INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

RESEARCH REPORT

NAGOYA, JAPAN

Stability of Oblique Modulation on Ion Acoustic Wave

Masashi Kako
and
Akira Hasegawa*

IPPJ-223 June 1975

Further communication about this report is to be sent to the Research Information Center, Institute of Plasma Physics, Nagoya University, Nagoya, Japan.

---

* Permanent Address: Bell Laboratories, Murray Hill, New Jersey, 07974

Abstract

Modulation on an ion acoustic wave is shown to be unstable in a direction oblique to that of the wave phase velocity.

In the general analysis of the dynamics of a complex amplitude of a plane wave with a form $\phi(\xi,\tau)e^{i(\overset{\circ}{k}x-\overset{\circ}{\omega}t)}$, Taniuti has shown that ϕ satisfies the nonlinear Schrödinger equation

$$i\frac{\partial\phi}{\partial\tau} + p\frac{\partial^2\phi}{\partial\xi^2} + q|\phi|^2\phi = 0 , \qquad (1)$$

where $p = \frac{1}{2}\left.\frac{\partial^2\omega}{\partial k^2}\right|_{k=k^\circ}$.

The amplitude function ϕ becomes unstable when the coefficients p and q satisfy $pq > 0$ (known as modulational instability).[2]

Using the above formulation, Shimizu and Ichikawa[3] have shown that the ion acoustic wave is modulationally stable. In fact there are many waves that are modulationally stable under this category.[4]

The purpose of this paper is to show that these modulationally stable waves can become modulationally unstable for a modulation applied in the direction oblique with respect to the direction of the phase velocity of the carrier wave. For example, if we consider a modulation in the x direction for a wave propagating in the x-y direction with the wavenumber k_x and k_y, $\phi(x)e^{i(k_x^\circ x+k_y^\circ y-\omega t)}$, p in Eq.(1) becomes $1/2\ \partial^2\omega/\partial k_x^2\,|_{k_x=k_x^\circ\ k_y=k_y^\circ}$. Consequently, the sign of p can in general be different from the case with $k_y=0$ (parallel modulation). Even if the sign of q may change when $k_y\neq 0$, the change of sign of p and q does not occur simulutaneously in general. This leads to the possibility that the sign of the product pq become positive for certain values of k_y/k_x and $|k|$.

The ion acoustic wave in a collisionless plasma with ions and hot electrons, for example, has an anisotropic expression for p while its dispersion relation is isotropic, i.e.,

$$(\frac{\omega}{k})^2 = 1 - \omega^2 \tag{2}$$

$$k^2 = k_x{}^2 + k_y{}^2$$

and $$p = \frac{1}{2\omega}(\frac{\omega}{k})^4[1 - (1 + 3\omega^2)\cos^2\theta] . \tag{3}$$

Here the wavenumber k and the frequency ω are normalized by the electron Debye wavenumber k_D and the ion plasma frequency ω_{pi} respectively and the angle θ is defined as $k_x = k \cos \theta$.

We can easily see from Eq.(2) that p changes its sign from negative to positive when θ exceeds a value $\arccos (1 + 3\omega^2)^{-1/2}$. Note that this value is 0° when ω is zero and is 60° when ω unity (the upper limit of ω). The upper curve in the Fig.1 shows this borderline.

The coefficient q of the nonlinear term in Eq.(1) is obtained straightforward even in the case of the oblique modulation, accoridng to Taniuti's general analysis.[1,2] The set of equations governing the ion acoustic wave is given as follows,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 , \tag{4}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathbf{E} = 0 , \tag{5}$$

$$\nabla\nu + \nu \mathbf{E} = 0 \quad , \tag{6}$$

$$\nabla\cdot\mathbf{E} - n + \nu = 0 \quad , \tag{7}$$

where $\nabla = (\partial/\partial x,\ \partial/\partial y)$, $\mathbf{v} = (v_x, v_y)$ and $\mathbf{E} = (E_x, E_y)$. This is just a two dimensional form of basic equations adopted by H. Washimi and T. Taniuti[5] to find the ion acoustic wave solitons. The characteristic quantities are $(T/M)^{1/2}$ for the ion velocity $\mathbf{v}$, Tk_D/e for the electric field $\mathbf{E}$ and n_0 for the ion density n and the electron density ν; space and time coordinates, x and t, are made dimensionless by the characteristic wavenumber and the frequency. We seek for a solution in the form

$$\varepsilon\phi\ (\xi,\tau)\exp[i(k_x x + k_y y - \omega t)] \ , \tag{8}$$

$$\xi = \varepsilon\ (x - \lambda t) \ , \tag{9a}$$

$$\tau = \varepsilon^2\ t, \tag{9b}$$

where ε indicates the magnitude of small but finite amplitude ϕ, which propagates with a group velocity λ, easily being given from Eq.(2)

$$\lambda = \left(\frac{\omega}{k}\right)^3 \cos\theta \ . \tag{10}$$

To the first order of Taniuti's general analysis of perturbation expansions in terms of Eqs.(8) and (9), one obtains

$$n^{(1)} = \phi \ , \qquad \underset{\sim}{k} \cdot \underset{\sim}{v}^{(1)} = \omega\phi$$

$$\underset{\sim}{k} \cdot \underset{\sim}{E}^{(1)} = - i\omega^2\phi \ , \qquad \nu^{(1)} = (\frac{\omega}{k})^2 \ \phi \ , \tag{11}$$

$$E_x^{(1)} = - i \frac{\omega^2}{k} \cos\theta \ \phi, \qquad v_x^{(1)} = \frac{\omega}{k} \cos\theta \ \phi$$

and the linear dispersion relation Eq.(2). We consider a plane wave with wavenumbers $k_x{}^\circ$ and $k_y{}^\circ$ as the carrier wave. We assume these wavenumbers are not very small so that the frequency determined by Eq.(2) is sufficiently large to preclude the appearance of harmonic modes of $k_x{}^\circ$ and $k_y{}^\circ$ as proper modes. The harmonics will appear virtually to higher orders.

To the second order, one obtains the second correction to quantities listed in Eqs.(11) in terms of a function $\phi^{(2)}(\xi,\tau)$ and $\partial\phi/\partial\xi$. The second harmonic mode of the carrier wave is also obtained in terms of ϕ^2. This comes from nonlinear self-interaction. Although we skip to list these expressions, one can easily see, that these are isotropic; hence are the same as in Ref.3. The zeroth harmonic mode also appears due to the nonlinear self-interaction of the modulated carrier wave. Its expression cannot be determined within the second order completely. One finds just

$$n_0^{(2)} = \nu_0^{(2)} \qquad \text{and} \qquad \underset{\sim}{k} \cdot \underset{\sim}{E}_0^{(2)} = 0 \ . \tag{12}$$

Proceeding to the third order, one obtains

$$n_0^{(2)} = -(1-\lambda^2)^{-1}\ (\frac{\omega}{k})^4 (k^2 + 2\cos^2\theta)\,|\phi|^2 \quad ,$$

$$\underset{\sim}{k}\cdot\underset{\sim}{v}_0^{(2)} = -(1-\lambda^2)^{-1}\ \omega\ \cos^2\theta\ (\frac{\omega^6}{k^4} + 2)\,|\phi|^2 . \qquad (13)$$

These expressions are anisotropic but agree with those in Ref.3 when θ is set zero.

To the third order in the fundamental mode, specified by $k_x{}^\circ$, $k_y{}^\circ$ and ω, the perturbation expansion analysis is closed to define an equation which governs the complex amplitude ϕ, being well-known called the nonlinear Schrödinger equation, Eq.(1). We obtain the nonlinear coefficient in the form

$$q \equiv q_2 + q_0 \qquad (14)$$

where q_2 and q_0 indicate the contributions from the second and the zeroth harmonics,

$$q_2 = -\frac{1}{3\omega}\ (1 + \frac{13}{2}\ \omega^2 - \frac{9}{4}\ \omega^4 + \frac{5}{4}\ \omega^6 - \frac{\omega^8}{2}) \ , \qquad (15a)$$

$$q_0 = \frac{\omega}{2}\ (1-\lambda^2)^{-1}\ [\omega^4(1-\omega^2) + 4\cos^2\theta(1+\omega^2-2\omega^4+\omega^6)] \ . \qquad (15b)$$

One readily sees that $q_2 < 0$ and $q_0 > 0$ since $\omega^2 \leq 1$. The lower curve in Fig.1 is that on which q becomes zero. One finds a wide domain where both p and q take the negative sign, i.e., a modulation on the ion acoustic wave is unstable when it propagetes obliquely with respect to the direction of the carrier wave making an angle within these two limits which depend on the frequency of the carrier wave.

It is expected consequently that the ion acoustic wave has an envelope soliton which propagates in this direction.

In particular, for the parallel modulation the coefficient p is always negative irrespective of the frequency while the coefficient q changes its sign from positive to negative when the frequency crosses a value $0.83\omega_{pi}$, which corresponds to a wavenumber $1.47k_D$. Even the parallel propagating modulation makes the ion acoustic wave unstable, if its frequency exceeds this value, under the condition of no collisions and no wave particle interaction.[6] However if ω exceeds $0.71\omega_{pi}$, the wave length becomes smaller than the Debye length; hence collective behavior of plasma will disappear and consequently the ion acoustic wave cannot actually exist.

We see two domains in ω-θ diagram, Fig.1, for a modulationally stable wave having an appropriate frequency ($0 < \omega < 0.83\omega_{pi}$). Note that for nearly parallel directions of the modulation the wave is stable due to the nonlinear self-interaction originating from the zeroth harmonic mode (or the slow mode), which is often referred to the so-called ponderamotive force, while the wave propagating nearly perpendicular to the direction of the modulation is stable due to the seocnd harmonic self-interaction since $p > 0$ there and q_2 is always negative. On the other hand, for a modulational instability the second harmonic mode is essential since $p < 0$ in the instability domain.

In the limit of small frequency we can obtain an analytical formula for modulational instabilities. From Eqs.(15a, b), we obtain

$$q_2 = -\frac{1}{3\omega}\left(1 + \frac{13}{2}\omega^2\right), \qquad (16a)$$

$$q_0 = 2\omega\cos^2\theta/(\sin^2\theta + 3\omega^2\cos^2\theta), \qquad (16b)$$

where the first two terms are retained with respect to ω^2. Then one obtains

$$q \propto 3\omega^2 - \left(1 + \frac{13}{2}\omega^2\right)\tan^2\theta \qquad (17)$$

and finds a critical angle θ_q, at which q is zero,

$$\theta_q \simeq \sqrt{3}\,\omega\left(1 - \frac{13}{4}\omega^2\right) \qquad (18)$$

which is written in radians. Also one obtains θ_c, at which p vanishes, in the same limit

$$\theta_c \simeq \sqrt{3}\,\omega\left(1 - \frac{3}{2}\omega^2\right). \qquad \text{(in radian)}$$

Consequently, we have modulational instabilities for ion acoustic waves propagating with an angle $\sqrt{3}\omega$ and its half width $\frac{7}{8}\sqrt{3}\omega^3$ with respect to the direction of modulation.

Because of the change of the basic property of the envelope function ϕ as a function of θ, the present result casts doubt on the applicability of the one dimensional inverse scattering method which are often used to solve the nonlinear Schrödinger equation[7].

The authors wish to thank Prof. Tosiya Taniuti for a valuable discussion. One of the authors (A.H.) is supported by the U. S. National Science Foundation grant No. OIP 74-12932.

Figure Caption

Fig.1 ω - θ diagram. Modulation on an ion acoustic wave is fixed on the direction 0° and the phase velocity of the ion acoustic wave is directed to an arbitrary direction between 0° and 90°. The curve, $p = p(\omega, \theta) = 0$, starts from the origin tangent to the line, $\theta = 0°$, up to the point ($\omega = 1, \theta = 60°$). The other curve, $q = q(\omega, \theta) = 0$, starts exactly in the same way as $p = 0$ curve from the origin and ends at the point ($\omega = 0.83, \theta = 0°$). The wave whose frequency and angle with respect to the direction of modulation are such in the domain bounded by the curves becomes unstable.

References

1. See for example a review on the reductive perturbation method, Prog. Theor. Phys. Suppl. No.55 (1974) prefaced by T. Taniuti.

2. V. I. Karpman and E. M. Krushkal, Zh. Eksp. i Teor. Fiz. 55, 530 (1968) [Soviet Phys. —— JETP 28, 277 (1969)]

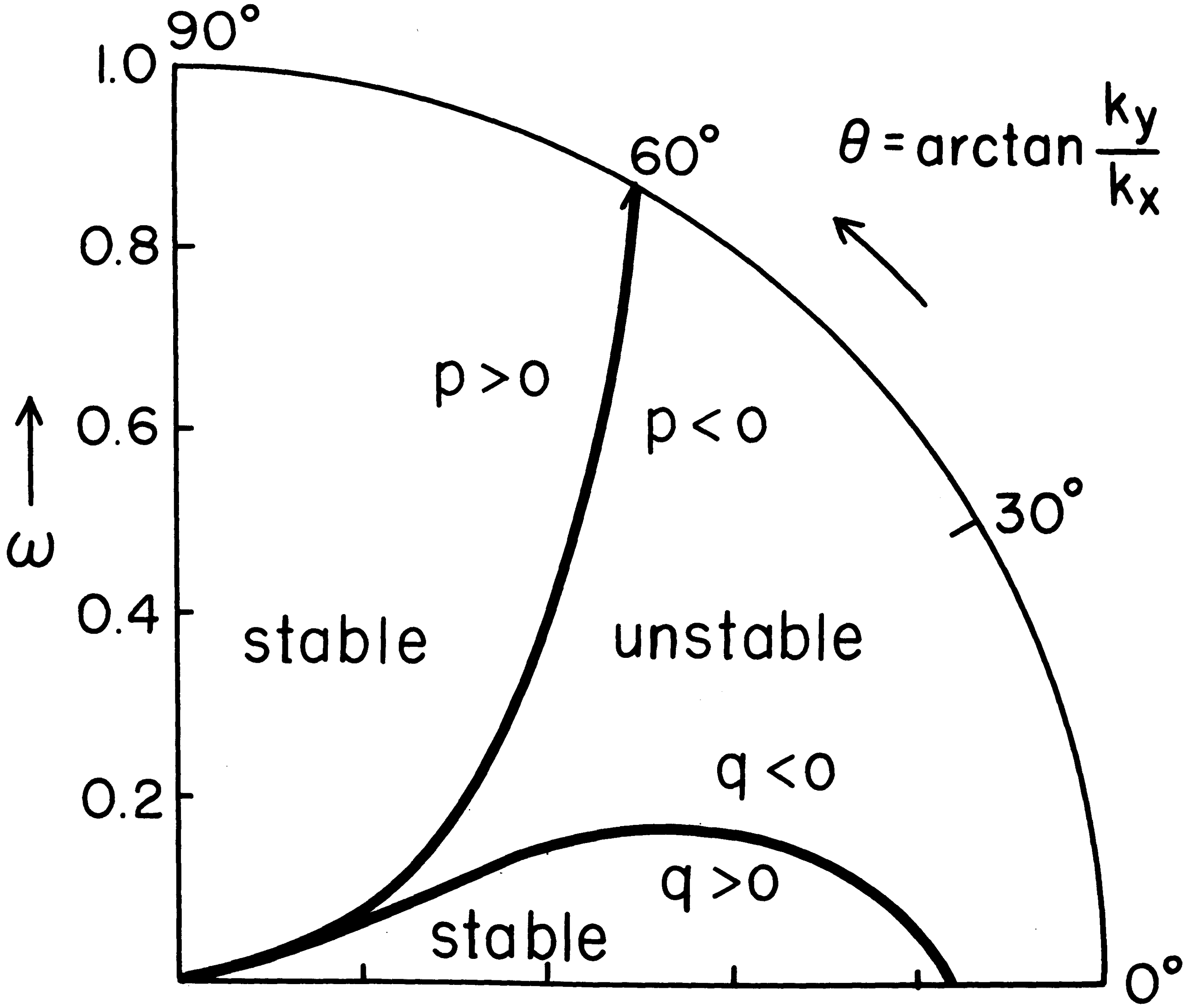
3. K. Shimizu and Y. H. Ichikawa, J. Phys. Soc. Japan 33, 789 (1972)

4. See for example a work in Ref. 1) p.120 by M. Kako

5. H. Washimi and T. Taniuti, Phys. Rev. Lett. 17, 996 (1966)

6. A conclusion by M. Kako in Ref. 4, p.131 was inexact. The coefficient q was computed to be positive just up to a value $k = 0.8\ k_D$ in Ref. 3.

7. V. E. Zakharov and A. B. Shabat, Zh. Eksp. i Teor. Fiz. 61, 118 (1971) [Soviet Phys. —— JETP 34, 62 (1972)]



1.0
0.8
0.6
0.4
0.2
ω
90°
60°
30°
0°
θ = arctan ky/kx
p > 0
p < 0
stable
unstable
q < 0
q > 0
stable