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Kink Instability of a Cylindrical Tokamak with an Arbitrary Current Distribution.

Kimitaka Itoh and Shoichi Yoshikawa

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan.

Permanent Address:

Department of Physics, Faculty of Science ,

University of Tokyo, Tokyo, Japan.

Abstract

A new method to analyze the MHD kink instability in the cylindrical approximation is presented. Using the nonlinear equilibrium equation with the helical perturbation method, an expression for the growth rate of instability of a perturbation is obtained. For m=1 mode, the growth rate and the stability-instability criterion, q(a)=1, are calculated analytically for an arbitrary current distribution. The growth rate is also calculated in the case of a uniform current.

Much work has been done previously about the MHD kink instability[1] in terms of the normal mode analysis[2] or of the energy principle method[3]. The former method is restricted to a particular current distribution, and the latter seems very complicated. Previous results imply that systems subject to kink instability have helical symmetry. On the other hand, small amplitude helical eqilibrium equations of a plasma which carries an arbitrary current in a cylindrical tokamak were obtained by means of MHD approximation[4]. By use of these equations, helical perturbations on the magnetic surface and the torsion of the plasma due to the perturbation can be calculated, once the current profiles are given. Our method presents a systematic way to analyze the MHD kink instability for an arbitrary current distribution. Especially in the case of the m=1 "free boundary" mode, the growth rate and the stability criterion have been obtained analytically. In the case of a uniform current, the growth rate has been also calculated. These results agree with the results obtained by Shafranov.

In the cylindrical approximation, the tokamak configuration can be considered as a plasma (radius a) in a perfectly conducting cylinder (radius b) with the periodicity $2\pi R$ (R is the major radius). By introducing the helical coordinates with $\varphi = l\theta + k_z z$ (r, θ, z are ordinary cylindrical coordinates), the φ component of the magnetic field B_φ and that of the vector potential A_φ are given as

$$B_\varphi = k_z r B_\theta - l B_z \tag{1}$$

$$A_\varphi = k_z r A_\theta - l A_z \equiv \psi \quad , \tag{2}$$

and the plasma equilibrium equation, $\vec{j}\times\vec{B}=\nabla p$, can be expressed as

$$(k_z^2+\frac{l^2}{r^2})\frac{\partial^2\psi}{\partial\varphi^2}+r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\psi+\frac{2}{r}\frac{l^2}{(k_z r)^2+l^2}\frac{\partial\psi}{\partial r}+\frac{2lk_zB_\varphi}{(k_z r)^2+l^2}=-\frac{d}{d\psi}(\frac{1}{2}B_\varphi^2)-\mu_0\frac{dp}{d\psi} \quad . \tag{3}$$

Since $(ak_z)^2=(a/R)^2 \ll 1$, in the limit of the strong troidal field B_t with $\beta=0$, Eq.(3) can be written as

$$\frac{l^2}{r^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r}-2k_zB_t=\mu_0 l J_z(\psi) \quad . \tag{3'}$$

The other components of the magnetic field and current density in the z direction are

$$k_z r B_z + l B_\theta = \frac{\partial\psi}{\partial r} \quad , \tag{4}$$

$$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial\varphi} \quad , \tag{5}$$

$$J_z(r) = \frac{1}{\mu_0}B_z\frac{\partial B_\varphi}{\partial\psi} \quad . \tag{6}$$

The boundary condition is such that B_r vanishes at r=b, or

$$\frac{\partial\psi}{\partial\varphi}=0 \qquad at \qquad r=b \; . \tag{7}$$

In order to solve Eqs.(3')&(7) by perturbation method, we expand $\psi(r,\varphi)$ as (inside the plasma)

$$\psi(r,\varphi) = \psi_0(r) + \alpha\psi_1(r)\cos\varphi + \alpha^2\psi_2(r) + \cdots , \tag{8}$$

where α is a free parameter. We substitute Eq.(8) into Eq.(3') , then we get

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi_0}{\partial r} = 2k_z B_t + \mu_0 l J_z(r) , \tag{9}$$

$$-\frac{l^2}{r^2}\psi_1 + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi_1}{\partial r} = \mu_0 l \frac{J_z'(r)}{\psi_0'}\psi_1 , \tag{10}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi_2}{\partial r} = \mu_0 l \frac{J_z'}{\psi_0'}\psi_2 + \frac{\mu_0 l(\psi_1)^2}{4(\psi_0')^3}\{J_z''\psi_0' - J_z'\psi_0''\} , \tag{11}$$

where ' denotes the derivative with respect to r. In vacuum, $\psi(r,\varphi)$ is expanded as

$$\psi_v(r,\varphi) = \psi_{v0}(r) + \varepsilon\psi_{v1}(r)\cos\varphi + \cdots , \tag{12}$$

and a set of equations for ψ_{v0}, ψ_{v1} is given as

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi_{v0}}{\partial r} = 2k_z B_t , \tag{13}$$

$$-\frac{l^2}{r^2}\psi_{v1} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi_{v1}}{\partial r} = 0 . \tag{14}$$

Then Eqs.(8)&(12) are combined on the surface of the plasma. From the continuity condition imposed on ψ on the surface of the plasma we obtain

$$\psi_{v1}(a)\,\varepsilon = \psi_1(a)\,\alpha \quad . \tag{15}$$

The displacement of the plasma surface, δ , is also determined by α (as shown below). By use of δ, we can calculate the magnetic energy change in the 2nd order of δ due to the perturbation. It shoud be noted that work done by the power supply due to the perturbation is two times the change of the magnetic energy, because the total current is not changed by the perurbation considered here (ordinary tokamak operation is under constant current condition). $\psi(r,\varphi)$ is constant on the surface $r=a+\delta\cos\varphi$ so $\psi(a+\delta\cos\varphi,\varphi)$ is independent of φ,

$$\psi(a+\delta\cos\varphi,\varphi) - \psi_0(a) = \psi_0'(a)\,\delta\cos\varphi + \psi_1(a)\,\alpha\cos\varphi = 0 ,$$

$$-\psi_0'(a)\,\delta = \psi_1(a)\,\alpha \quad . \tag{16}$$

ψ_1' is not continuous in the first order of α. This means that there is the 1st order surface current. Work ΔW_s is done due to the torsion of the plasma column because of $\vec{j}\times\vec{B}$ force from the outside to the inside. ΔW_s is given as

$$\Delta W_s = \frac{1}{2l\mu_0}\int_0^{2\pi} 2\psi_0'(a)\,(\psi_v'(a) - \psi'(a))\,a\,\delta\cos\varphi\,d\varphi \quad . \tag{17}$$

Let the change of B^2 due to this perturbation be ΔB^2, then the magnetic energy change in the plasma per unit length, ΔW_p , is

$$\Delta W_p = \frac{1}{2l\mu_0}\int_0^{2\pi}\int_0^{a+\delta\cos\varphi}\Delta B^2\, r\, dr\, d\varphi \;, \tag{18}$$

and that in vacuum per unit length, ΔW_v , is

$$\Delta W_v = \frac{1}{2l\mu_0}\int_0^{2\pi}\int_{a+\delta\cos\varphi}^{b}\Delta B^2\, r\, dr\, d\varphi \;. \tag{19}$$

The total magnetic energy change per unit length, ΔW_m , is equal to $\Delta W_p + \Delta W_v$.

From Eqs.(1),(4)&(5), we obtain

$$B^2 = \frac{1}{l^2}\left\{ B_\varphi^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right\} + \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\varphi}\right)^2 \;. \tag{20}$$

In the plasma, B_φ is expanded as

$$B_\varphi(\psi) = B_\varphi(\psi_0) + \left.\frac{\partial B_\varphi}{\partial\psi}\right|_{\psi_0}\alpha\psi_1\cos\varphi + \left.\frac{\partial B_\varphi}{\partial\psi}\right|_{\psi_0}\alpha^2\psi_2 + \frac{1}{2}\left.\frac{\partial^2 B_\varphi}{\partial\psi^2}\right|_{\psi_0}\alpha^2\psi_1^2\cos^2\varphi + \cdots \;. \tag{21}$$

Then

$$\Delta B^2 = \left[2B_\varphi\frac{\partial B_\varphi}{\partial\psi}\psi_1 + 2\frac{\partial\psi_0}{\partial r}\frac{\partial\psi_1}{\partial r}\right]\frac{1}{l^2}\alpha\cos\varphi + \left[\frac{1}{l^2}\left\{2B_\varphi\frac{\partial B_\varphi}{\partial\psi}\psi_2 + \frac{\partial}{\partial\psi}\left(B_\varphi\frac{\partial B_\varphi}{\partial\psi}\right)\psi_1^2\cos^2\varphi + 2\frac{\partial\psi_0}{\partial r}\frac{\partial\psi_2}{\partial r} + \left(\frac{\partial\psi_1}{\partial r}\right)^2\cos^2\varphi\right\} + \frac{\psi_1^2}{r^2}\sin^2\varphi\right]\alpha^2 + \cdots \;. \tag{22}$$

In vacuum B_φ is constant, and

$$\Delta B^2 = \frac{2}{l^2}\frac{\partial\psi_{v0}}{\partial r}\frac{\partial\psi_{v1}}{\partial r}\varepsilon\cos\varphi + \left\{\frac{1}{l^2}\left(\frac{\partial\psi_{v1}}{\partial r}\right)^2\cos^2\varphi + \frac{\psi_{v1}^2}{r^2}\sin^2\varphi\right\}\varepsilon^2 + \cdots \;. \tag{23}$$

$\alpha, \delta, \varepsilon$ are related to each other through Eqs.(15)&(16).

Substituting Eq.(22) into Eq.(18), and Eq.(23) into Eq.(19), ΔW_m is calculated. From the energy conservation $2\Delta W_m$ is equal to $\Delta W_m + \Delta W_s + \Delta W_k$, i.e.

$$\Delta W_k = \Delta W_m - \Delta W_s \quad . \tag{24}$$

The expression for the growth rate γ is given as

$$\gamma^2 \delta^2 = \frac{2}{\pi a^2 \langle \rho \rangle} \Delta W_k \quad . \tag{25}$$

where $\langle \rho \rangle$ is the averaged density.

We next solve the case of the m=1 mode. Let $\ell = 1$, and the solutions for Eqs.(10)&(11) are given as

$$\psi_1(r) = \frac{d\psi_0}{dr} \quad , \tag{26}$$

$$\psi_2(r) = \frac{1}{4}(\mu_0 J_z(r) + 2k_z B_t) . \tag{27}$$

Eq.(27) is not a general solution of Eq.(13), but it is adequate for this analysis, for only the 2nd order perturbation generated by ψ_1 is significant. The kinetic energy change, ΔW_k, due to this perturbation is

$$\Delta W_k = \frac{\pi}{\mu_0} a^2 k_z^2 B_t^2 \frac{1}{q^2} \left\{ 3(1-q) - \frac{3}{1-\frac{a^2}{b^2}} (1-q)^2 \right\} . \tag{28}$$

The values of γ^2 are obtained for various values of a/b and are shown in Fig.1. From Eq.(28) it is evident that the stability criterion of m=1 mode is q(a)=1 for an arbitrary distribution of current.

For the case of a uniform current, ΔW_k for any m mode can be obtained by this method. Fig.2 shows the results of our calculation up to m=3.

Higher mode instability for any current profile is being analyzed numerically by use of this method[5].

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Figure Captions

Fig.1. Growth rates of m=1 mode perturbation for an arbitrary current distribution and various values of a/b. It is evident that the stability criteria are q(a)=1. As the value of a/b is increased, the unstable region becomes narrower and the growth rate of instability approaches zero. 1)a/b=0, 2)a/b=0.6, 3)a/b=0.8.

Fig.2. Growth rates of perturbation for the uniform distribution of current density and for the various values of a/b. 1)a/b=o, 2)a/b=0.6, 3)a/b=0.8.

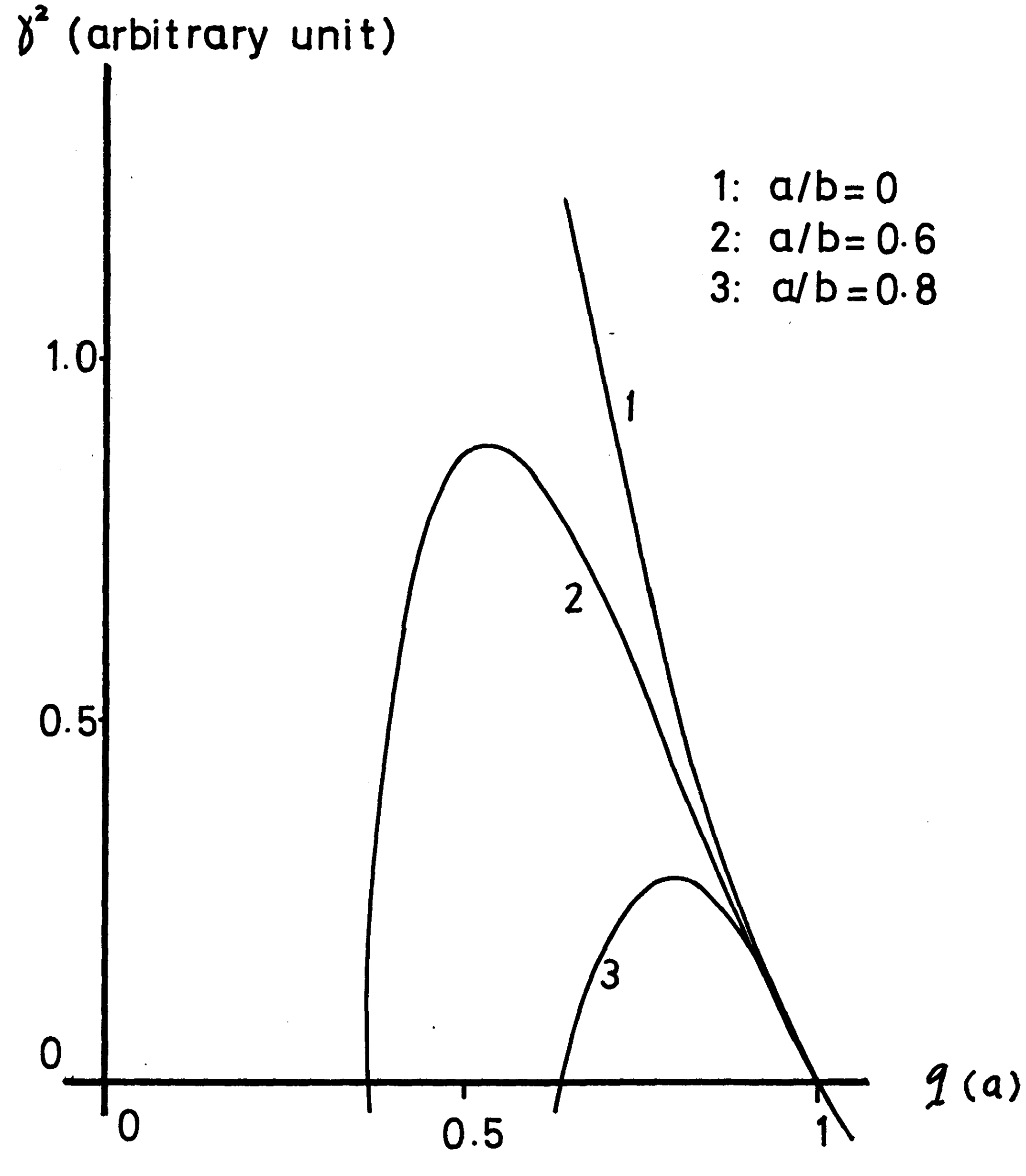

Fig. 1

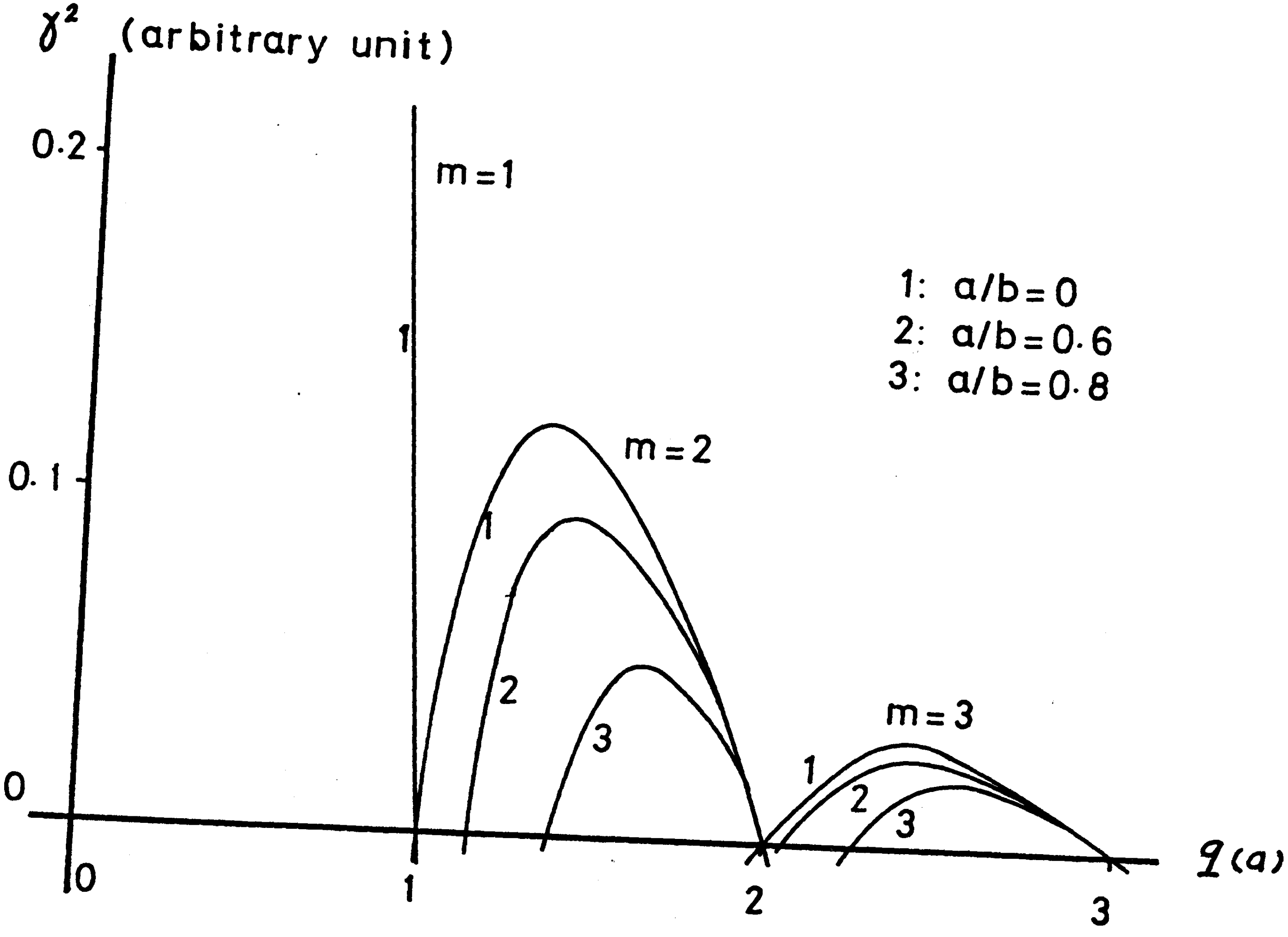

Fig. 2