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RESEARCH REPORT

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Modified Clemmow-Mullaly-Allis Diagram
for Large-Amplitude Electromagnetic
Waves in Magnetoplasmas

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Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan.

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ABSTRACT

A possible modification to the well known Clemmow=Mullaly-Allis diagram is analysed taking into account the radiation pressure force due to a large-amplitude electromagnetic field $\mathbf{E}$ in magnetoplasmas.

We restrict ourselves here to the propagations parallel (the right and left-hand circularly polarized waves) and/or perpendicular (the ordinary and extraordinary modes) to the static magnetic field $\mathbf{B}_o$. We analyse electromagnetic waves incident normally on a semi-infinite uniform plasma, on which $\mathbf{B}_o$ is applied parallel and/or perpendicular to the surface. Considerations are limited to a cold collisionless plasma where the incident waves are evanescent.

Simple expressions are obtained for the cut-off conditions of the waves except the extraordinary mode. In the latter case, the cut-off condition is calculated numerically solving an integral equation. The results are demonstrated in the usual Clemmow-Mullaly-Allis diagram for the various values of $b = 2E_i^2e^2/m\omega^2\kappa T_e$, where E_i and ω are, respectively, the amplitude and the angular frequency of the incident wave. The cut-off lines are shown to move towards the higher densities with increasing b.

I. INTRODUCTION

In recent years, nonlinear phenomena caused by the electromagnetic waves (EM waves, hereafter) of large amplitudes in plasmas have been pointed out by many authors. They include the parametric decay processes[1], the particle trapping[2], the modulational instabilities[3], self-focusing[4,5] and so on.

It is well known that the characteristic features of the EM waves in cold magnetoplasmas are expressed in the Clemmow-Mullaly-Allis diagram (CMA diagram, hereafter)[6]. We consider a coordinate system in which the scale lengths in the different directions are proportional to the parameters of the plasma, such as electron density, static magnetic field strength, percentage composition by the ion species. This parameter space is divided into a number of volumes. Inside each of these volumes, the topological genera of the wave normal surface[7] are conserved. The diagram is extremely useful to understand the wave phenomena in laboratory and/or ionospheric plasmas.

However, it has been based on the linear treatment of wave analysis. Thus, in principle, the diagram can be applied only for the waves of small amplitudes. Questions arise here: what change will happen in the CMA diagram, when the wave amplitudes increase? Is it possible to assess some expected nonlinear processes, even qualitatively, in the diagram? Although these questions are left, in general, unsolved at this moment, a possible modification in the diagram is presented in this article. We analyse a modification of the CMA diagram caused by the radiation pressure of the high-power EM waves. Here, the term "radiation pressure" means the time-averaged force acting on the electrons in the EM field, when it is spatially inhomogeneous.

As is well known, a traveling wave creates no radiation pressure in its direction of propagation. In order the radiation pressure to be effective, the wave must create a quasi-potential barrier due to the radiation pressure in the laboratory frame, whereas the traveling waves create the potential barrier in the wave frame[8]. In other words, the resonant electrons whose velocities are close to the wave phase velocity can be trapped by the potential barrier[2]. However, we do not take into account the trapping phenomena in this article. We confine ourselves to a cold plasma where no resonant electrons are present.

The quasi-potential due to the radiation pressure can be important in the following cases:

(I) A plane standing EM field, which can be decomposed into two propagating waves in opposite directions one another, can cause the potential in the direction of propagations.

(II) A propagating plane EM wave can create the potential in a direction perpendicular to that of the propagation[5]. In microwave terminology, waveguide solutions can exist in the plasma.

(III) When an incident plane EM wave on a plasma is evanescent, the EM field in the plasma due to the skin effect[9] can cause the quasi-potential in the direction of incidence.

In this article, we analyse the case (III), which results in a modification of the CMA diagram.

II. OUTLINE OF THE FORMULATION

Consider a plane EM wave incident normally on a semi-infinite uniform plasma. Considerations are limited to a high-density plasma such that the incident wave is evanescent. Then the EM field $\mathbf{E}(\mathbf{r})\exp(-i\omega t)$ in the plasma does not propagate. The effect of collisions between plasma constituents are ignored. Ions are assumed to be a continuous medium which recovers the neutrality in the plasma. The plasma is assumed to be cold, i.e.,

$$d \gg v_{te}/\omega, \quad v_d/\omega,$$

where $v_{te} = (2\kappa T_e/m)^{1/2}$, $v_d = e|\mathbf{E}|/m\omega$ and d is the skin depth which is the characteristic distance of the spatial variation of $\mathbf{E}(\mathbf{r})$. Landau and Lifshitz[10] derived an expression of radiation pressure of the EM field acting on the electrons. Pitaevskii[11] extended their analysis to a magnetized plasma. We write in our notation the radiation pressure force $\mathbf{F}(\mathbf{r})$ as,

$$\mathbf{F}(\mathbf{r}) = -\frac{e^2}{2m\omega^2}\beta_{ik}\nabla E_i{}^* E_k \tag{1}$$

where

$$\beta_{ik} = \begin{bmatrix} \dfrac{\omega^2}{\omega^2-\omega_c{}^2} & -\dfrac{i\omega\omega_c}{\omega^2-\omega_c{}^2} & 0 \\ \dfrac{i\omega\omega_c}{\omega^2-\omega^2} & \dfrac{\omega^2}{\omega^2-\omega_c{}^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

and ω_c is the electron cyclotron frequency. Furthermore, the averaged electron density is assumed to change by an amount of the Boltzmann factor caused by the quasi-potential, if

$$d \gg v_{te}\tau\ ,\ v_{ti}\tau\ ,$$

where $v_{ti} = (2\kappa T_i/M)^{1/2}$ and τ is the characteristic period that $\mathbf{E}(\mathbf{r})$ is present in the plasma. The radiation pressure acting on the ions is neglected, since it is m/M times smaller than that on the electrons as is shown in (1). The change of the electron density causes a charge separation which results in a self-consistent potential ψ in the plasma. Then the densities of electrons and ions, N_e and N_i, are given respectively as

$$N_e = N_0 \exp\left(\frac{e\psi - \dfrac{e^2}{2m\omega^2}\beta_{ik}E_i^* E_k}{\kappa T_e} \right), \tag{3}$$

$$N_i = N_0 \exp\left(- \frac{e\psi}{\kappa T_i} \right), \tag{4}$$

where N_o is the plasma density in the region where no EM fields are present. Assuming that the skin depth d is much greater than the Debye length, we impose the quasi-neutral condition, i.e. $N_e = N_i = N'$. Using (3) and (4), one obtains,

$$e\psi = \frac{e^2}{2m\omega^2(1+T_e/T_i)}\beta_{ik}E_i^* E_k\ .$$

Substituting $e\psi$ into (3) or (4), the density N' is given as

$$N' = N_o \exp\left[-\frac{e^2\beta_{ik}E_i{}^*E_k}{2m\omega^2\ (T_e+T_i)}\right]. \qquad (5)$$

We assume $T_e >> T_i$ hereafter for simplicity. Substituting (2) into (5), the normalized electron density is expressed[11] as follows,

$$X' = X\exp\left\{-\frac{e^2}{2m\omega^2\kappa T_e}\left[|E_{\parallel}|^2+\frac{|E_{\perp}|^2}{1-Y^2}-\frac{iY}{1-Y^2}(E_x{}^*E_y-E_xE_y{}^*)\right]\right\}. \qquad (6)$$

where $X' = (\omega_p{}'/\omega)^2$, $X = (\omega_p/\omega)^2$, $Y = \omega_c/\omega$, $\omega_p{}'^2 = eN'/\varepsilon_o m$ and $\omega_p{}^2 = eN_o/\varepsilon_o m$. Using (6), we calculate the dielectric tensor ε_{ik}' of the plasma as,

$$\epsilon_{ik}' = \delta_{ik} + \frac{\partial\epsilon_{ik}}{\partial X'} = \delta_{ik} + X'\beta_{ik},$$

i.e.

$$[\epsilon'] = \begin{bmatrix} \epsilon_1 & \epsilon_2 & 0 \\ -\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \qquad (7)$$

where

$$\epsilon_1 = 1-\frac{X'}{1-Y^2},\quad \epsilon_2 = \frac{iX'Y}{1-Y^2}\quad \text{and}\quad \epsilon_3 = 1-X'.$$

The structure of (7) is the same as the dielectric tensor well known in the cold plasma theory[6], except that X is replaced by X' in (7).

The Maxwell equation for $\mathbf{E}(\mathbf{r})$ is written as,

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = \frac{\omega^2}{c^2}[\epsilon']\mathbf{E} \, , \tag{8}$$

where c is the velocity of light in vacuum. The field **E** in (8) can be generally different from the EM field in $[\varepsilon']$ which causes the quasi-potential in the plasma. In other words, (8) can be a linear equation regarding **E** with a small amplitude in a plasma where another strong EM field is present. However, we restrict ourselves here in such a case that **E** in (8) and that in $[\varepsilon']$ are the same one. In this case, **E** should be determined self-consistently from a set of nonlinear equations (6), (7) and (8).

III. PARALLEL PROPAGATIONS ($\mathbf{B}_o // \mathbf{k}$)

We choose z axis along the applied magnetic field $\mathbf{B}_o$, which is perpendicular to the plasma surface. The plasma is assumed to occupy the half-space $z \geq 0$. In this case, the EM field in the plasma is decomposed into the longitudinal and the transverse components, $E_\parallel$ and $E_\perp$, respectively, as is seen from (7) and (8). The former one is the electron plasma wave propagating in a direction of $\mathbf{B}_o$. We put $E_\parallel = 0$, since the purely longitudinal plasma waves do not concern us in this section.

$$\begin{aligned} &\frac{d^2E_x}{dz^2} + \frac{\omega^2}{c^2}(\epsilon_1 E_x + \epsilon_2 E_y) = 0 \, , \\ &\frac{d^2E_y}{dz^2} + \frac{\omega^2}{c^2}(-\epsilon_2 E_x + \epsilon_2 E_y) = 0 \, . \end{aligned} \tag{9}$$

We transform E_x and E_y into the rotating vector components, E_r and E_ℓ defined by

$$E_1 = \frac{E_x + iE_y}{\sqrt{2}} , \quad E_r = \frac{E_x - iE_y}{\sqrt{2}} , \tag{10}$$

where the suffixes r and ℓ denote, respectively, the right and left-hand circularly polarizations. Then one obtains the following equations instead of (9), as

$$\begin{aligned} \frac{d^2E_1}{dz^2} + \frac{\omega^2}{c^2}\epsilon_1 E_1 &= 0 , \\ \frac{d^2E_r}{dz^2} + \frac{\omega^2}{c^2}\epsilon_r E_r &= 0 , \end{aligned} \tag{11}$$

where

$$\begin{aligned} \epsilon_1 &= \epsilon_1 - i\epsilon_2 = 1 - \frac{X'}{1+Y} , \\ \epsilon_r &= \epsilon_1 + i\epsilon_2 = 1 - \frac{X'}{1-Y} . \end{aligned} \tag{12}$$

Using (10), E_x and E_y are expressed, respectively, as $E_x = (E_\ell + E_r)/\sqrt{2}$ and $E_y = (E_\ell - E_r)/\sqrt{2}i$. Substituting these relationship into (6), one obtains

$$X' = X\exp\left[-\frac{e^2}{2m\omega^2\kappa T_e}\left(\frac{|E_r|^2}{1-Y} + \frac{|E_1|^2}{1+Y}\right)\right] \tag{13}$$

When there is only one of the right- (R wave, hereafter) and the left- (L wave, hereafter) hand circularly polarized waves in the plasma, (13) is simplified. For simplicity, we define $Y > 0$ and $Y < 0$ for R and L waves, respectively. Then, the above expressions are rewritten into a single form, as

$$X' = X\exp\left[-\frac{|E_s|^2}{E_c^2(1-Y)}\right], \tag{14}$$

where $E_c^2 = 2m\omega^2\kappa T_e/e^2$ and the suffix s denotes r or ℓ. Then, (12) is expressed as,

$$\epsilon_s = 1 - \frac{X'}{1-Y}. \tag{15}$$

Using (14) and (15), the Maxwell equation (8) is written[8] as,

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\left\{1 - \frac{X}{1-Y}\exp\left[-\frac{|E|^2}{E_c^2(1-Y)}\right]\right\}E = 0, \tag{16}$$

where suffix s is dropped for simplicity. We multiply the above equation by 2dE/dz and integrate, obtaining

$$\left(\frac{dE}{dz}\right)^2 + k_o^2E^2 - k_o^2\frac{(\omega_p/\omega)^2}{1-\omega_c/\omega}\int_0^{E^2}\exp\left[-\frac{t}{E_c^2(1-\omega_c/\omega)}\right]dt = 0, \tag{17}$$

where k_o is the wave number in vacuum. Here the boundary condition for the evanescent wave, that E and dE/dz tend, respectively, to zero as z approaches infinity, has been used to derive (17). Next, we put z = 0 in (17) in order to obtain the cut-off condition of the incident CP wave. The boundary condition at z = 0 that the tangential components of the electric and the magnetic fields should be, respectively, continuous, takes the form[12],

$$E(0) = 2E_i \cos(\theta/2) \ , \quad dE(0)/dz = -2k_o E_i \cos(\theta/2) \ , \tag{18}$$

where θ and E_i are, respectively, the phase angle of reflection coefficient and the amplitude of the incident circularly polarized wave. Substituting these expressions into (17), and putting $\theta = +0$, one obtains the cut-off condition as,

$$(\omega_p/\omega)^2 = b/\left\{1 - \exp\left[-b/(1-\omega_c/\omega)\right]\right\}, \tag{19}$$

where

$$b = 4(E_i/E_c)^2.$$

Equation (19) becomes $(\omega_p/\omega)^2 = b$ for $b >> 1$. Equation (19) reduces to the linear relationship[6]

$$(\omega_p/\omega)^2 = 1 - \omega_c/\omega \ , \tag{20}$$

when b approaches zero.

The cut-off conditions calculated from (19) for various b's are shown in Fig.1. The upper and lower half planes correspond, respectively, to R wave and L wave. The plane is divided into three regions, namely A, B and C. They correspond, respectively, to the whistler waves, the evanescent region and the fast electromagnetic waves. The border line between B and C is the cut-off line which is shown, in Fig.1, by the dashed line in the linear limit, i.e.

(20). Equation (19) predicts that the cut-off line moves towards the higher densities with increasing b.

IV. A PERPENDICULAR PROPAGATION ($\mathbf{B}_o \perp \mathbf{k}$)

In this case, the magnetic field $\mathbf{B}_o$ in z direction is applied parallel to the plasma surface. The plasma is assumed to occupy a region where $x \geq 0$. A plane EM wave is incident normally on the plasma surface. The direction of polarization of the incident wave is assumed to be in y direction[13]. Such a polarization can excite the extraordinary modes in the plasma. We again in this Section assume that the EM field is evanescent in the plasma.

We substitute (7) into (8), and put $E_{\parallel} = 0$ and $\partial/\partial y = \partial/\partial z = 0$. Then, we obtain the following equations regarding E_x and E_y in the plasma, as

$$\frac{d^2 E_y}{dx^2} + k_o^2\left[1 - \frac{X'(1-X')}{1-Y^2-X'}\right] E_y = 0 , \tag{21}$$

$$E_x = -\frac{iX'Y}{1-Y^2-X'} E_y . \tag{22}$$

Substituting (22) into (6), one obtains the expression for X' as

$$X' = X\exp(-bT) , \tag{23}$$

where

$$T = \frac{X'^2+(1-Y^2)(1-2X')}{(1-Y^2-X')^2} , \tag{24}$$

and $b' = 4(E_y/E_c)^2$. The analysis is analogous to that described in Section III. We multiply (21) by $2dE_y/dx$. and integrate, obtaining

$$\left(\frac{dE_y}{dx}\right)^2 + k_o{}^2E_y{}^2 - k_o{}^2\int_0^{E_y^2} \frac{X'(1-X')}{1-Y^2-X'}\, dE_y{}^2 = 0\,. \tag{25}$$

Here, we have used the boundary condition at x infinity for evanescent waves to obtain (25). The boundary conditions at $x = 0$ are $E_y(0) = 2E_i\cos(\theta/2)$ and $dE_y(0)/dx = -2k_oE_i\sin(\theta/2)$ which are quite similar to (18). Substituting above expressions into (25), one obtains the following equation as

$$1 - \cos^2(\theta/2)\cdot\int_0^1 \frac{X'(1-X')}{1-Y^2-X'}\, d\xi = 0\,, \tag{26}$$

where

$$X' = X\exp(-b\xi T\cos^2\frac{\theta}{2})\,,\quad b = 4(E_i/E_c)^2. \tag{27}$$

To get the cut-off condition, we put $\theta = +0$ in (26), obtaining

$$\int_0^1 \left[1 - \frac{X'(1-X')}{1-Y^2-X'}\right] d\xi = 0\,, \tag{28}$$

where

$$X' = X\exp(-b\xi T)\,. \tag{29}$$

Then, the cut-off line in the CMA diagram can be obtained, if one finds the set of plasma parameters X, Y for a given b, which satisfies (28), (29) and (24). Equation (28) reduces to the linear relationship

$$(\omega_p/\omega)^2 = 1 \pm \omega_c/\omega , \tag{30}$$

when b approaches zero. Inspecting above three equations, one can easily find that the cut-off line moves towards higher $(\omega_p/\omega)^2$ with increasing b, as long as $T > 0$. Next, we solve these equations numerically for $b \ll 1$, since they are too complicated to find analytical solutions. In Fig.2, an example is shown in the CMA diagram for $b = 0.1$. The upper and lower half planes correspond, respectively, to the extraordinary and ordinary modes.

V. DISCUSSIONS

The modifications of the CMA diagram described in this article are attributed to a density-rarefaction of the plasma due to the radiation pressure of the evanescent waves. It should be noted, however, that other nonlinear processes, which are apparently different from the density-rarefaction, may be existing in the real plasma. They are, for example, the heating of the plasma through a parametric decay instability[14], the stimulated Raman and/or Brillouin scatterings[15] in the under-dense region at the plasma boundary. Nevertheless, we expect that the modified CMA diagram may be valid qualitatively, in such a plasma where the kinetic pressure is steadily balancing the radiation pressure due to the large amplitude electromagnetic waves.

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13. The polarization in z direction is another choice of the perpendicular propagations. In this case, the ordinary modes can be excited in the plasma. However, we omit

the ordinary modes in this article, since they are trivial. In fact, the effect of B_0 disappears for ordinary modes in the cold plasmas. It should be noted that the expressions for the ordinary modes can be obtained, putting Y = 0 in the formulations in the Sections III and/or IV.

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FIGURE CAPTIONS

Fig.1 Modified CMA diagram for the parallel propagations. The parameter b is proportional to the incident power on the plasma surface.

Fig.2 Modifications of the cut-off lines for the perpendicular propagations.

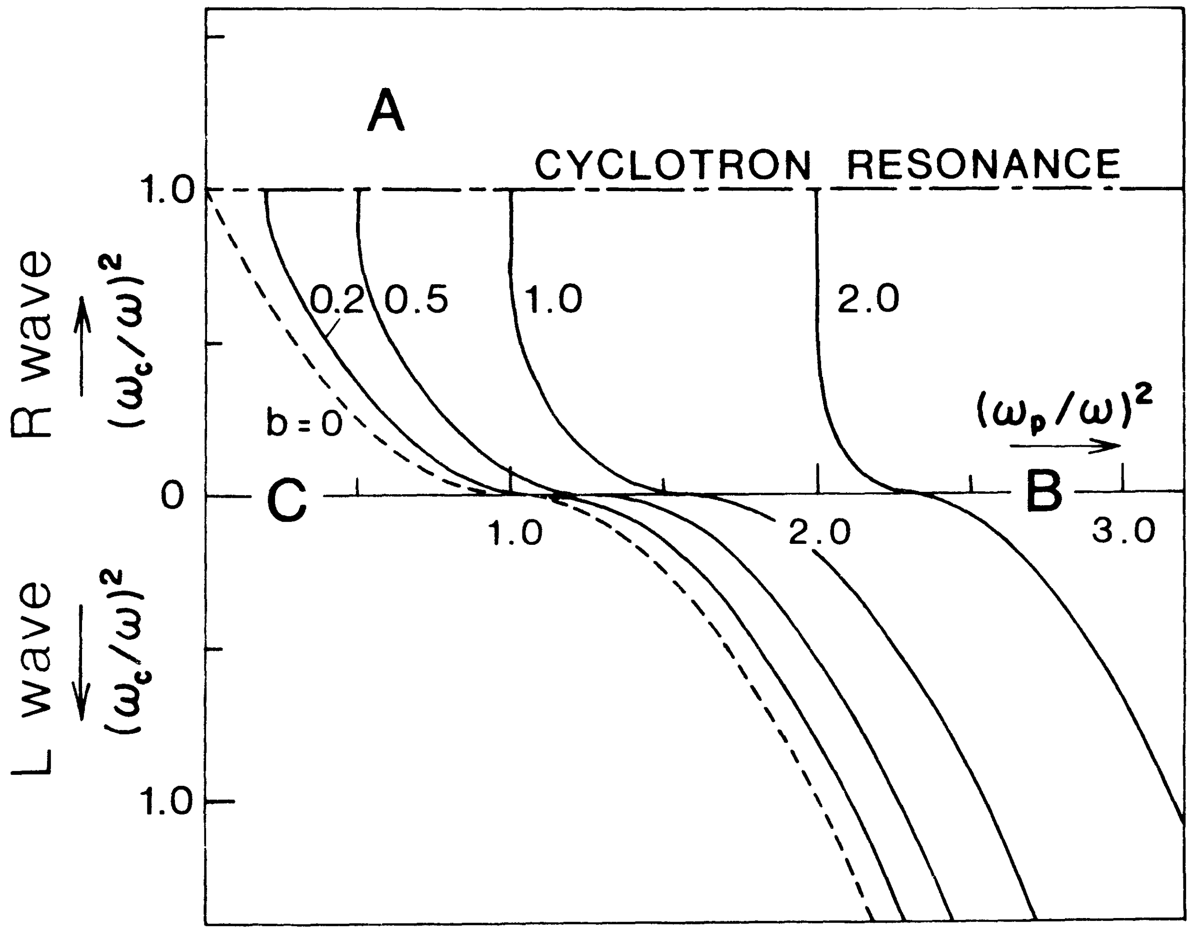
A
CYCLOTRON RESONANCE
R wave
$(\omega_c/\omega)^2$
L wave
$(\omega_c/\omega)^2$
$(\omega_p/\omega)^2$
B
C
b = 0
0.2
0.5
1.0
2.0
1.0
0
1.0
1.0
2.0
3.0

Fig. 1

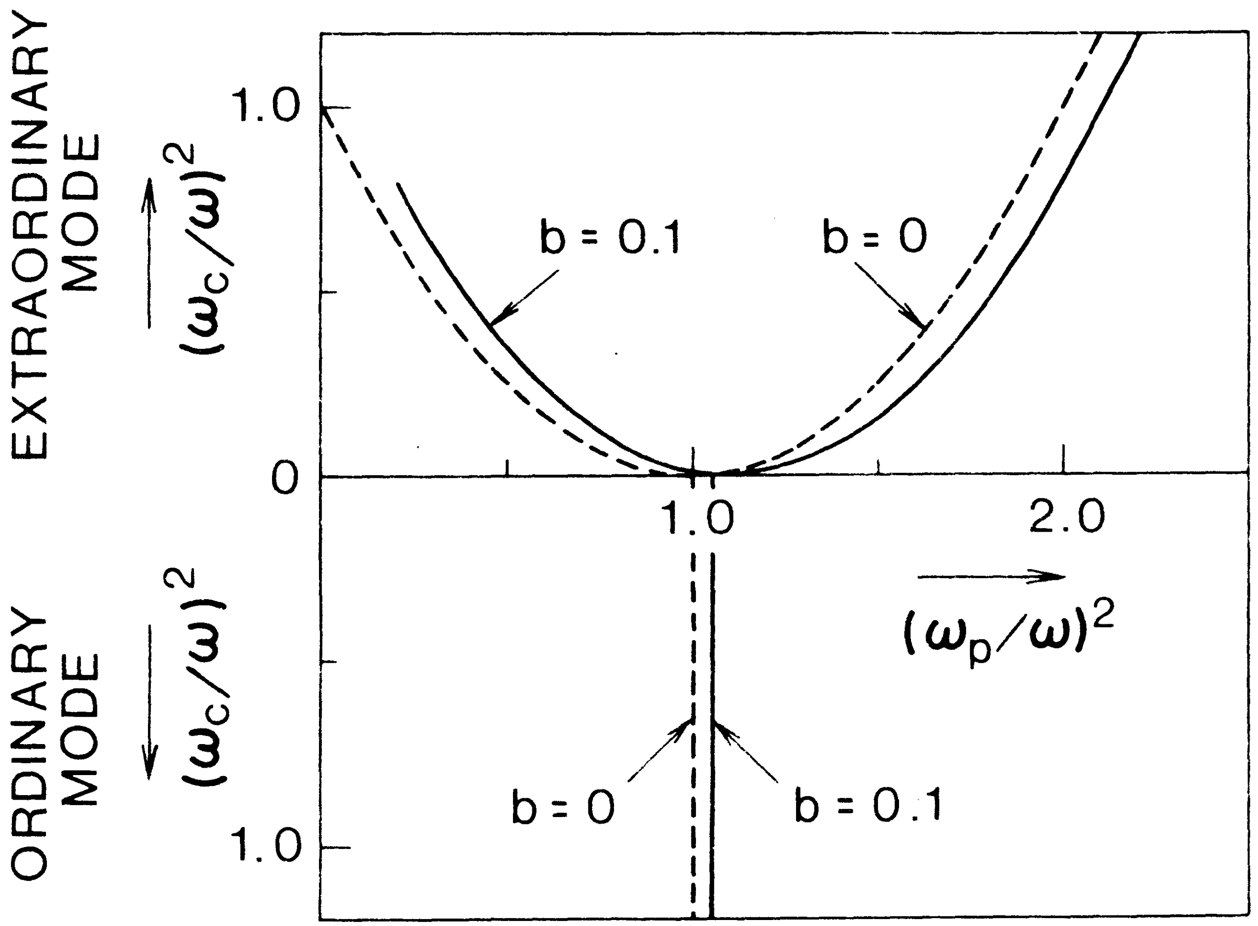
EXTRAORDINARY MODE
$(\omega_c/\omega)^2$
ORDINARY MODE
$(\omega_c/\omega)^2$
1.0
0
1.0
1.0
2.0
$(\omega_p/\omega)^2$
b = 0.1
b = 0
b = 0
b = 0.1

Fig. 2