$- \frac{\partial^2}{\partial \xi^2} \phi^{(1)} - \frac{\partial^2}{\partial \eta^2} \phi^{(1)} = n^{(2)} - (\phi^{(2)} + \frac{1}{2} \phi^{(1)} \phi^{(1)})$$

Eliminating $n^{(2)}$, $u^{(2)}$ and $v^{(2)}$ gives

$$(V^{-1} - V) \frac{\partial^2}{\partial \xi^2} \phi^{(2)} + V^{-1} \frac{\partial^2}{\partial \eta^2} \phi^{(2)} = \cdots\cdots ,$$

where the right hand side includes only the lowest order quantities. Carrying out the transformation (22) and substituting eq.(24) one finds

$$4(V^{-1} - V) \frac{\partial^2}{\partial \alpha \partial \beta} \phi^{(2)} = T\{f(\alpha)\} + T\{g(\beta)\} + S\{f(\alpha), g(\beta)\} \tag{26}$$

One can directly integrate with respect to β, i.e.,

$$4(v^{-1}-V)\ \frac{\partial}{\partial\alpha}\ \phi^{(2)} = \beta T_\alpha + \int T_\beta\, d\beta + \int S_{\alpha\beta}\, d\beta$$

Since $\phi^{(2)}$ must be finite, one has a requirement

$$T_\alpha = T\ \{f(\alpha)\} = 0$$

Similarly, there follows

$$T_\beta = T\ \{g(\beta)\} = 0$$

These two are nothing but Korteweg-de Vries equations

$$\frac{\partial f}{\partial\tau} + Vf\ \frac{\partial f}{\partial\alpha} + \frac{V^3}{2}\ \frac{\partial^3 f}{\partial\alpha^3} = 0$$

$$\frac{\partial g}{\partial\tau} + V\ g\ \frac{\partial g}{\partial\beta} + \frac{V^3}{2}\ \frac{\partial^3 g}{\partial\beta^3} = 0 \qquad (27)$$

Here one finds equations which govern two-dimensional propagation of the ion-acoustic waves. The wave profile is given by an addition of two functions, each of which is a solution of the Korteweg-de Vries equation involving a complex argument, α and β defined by (22).

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