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MODULATIONAL INSTABILITY AND ENVELOPE-SOLITONS FOR NONLINEAR ALFVÉN WAVES PROPAGATING ALONG THE MAGNETIC FIELD IN COLD PLASMAS

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Abstract

The modulational instability and envelope-solitons for the Alfvén waves propagating along the static magnetic field in cold collisionless plasmas are analyzed using the modified nonlinear Schrödinger equation derived previously by the authors. It is shown that the modulational instability exists for a small amplitude but does not for a amplitude larger than the critical value in the case of the left-hand circularly polarized Alfvén wave (left Alfvén wave). On the other hand, the modulational instability does not exist for the right=hand circularly polarized Alfvén wave (right Alfvén wave). Without the modulational instability, the rarefactive and compressive envelope-solitons exist for the left Alfvén wave and the two types of the rarefactive envelope-solitons for the right Alfvén wave.

§1. Introduction

In recent years, study of nonlinear wave phenomena has been one of the most important problems in plasma physics. It is known that the complex amplitude Ψ of various waves such as the electron plasma[1] and electron cyclotron waves[2] in plasmas is expressed by the nonlinear Schrödinger equation[1-5] (N.S. equation), namely

$$i\frac{\partial\Psi}{\partial t} + P\frac{\partial^2\Psi}{\partial x^2} + Q|\Psi|^2\Psi = 0,$$

where P is proportional to the derivative of the group velocity of the wave by the wave number and Q is the coefficient of the nonlinear frequency shift. The N.S. equation has been investigated in detail by many authors.[3-7] The modulational instability exists for $PQ > 0$ and does not for $PQ < 0$. Without the modulational instability, the envelope=soliton solution is obtained.[3] Hasegawa investigated the modulational instability of the electron cyclotron wave by both the analysis and computer simulation.

It was shown, in our previous paper,[8] that the nonlinear Alfvén waves propagating along the static magnetic field were described by the modified nonlinear Schörinder equation (M.N.S. equation). It can be used even in the case that the N.S. equation is not valid, since the former includes the higher order term in addition to the latter. In the present paper, the M.N.S. equation is solved to study the modulational instability and envelope-solitons. The results considerably differ from those[3,4] expected by the usual N.S. equation.

It is shown that the modulational instability does not always exist even for $PQ > 0$. Moreover, the envelope-soliton solutions for the left and right-hand circularly polarized Alfvén waves (left and right Alfvén waves) are obtained, when the modulational instability does not exist.

In §2, the results previously developed by the authors are summarized as the starting point of the present analysis. In §3, the analytical results of the modulational instability are described in some detail. The Korteweg-de Vries equation for the envelope of the Alfvén waves is derived in §4. In §5, the rarefactive and compressive envelope-solitons are classified for the left and right Alfvén waves.

§2. Previous Results by the Authors[8)]

The nonlinear Alfvén waves propagating along the magnetic field are described by the following equation,

$$\frac{\partial \psi}{\partial t} + \frac{1}{4}\frac{\partial}{\partial x}(4 + |\psi|^2)\psi - \frac{iu_0'}{2}\frac{\partial^2 \psi}{\partial x^2} = 0. \tag{1}$$

We introduce the following variables,

$$\psi = \Psi e^{i(k_0 x - \omega_0 t)} \tag{2a}$$

$$\Psi = a e^{i\phi} \tag{2b}$$

$$\omega_0 = k_0 + \frac{u_0'}{2} k_0^2 + \left(\frac{\partial \omega}{\partial a^2}\right)_0 a_0^2 , \tag{2c}$$

where k_o and ω_o are, respectively, the wave nmber and the frequency of the carrier wave. Using Eq. (2a), Eq. (1) is rewitten regarding Ψ as,

$$\Psi_t + u_0\Psi_x - i\frac{u_0'}{2}\Psi_{xx} + i\left(\frac{\partial\omega}{\partial a^2}\right)_0 \left(|\Psi|^2 - |\Psi_0|^2\right)\Psi - \frac{1}{4}\left(|\Psi|^2\Psi\right)_x = 0, \tag{3}$$

where $|\Psi_0| \equiv a_0$, $u_0 = 1 + u_0'k_0$, $(\partial\omega/\partial a^2)_0 = k_0/4$ and $u_0' = \mp 2\mu = \mp(1/\omega_{ci} - 1/\omega_{ce})$. Here the upper and lower signs correspond to the left and right Alfvén waves, respectively. Equation (3) was named the modified nonlinear Schrödinger eqation (M.N.S. equation).[8] Subsituting Eq.(2b) into Eq.(3), two equations for the amplitude a and phase ϕ are derived from the real and imaginary parts

$$a_t + u_0 a_x + \frac{u_0'}{2}\left(2a_x\phi_x + a\phi_{xx}\right) + \frac{3}{4}a^2 a_x = 0, \tag{4a}$$

$$\phi_t + u_0\phi_x + \frac{u_0'}{2}\phi_x^2 + \left(\frac{\partial\omega}{\partial a^2}\right)_0 (a^2 - a_0^2) - \frac{u_0'}{2a}a_{xx} + \frac{1}{4}a^2\phi_x = 0. \tag{4b}$$

§3. Modulational Instability

A small perturbation for the amplitude of Alfvén waves can grow in time when a finite, constant amplitude a_0 and a wave number k_0 are initially given. It is called the modulational instability.[2] In this section, the modulational instability is studied using Eq.(4). The variables are separated into the constant a_0 and the perturbations $\tilde{a}$ and $\tilde{\phi}$, which vary slowly in time and space rather than those of carrier wave,

$$a = a_0 + \tilde{a} \quad , \quad \phi = \tilde{\phi} . \tag{5a}$$

The perturbations in amplitude are written in a form of the plane wave as,

$$\tilde{a}\,,\ \tilde{\phi}\ \propto\ e^{i(K\chi-\Omega t)}\quad, \tag{5b}$$

where K and Ω are, respectively, the wave number and frequency of the envelope. Then, the dispersion equation is obtained by linearizing Eq.(4),

$$\Omega = \left(u_0 + \frac{a_0^2}{2}\right)K \pm K\left[\frac{a_0^4}{16} + \left(\frac{\partial\omega}{\partial a^2}\right)_0 u_0' a_0^2 + \frac{u_0'^2}{4}K^2\right]^{\frac{1}{2}}$$

$$= \beta_2 K - \beta_3 K^3 + O(K^5)\ , \tag{6}$$

where

$$\beta_1 = \left[\frac{a_0^4}{16} + \left(\frac{\partial\omega}{\partial a^2}\right)_0 u_0' a_0^2\right]^{\frac{1}{2}} \tag{7a}$$

$$\beta_2 = u_0 + \frac{a_0^2}{2} \pm \beta_1 \tag{7b}$$

$$\beta_3 = \mp\ \frac{u_0'^2}{8\beta_1}\ . \tag{7c}$$

The modulational instability does not exist under the conditions of

$$\beta_1^2 = \frac{a_0^4}{16} + \left(\frac{\partial\omega}{\partial a^2}\right)_0 u_0' a_0^2 > 0 \quad \text{namely} \quad a_0^2 > -16\left(\frac{\partial\omega}{\partial a^2}\right)_0 u_0'. \tag{8}$$

The right Alfvén wave is always stable for the modulation because of $(\partial\omega/\partial a^2)_0 u_0' = k_0 u_0'/4 > 0$. On the other hand, the left Alfvén wave is stable for $a_0^2 > -4k_0u_0'$ but unstable for $a_0^2 < -4k_0u_0'$. In other words, in the latter case there is a range of K where the internal term of the square root is negative in Eq.(6). Hereafter in this section, the interest is restricted to the modulational instability for the left Alfvén wave.

From Eq.(6) the growth rate Γ of the modulational instability is obtained as,

$$\Gamma = \mathrm{Im}(\Omega) = \frac{1}{2} K \left[2\mu k_o a_o^2 - \frac{a_o^4}{4} - 4\mu^2 K^2 \right]^{\frac{1}{2}} . \tag{9}$$

It has the maximum value of

$$\Gamma_m = \frac{a_o^2}{8\mu} \left(\mu k_o - \frac{a_o^2}{8} \right) \tag{10a}$$

at

$$K_m = \frac{a_o}{2\mu} \left(\mu k_o - \frac{a_o^2}{8} \right)^{\frac{1}{2}} . \tag{10b}$$

Furthermore, Γ_m has the maximum value of

$$\Gamma_{mm} = \frac{1}{4} \mu k_o^2 \tag{11a}$$

at

$$a_{om} = 2 (\mu k_o)^{\frac{1}{2}} . \tag{11b}$$

The dependence of Γ on K is shown in Fig.1 for the constant value a_o, and Γ_m vs. a_o is shown in Fig.2. The broken line shows the dependence of Γ_m on a_o obtained from the usual N.S. equation, namely $\Gamma_m = k_o a_o^2/8$. As is shown in Fig.2, Γ_m obtained from the M.N.S. equation increases for $0 < a_o < a_{om}$ but decreases for $a_o > a_{om}$, as a_o increases.

Then, Γ_m equals zero at $a_o = \sqrt{2} a_{om}$. From Eq.(9), the modulational exists in the following ranges,

$$0 < K < \sqrt{2} K_m \qquad \text{for } K \tag{12}$$

and

$$\beta_- < a_0^2 < \beta_+ \qquad \text{for } a_0 \tag{13a}$$

$$\beta_\pm = 4\mu \left[k_0 \pm (k_0^2 - K^2)^{\frac{1}{2}} \right] . \tag{13b}$$

Although $PQ = -(\partial\omega/\partial a^2)_o u_o'/2 = \mu k_o/4 > 0$ holds for the left Alfvén wave, it is stable for the modulation under the condition of $a_o \geq a_{om}$. The result cannot be explained from the N.S. equation.[3)]

§4. Derivation of Korteweg-de Vries Equation

The M.N.S. equation is analyzed to obtaine a steady-state solution, namely the evnelope-soliton. The Korteweg-de Vries equation (K-dV equation) can be derived from the M.N.S. equation by the similar method to that for N.S. equation.[3)] Another method to derive the K-dV equation developed by the authors is described in the Appendix A. First, Ψ in Eq.(3) is replaced by,

$$\Psi = \rho^{\frac{1}{2}} e^{i \int \frac{\sigma}{u_0'} dx} , \tag{14}$$

then, the following two equations are derived from the real and imaginary parts of Eq.(3),

$$\rho_t + u_0 \rho_x + (\sigma + \frac{3}{4}\rho)\rho_x + \rho\sigma_x = 0 \tag{15a}$$

$$\sigma_t + u_0\sigma_x + \frac{1}{4}(\sigma + k_0 u_0')\rho_x + (\sigma + \frac{\rho}{4})\sigma_x - \frac{u_0'^2}{4}\frac{\partial}{\partial x}\left[\rho^{-\frac{1}{2}}\frac{\partial}{\partial x}(\rho^{-\frac{1}{2}}\rho_x)\right] = 0 . \tag{15b}$$

The following perturbations of a plane wave are substituted into Eq.(15).

$$\rho = \rho_0 + \tilde{\rho} , \sigma = \tilde{\sigma} \quad \text{and} \quad \tilde{\rho}, \tilde{\sigma} \propto e^{i(Kx - \Omega t)} . \tag{16}$$

Linearizing the equations, one obtaines the dispersion relation, regarding K and Ω, which is the same as Eq.(6), if a_0^2 is replaced by ρ_0.

Since a modulated wave is stable for $\beta_1^2 > 0$, the reductive perturbation method[9] can be applied to Eq.(15). It is arranged in a form,

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} \left(H_\alpha^\beta \frac{\partial}{\partial t} + K_\alpha^\beta \frac{\partial}{\partial x} \right) \cdot U = 0 \tag{17}$$

where s = 1, p = 3 and

$$U = \begin{bmatrix} \rho \\ \sigma \end{bmatrix} , \quad A = \begin{bmatrix} u_0 + \frac{3}{4}\rho + \sigma & \rho \\ \frac{\sigma}{4} + \frac{k_0}{4}u_0' & u_0 + \frac{\rho}{4} + \sigma \end{bmatrix} . \tag{18}$$

Next, the following coordinate stretching and the expansions are introduced

$$\xi = \epsilon^{\frac{1}{2}}(x - \lambda t) , \quad \tau = \epsilon^{\frac{3}{2}} t$$
$$\rho = \rho_0 + \sum_{l=1} \epsilon^l \rho^{(l)} , \quad \sigma = \sum_{l=1} \epsilon^l \sigma^{(l)} , \tag{19}$$

where ε is a small parameter. From the determinant $|A - \lambda I| = 0$,

$$\lambda = u_0 + u_0'\sigma + \frac{\rho}{2} \pm \left(\frac{\rho^2}{16} + \frac{k_0}{4} u_0' \rho + \frac{1}{4} u_0' \sigma \rho \right)^{\frac{1}{2}} \tag{20}$$

is obtained, where $\lambda_0 = \beta_2$ and I means the unit tensor. Under the condition that $\rho^{(1)}$ and $\sigma^{(1)}$ approach, respectively, zero for $\xi \to \pm\infty$, the equation of $\varepsilon^{3/2}$-order in Eq.(15) are written as

$$(A_0 - \lambda_0 I)\begin{bmatrix} \rho^{(1)} \\ \sigma^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{\rho_0}{4} \mp \beta_1 & \rho_0 \\ \frac{k_0}{4} u_0' & -\frac{\rho_0}{4} \mp \beta_1 \end{bmatrix} \begin{bmatrix} \rho^{(1)} \\ \sigma^{(1)} \end{bmatrix} \equiv \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \rho^{(1)} \\ \sigma^{(1)} \end{bmatrix} = 0 \tag{21}$$

Moreover, the K-dV equation is obtained from the $\varepsilon^{5/2}$-order, as

$$\frac{\partial \rho^{(1)}}{\partial \tau} + \beta_4 \rho^{(1)} \frac{\partial \rho^{(1)}}{\partial \xi} + \beta_3 \frac{\partial^3 \rho^{(1)}}{\partial \xi^3} = 0 , \tag{22}$$

where

$$\beta_4 = \nabla_U \lambda_0 \cdot R = \frac{3}{2} \left(\frac{1}{4} \pm \frac{\beta_1}{\rho_0} \right) , \tag{23a}$$

$$\beta_3 = \frac{(L \cdot K_0 \cdot R)}{(L \cdot R)} = \mp \frac{u_0'^2}{8\beta_1} , \tag{23b}$$

$$L = (1, -d_{11}/d_{21}), \ R = \begin{bmatrix} 1 \\ -\frac{d_{11}}{d_{12}} \end{bmatrix}, \ K_0 = K_1' K_2' K_3' = \begin{bmatrix} 0 & 0 \\ -\frac{u_0'^2}{4\rho_0} & 0 \end{bmatrix} . \tag{23c}$$

The perturbation of the local wave number $k^{(1)}$ is given by the following expression from Eqs.(14) and (21) as,

$$k^{(1)} = \frac{\sigma^{(1)}}{U_0'} = -\frac{d_{11}}{U_0' d_{12}}\rho^{(1)} = -\frac{1}{U_0'}\left(\frac{1}{4} \mp \frac{\beta_1}{\rho_0}\right)\rho^{(1)} \equiv \beta_5 \rho^{(1)} . \tag{24}$$

Thus the small perturbations of the envelope and wave number vary according to the K-dV equation.

§5. Envelope-Soliton Solution

As is well known, the K-dV equation has the soliton solutions. One-soliton solution[10)] from Eq.(22) is given by

$$\rho^{(1)} = \frac{3\lambda^{(1)}}{\beta_4} \operatorname{sech}^2 \frac{1}{2}\left(\frac{\lambda^{(1)}}{\beta_3}\right)^{\frac{1}{2}} (\xi - \lambda^{(1)}\tau) , \tag{25}$$

where $\lambda^{(1)}$ is an optional constant expressing the velocity of the soliton. Therefore, an envelope-soliton solution for ψ is written as,

$$\psi = (\rho_0 + \tilde{\rho})^{\frac{1}{2}} \exp\left[i \int \tilde{k}\, dx + i(k_0 x - \omega_0 t) \right] \tag{26a}$$

$$\tilde{\rho} = \frac{3\Delta\lambda}{\beta_4} \operatorname{sech}^2 \frac{1}{2}\left(\frac{\Delta\lambda}{\beta_3}\right)^{\frac{1}{2}} \left[x - (\beta_2 + \Delta\lambda) t \right] , \tag{26b}$$

where the relations, $\tilde{k} = \beta_5\tilde{\rho}$, $\Delta\lambda = \varepsilon\lambda^{(1)}$, $\tilde{k} = \varepsilon k^{(1)}$ and $\tilde{\rho} = \varepsilon\rho^{(1)}$ are used. When the amplitude of the soliton $\Delta\rho_0 \equiv 3\Delta\lambda/\beta_4$ is given, Eq.(26b) is rewritten as

$$\tilde{\rho} = \Delta\rho_0 \operatorname{sech}^2 \frac{1}{2}\left(\frac{\Delta\rho_0 \beta_4}{3\beta_3}\right)^{\frac{1}{2}} \left[x - \left(\beta_2 + \frac{\Delta\rho_0\beta_4}{3}\right) t \right] . \tag{26b'}$$

As is shown in Eq.(26b), $\Delta\lambda/\beta_3 > 0$ is a necessary condition to obtain the soliton solution. It should be noted that the compressive soliton exists for $\Delta\rho_o = 3\Delta\lambda/\beta_4 > 0$, and the rarefactive soliton for $\Delta\rho_o < 0$. When $\beta_5 > 0$, the local wave number $k_o + \tilde{k}$ is also large in the portion where $\tilde{\rho}$ is large. On the other hand, when $\beta_5 < 0$, $k_o + \tilde{k}$ is small in the portion of large $\tilde{\rho}$.

Now, the envelope-soliton for the left and right Alfvén waves are investigated in detail. When the time, velocity, space and magnetic field are normalized, respectively, by ω_{ci}^{-1}, the Alfvén velocity V_A, V_A/ω_{ci} and the static magnetic field B_o to simplify the representation, the relations

$$u_0 = 1 \mp k_0 \,, \quad u_0' \doteqdot \mp 1 \quad \text{and} \quad \beta_1 = \left(\frac{\rho_0^2}{16} \mp \frac{k_0}{4} \rho_0 \right)^{\frac{1}{2}} \tag{27}$$

are obtained,[8] where the upper and lower signs correspond to the left and right Alfvén waves, respectively.

The results are summarized in Table I. Moreover, the envelope-soliton profiles are shown in Fig.3. In the reference frame moving with the velocity of β_2, the rarefactive envelope=soliton propagating in left direction and the compressive envelope-soliton propagating in right direction exist for the left Alfvén wave without the modulational instability. For each of the envelope-solitons, the local wave number is also large in the portion where the amplitude is large. Since the modulational instability does not exist for the right Alfvén wave, the rarefactive envelope-solitons propagating in left and right directions exist. The local wave number is large for the former but small for the latter in the portion where the amplitude is large. The velocity of envelope-soliton in given by $\beta_2 + \Delta\lambda$ in the rest frame.

In addition, for the left Alfvén wave, the envelope= soliton solved from the N.S. equation and the limiting case that β_1 goes to zero are discussed, respectively, in the Appendixes B and C.

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Appendix A Another Derivation of K-dV Equation by the Reductive Perturbation Method

The reductive perturbation method given by Taniuti and Wei[9] cannot be directly applied to Eq.(4) because it is not arranged to the type of Eq.(17). However, the method can be applied to Eq.(4), if the coordinate stretching and expansions are determined in the following procedure.

Taking into account the linear dispersion equation (6), the following coordinate stretching is introduced,

$$\xi = \epsilon^{\alpha}(x-\beta_2 t)\ ,\ \tau = \epsilon^{3\alpha} t\ ,\quad \alpha > 0\ ,\ 0 < \epsilon < 1 \tag{A.1}$$

Expansions for a and ϕ are written as

$$a = a_0 + \check{a} = a_0 + \epsilon^{m_1}(a^{(1)} + \epsilon a^{(2)} + \cdots)\ , \tag{A.2a}$$

$$\phi = \check{\phi} = \epsilon^{m_2}(\phi^{(1)} + \epsilon\phi^{(2)} + \cdots)\ , \tag{A.2b}$$

where $0 < m_1,\ m_2 \leq 1$. Substituting Eq.(A.2) into Eq.(4) and using Eq.(A.1), $\{\alpha;\ m_1,\ m_2\}$ can be determined from a criterion that the evaluation of $r = m_1 + m_2$ has the minimum value, namely

$$\{\alpha\ ;\ m_1,\ m_2\} = \{\tfrac{1}{2}\ ;\ 1\ ,\ \tfrac{1}{2}\}\ . \tag{A.3}$$

Now, since the coordinate stretching and the expansions are uniquely determined in Eqs.(A.1) and (A.2), Eq.(4) is arranged in the order of ϵ. Consequently, the K-dV equation is obtained in the order of $\epsilon^{5/2}$ as,

$$\frac{\partial a^{(1)}}{\partial \tau} + \beta_6 a^{(1)} \frac{\partial a^{(1)}}{\partial \xi} + \beta_3 \frac{\partial^3 a^{(1)}}{\partial \xi^3} = 0 , \tag{A.4}$$

where

$$\beta_6 = 3a_0 \left(\frac{1}{4} \pm \frac{\beta_1}{a_0^2} \right) , \tag{A.5}$$

and β_3 is given by Eq.(7c). The K-dV equation (A.4) is essentially the same as Eq.(22). The small difference between β_4 and β_6 comes from the two transformations of Eqs.(2b) and (14).

Appendix B Envelope-Soliton Solved from the Nonlinear Schrödinger Equation

The last term can be ignored in Eq.3, when the characteristic length of the envelope is sufficiently long. Using a coordinate transformation in space and time,

$$\xi = x - u_0 t \quad , \quad \tau = t \tag{A.6}$$

the following N.S. equation is obtained for $|\Psi_0| = 0$ from Eq.3 without the last term,

$$i \Psi_\tau + \frac{u_0'}{2} \Psi_{\xi\xi} - \left(\frac{\partial \omega}{\partial a^2}\right)_0 |\Psi|^2 \Psi = 0 . \tag{A.7}$$

Since the two coefficients $u_0'/2 = -\mu$ and $-(\partial\omega/\partial a^2)_0 = -k_0/4$ take the same negative sign, the solution of Eq.(A.7) which goes to zero for $|\xi| \to \infty$ is a solitary wave,[3,6]

$$\Psi = \left(\frac{8\omega_1}{k_0}\right)^{\frac{1}{2}} \operatorname{sech} \left[\left(\frac{\mu}{\omega_1}\right)^{-\frac{1}{2}} \xi \right] e^{-i\omega_1 \tau} , \tag{A.8}$$

where ω_1 is an optional constant. Therefore, envelope= soliton solution for the left Alfvén wave is written as

$$\psi = \left(\frac{8\omega_1}{k_0}\right)^{\frac{1}{2}} \operatorname{sech}\left[\left(\frac{\mu}{\omega_1}\right)^{-\frac{1}{2}}(x-u_0 t)\right] e^{i[k_0 x-(\omega_0+\omega_1)t]} \quad \text{(A.9)}$$

For this envelope-soliton, the wave number is always constant.

Appendix C Limiting Case for the Left Alfvén Wave

The left Alfvén wave can satisfy the condition $\beta_1 = 0$, when ρ_o is equal to $-4k_o u_o'$. Equation (22) is not valid in such a case, because β_3 goes to infinity. It is also understood from that the two eigenvalues of λ_o degenerate for $\beta_1 = 0$. In that exceptional case, the dispersion equation (6) is rewritten as

$$\Omega = \left(u_0 + \frac{\rho_0}{2}\right) K \pm \frac{|u_0'|}{2} K^2 . \quad \text{(A.10)}$$

Hence, following coordinate stretching and the expansions are introduced

$$\xi = \epsilon(x - \lambda_0 t) , \quad \tau = \epsilon^2 t$$

$$\rho = \rho_0 + \sum_{\ell=1} \epsilon^{\ell} \rho^{(\ell)} , \quad \sigma = \sum_{\ell=1} \epsilon^{\ell} \sigma^{(\ell)} , \quad \text{(A.11)}$$

where $\lambda_o = u_o + \rho_o/2 = u_o + 4k_o$. Using Eq.(A.11), three equations are obtained by arranging the ε^2, ε^3 and ε^4-orders in Eq.(15) assuming that $\rho^{(1)}$, $\rho^{(2)}$, $\rho^{(1)}$ and $\sigma^{(2)}$ approach, respectively, zero for $\xi \to +\infty$, namely

$$\frac{1}{4} \rho^{(1)} + \sigma^{(1)} = 0 \quad \text{(A.12a)}$$

$$\frac{\partial \rho^{(1)}}{\partial \tau} + \frac{1}{4}\rho^{(1)}\frac{\partial \rho^{(1)}}{\partial \xi} + \rho_0 \frac{\partial f^{(2)}}{\partial \xi} = 0 \tag{A.12b}$$

$$\frac{\partial f^{(2)}}{\partial \tau} + \frac{1}{4}\frac{\partial}{\partial \xi}\rho^{(1)} f^{(2)} - \frac{u_0'^2}{4\rho_0}\frac{\partial^3 \rho^{(1)}}{\partial \xi^3} = 0 , \tag{A.12c}$$

where $\qquad f^{(2)} \equiv \rho^{(2)}/4 + \sigma^{(2)}$.

A steady-state solution is examined along the characteristic line of $\zeta = (\xi - c^{(1)}\tau)$. In that case, Eqs.(A.12b) and (A.12c) are equivalent to the following equation,

$$\frac{\partial^2 \rho^{(1)}}{\partial \tau^2} + \frac{3}{8}\frac{\partial^2}{\partial \tau \partial \xi}\rho^{(1)\,2} + \frac{1}{32}\frac{\partial^2}{\partial \xi^2}\rho^{(1)\,3} + \frac{u_0'^2}{4}\frac{\partial^4 \rho^{(1)}}{\partial \xi^4} = 0 . \tag{A.13}$$

Using $\zeta = (\xi - c^{(1)}\tau)$ and integrating Eq.(A.13) twice by ζ under the condition of $\rho^{(1)} \to 0$ for $\zeta \to +\infty$, the following equation is obtained,

$$16\, u_0'^2 \left(\frac{\partial \rho^{(1)}}{\partial \zeta}\right)^2 + \Phi(\rho^{(1)}) = 0 , \tag{A.14a}$$

$$\Phi(\rho^{(1)}) = \rho^{(1)\,2}(\rho^{(1)} - 8c^{(1)})^2 . \tag{A.14b}$$

The value of $\rho^{(1)}$ must be equal to zero or $8c^{(1)}$, when $\partial\rho^{(1)}/\partial\zeta$ has a real value in Eq.(A.14a). This means that the wave with the amplitude perturbation of $\tilde{\rho} = 8c$ $(c > 0)$ propagates with the velocity of c in the reference frame moving with the constant velocity λ_0, where the relations of $\tilde{\rho} = \varepsilon\rho^{(1)}$ and $c = \varepsilon c^{(1)}$ are taken into account. Moreover, since $\tilde{k} = -\tilde{\rho}/4u_0'$ $= \tilde{\rho}/8\mu$ is obtained from Eq.(26c), the wave number $k_0 + \tilde{k}$ is also large when the amplitude is large. The result in this appendix corresponds to the limiting case that β_1 goes to zero in the envelope-soliton solution described by Eq.(26a). For the case $c < 0$, the modulational instability occurs because of $\rho_o + 8c < 4k_o$.

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Figure and Table Captions

Fig.1 The growth rate Γ vs. the wave number K in the modulational instability.

Fig.2 The maximum growth rate Γ_m vs. the amplitude a_o in the modulational instability.

Fig.3 Envelope-soliton profiles and propagation direction in the reference frame moving with β_2. The numbers correspond to those in Table1, respectively.

Table1. Classification of envelope-solitons for Alfvén waves, where $\beta_1 = (\rho_o{}^2/16 + (k_o/4)u_o'\rho_o)^{\frac{1}{2}}$ and $u_o' = -1$ and 1, respectively, for the left and right Alfvén waves.

	left Alfvén wave		right Alfvén wave	
	(1)	(2)	(3)	(4)
β_2	$u_o+\frac{1}{2}\rho_o+\beta_1$	$u_o+\frac{1}{2}\rho_o-\beta_1$	$u_o+\frac{1}{2}\rho_o+\beta_1$	$u_o+\frac{1}{2}\rho_o-\beta_1$
β_3	$\frac{-(u_o')^2}{8\beta_1}<0$	$\frac{(u_o')^2}{8\beta_1}>0$	$-\frac{(u_o')^2}{8\beta_1}<0$	$\frac{(u_o')^2}{8\beta_1}>0$
β_4	$\frac{3}{2}(\frac{1}{4}+\frac{\beta_1}{\rho_o})>0$	$\frac{3}{2}(\frac{1}{4}-\frac{\beta_1}{\rho_1})>0$	$\frac{3}{2}(\frac{1}{4}+\frac{\beta_1}{\rho_o})>0$	$\frac{3}{2}(\frac{1}{4}-\frac{\beta_1}{\rho_1})<0$
β_5	$\frac{-1}{u_o'}(\frac{1}{4}-\frac{\beta_1}{\rho_o})>0$	$\frac{-1}{u_o'}(\frac{1}{4}+\frac{\beta_1}{\rho_o})>0$	$\frac{-1}{u_o'}(\frac{1}{4}-\frac{\beta_1}{\rho_o})>0$	$\frac{-1}{u_o'}(\frac{1}{4}+\frac{\beta_1}{\rho_o})<0$
$\Delta\lambda$	-	+	-	+

Table 1.

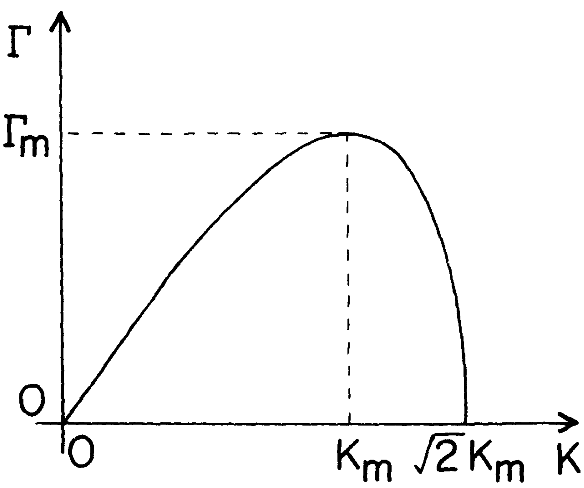


Fig. 1

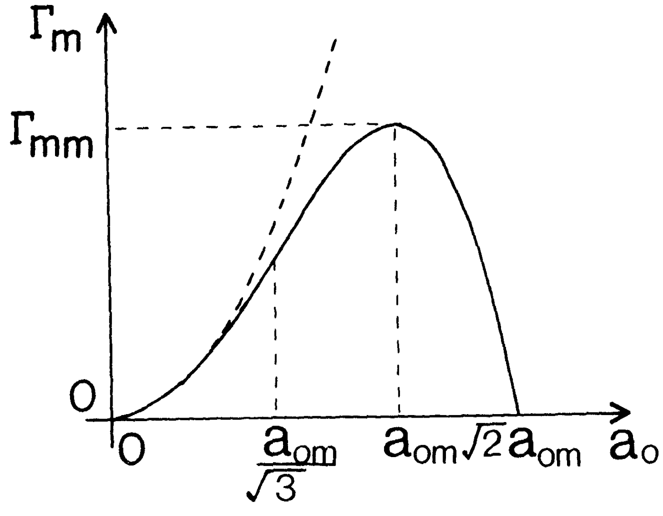
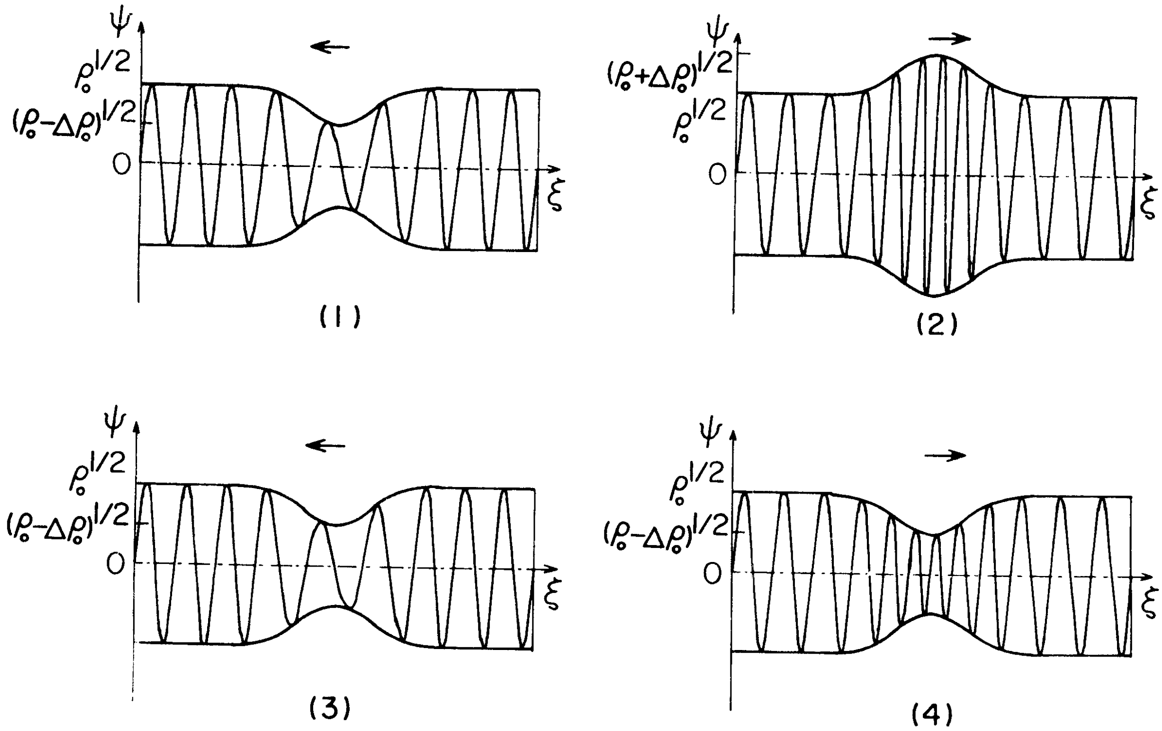


Fig. 2



ψ
$\rho_o^{1/2}$
$(\rho_o-\Delta\rho_o)^{1/2}$
0
ξ
(1)
ψ
$(\rho_o+\Delta\rho_o)^{1/2}$
$\rho_o^{1/2}$
0
ξ
(2)
ψ
$\rho_o^{1/2}$
$(\rho_o-\Delta\rho_o)^{1/2}$
0
ξ
(3)
ψ
$\rho_o^{1/2}$
$(\rho_o-\Delta\rho_o)^{1/2}$
0
ξ
(4)

Fig. 3