INSTITUTE OF PLASMA PHYSICS

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RESEARCH REPORT

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Study of Helical Plasma Equilibria

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Abstract

The equilibrium of the plasma limited by the material limiter or the separatrix in the linear stellarator system was solved numerically by the finite difference method.

The expansion and the deformation of the plasma column due to the longitudinal plasma current and the plasma pressure were obtained and were in agreement with the predicted values in the simple case where the analytical solutions were already found.

1. Introduction

In recent years, the axisymmetric equilibria of the plasma limited by the material limiter or the separatrix have been solved numerically by the various authors.[1] These calculations are indispensable for designing the system of the poloidal divertor and the apparatus of the non-circular tokamak, the shape of which is controlled by the external coil system.[2] In the toroidal stellarator experiment, the information about the deformation of the magnetic surface due to the plasma pressure or due to the toroidal plasma current is necessary. The computational study of the helical equilibria of $\beta = 1$ plasma with free boundary was done by N. Friedman.[3] His calculation was qualitatively applicable to the Scyllac experiment.

In this paper, the helical equilibrium of the plasma with free and diffuse boundary was solved numerically as the zero order approximation of low β plasma in a toroidal stellarator. As a special case of this analysis, we can obtain free boundary axisymmetric equilibria. In the case of quasi-homogeneous plasma current, the theoretical analysis by P. Barberio-Corsetti[4] and the one extended in this paper can give satisfactory agreement to the result obtained by the finite diference method.

2. Approximate analysis of free boundary equilibria

We use the differential equation of the helical plasma equilibria in the cylindrical system, r, θ, z[5]

$$L\psi + \frac{2\ell k}{\ell^2+k^2r^2} B_\phi = -\mu_o j_\phi \tag{1}$$

$$L = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{2\ell^2}{\ell^2+k^2r^2}\frac{\partial}{\partial r} + \frac{\ell^2+k^2r^2}{r^2}\frac{\partial^2}{\partial\phi^2}, \tag{2}$$

$$\mu_o J_\phi = \mu_o(\ell^2+k^2r^2)\frac{dp}{d\psi} + B_\phi\frac{dB_\phi}{d\psi}, \tag{3}$$

$$J_\phi = krj_\theta - \ell j_z, \tag{4}$$

$$B_\phi = krB_\theta - \ell B_z, \tag{5}$$

$$\phi = \ell\theta + kz, \tag{6}$$

where ψ is the stream function of the magnetic field and the function of r and $\phi = \ell\theta + kz$ only.

Because of the helical symmetry, the following relations are also available

$$\frac{\partial\psi}{\partial r} = krB_z + \ell B_\theta, \tag{7}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\phi} = -B_r. \tag{8}$$

In the simple case of $B_\phi = B_{\phi o} + s\psi$, s = const and $\frac{dp}{d\psi} = 0$, the general solutions of equation (1) in the plasma ψ_p, and in the vacuum ψ_v, were obtained by Chandrasekhar et al.[6)]

$$\psi_p = -\frac{B_{\phi o}}{s} + \frac{B_{\phi o}}{s}\{J_o(sr) - \frac{kr}{\ell} J_1(sr)\}$$

$$+ \sum^{\infty} An[sI_{n\ell}(k_n r) + \frac{k}{\ell} k_n r \, I'_{n\ell}(k_n r)] \cos n\phi \ , \qquad (9)$$

where $\quad k_n{}^2 = n^2 k^2 - s^2, \qquad (10)$

$$\psi_V = \frac{k}{2} B_{zo} \, r^2 + c_2 \log r + \{b_o r \, I_\ell{}' \, (kr) + b_1 r \, K_\ell{}' \, (kr)\}\cos\phi$$

$$+ \sum_{n=2}^{\infty} \{B_n r \, I'_{n\ell}(nkr) + C_n r \, K'_{n\ell}(nkr)\}\cos n\phi. \qquad (11)$$

Now we are going to apply these equations to a specific case of helical plasma equilibria. Figure 1 shows the schematics of the linear stellarator system under consideration. The vacuum stellarator field is produced by the helical sheet coil which is placed at $r = a_c$. The current distribution on the sheet coil is given by the following equation

$$i = i_o \cos(\ell\theta + kz). \qquad (12)$$

The plasma boundary is limited by the separatrix or the material limiter, of which diameter is 2b. The B_z field is produced by the coil which is placed outside the helical sheet coil. The helical coil current I_h, which flows in one direction, is given by

$$I_h = \int_{-\frac{\pi}{2\ell}}^{\frac{\pi}{2\ell}} i_o \cos(\ell\theta) \times a_c d\theta = 2\frac{a_c}{\ell} i_o \ , \qquad (13)$$

and the stream function ψ_h due to the helical sheet current is, in the term of I_h

$$\psi_h = -\frac{\mu_o \ell}{2} \frac{k^2 a_c}{\sqrt{\ell^2+k^2a_c^2}} K_\ell'(ka_c)\cdot r \cdot I_\ell'(kr)\cos\phi \times I_h. \quad (14)$$

If we confine ourselves to the analysis of the deformation of the magnetic surface and the shift of the separatrix due to the presence of the plasma, we may neglect the higher order terms which contain $\cos(n\ell\phi)$, $n \geqq 2$. Also, we may neglect the presence of the conductive shell placed at $r = r_c$ for convenience.

Then we can assume that

$$A_n = 0 \qquad n \geqq 2 , \qquad (15)$$

$$B_n = 0, \qquad C_n = 0 \qquad n \geqq 2 . \qquad (16)$$

From the boundary condition of $r = a_c$, we can get the equation (17),

$$b_o = -\frac{\mu_o \ell}{2} I_h \frac{k^2 a_c}{\sqrt{\ell^2+k^2a_c^2}} K_\ell'(ka_c) . \qquad (17)$$

The boundary condition at the plasma surface Σ is

$$B_\phi = B_\phi , \qquad (18)$$

$$\frac{\partial \psi_p}{\partial r} = \frac{\partial \psi_v}{\partial r} , \qquad (19)$$

$$\frac{\partial \psi_p}{\partial \theta} = \frac{\partial \psi_v}{\partial \theta} . \qquad (20)$$

The shape of Σ is generally not circular, so the determination of the coefficient is a complex problem.

If we assume that the deviation of the shape of the separatrix from circle is small, we can connect the equations (14) and (15) till the accuracy of $O(\frac{\delta}{R})^1$ by the method[4] adopted by P. Barberio-Corsetti, where δ is the deviation of the separatrix from the circle of the radius R_o. We define the function p_p, q_p, p_v, q_r by

$$\Psi_p(r, \theta) = p_p(r) + \varepsilon\, q_p(r, \theta) \ , \tag{21}$$

$$\Psi_v(r, \theta) = p_v(r) + \varepsilon\, q_v(r, \theta) \ , \tag{22}$$

and assume that $\varepsilon << 1$. Then to order one

$$B_{\phi o} + s\, p_p(R_o) = kc_2/\ell - \ell B_{zo} \tag{23}$$

$$p_p'(R_o) = p_v'(R_o) \ . \tag{24}$$

Since, the deviation of the plasma boundary from the circle δ is $\delta \approx -\varepsilon q/p'$ to order ε, we get

$$q_p(R_o) = q_v(R_o) \ , \tag{25}$$

$$q_p' - q_p p_p''/p_p' = q_v' - q_v p_v''/p_v' \ , \tag{26}$$

where the prime means $\frac{d}{dr}$ and everything is calculated at $r = R_o$. From (23) and (24) we can get neglecting $(\alpha r)^2$ and

$(sr)^2$,

$$B_{\phi o} = -\ell B_{zo} \,, \tag{27}$$

$$c_2 = +\frac{B_{\phi o}}{2} sR_o^2 \,, \tag{28}$$

For $\ell = 2$ system, from equations (25), (26) we have, expanding for small radius,

$$A_1 = \frac{2b_o}{(k^2-s^2)(1+\frac{s}{2k})} \,. \tag{29}$$

Substituting (26) into (9), we obtain the quation for the magnetic surfaces near the center,[4)]

$$r^2\{1 - \frac{2b_o}{B_{\phi o}(1+s/2k)} \cos 2\theta\} = \text{const} \,. \tag{30}$$

For $\ell = 3$ system, we have

$$A_1 = + 3b_o \frac{(k+\frac{3}{2}s)^2}{(k+s)^2} \frac{k^2}{k_1^3} \,. \tag{31}$$

Substituting (31) into (9) and by differentiating (9), we have the equation of the distance of hyperbolic singular point to the axis, in the presence of plasma current

$$r_s = \left| \frac{16}{9} (1+\frac{s}{k}) \frac{B_{\phi o}}{b_o} \cdot \frac{1}{k} \right| . \tag{32}$$

Although (32) was obtained for $\varepsilon \ll 1$, equation (32) seems to be applicable for $\varepsilon \approx 1$, judging from the fact that equation

(29) gives the exact distance of the hyperbolic point in the case of s = 0.

3. Numerical scheme

Equation (1) can be solved numerically more straightforward by introducing the function $S(\psi)$ as was done in the case of the axisymmetric equilibria.[1)]

Then we can rewrite Equation (1)

$$L\psi + \frac{2\ell k}{\ell^2+k^2r^2} B_{\phi 1} = -S(\psi)*\{\mu_o(\ell^2+k^2r^2)p' + B_\phi \frac{dB_\phi}{d\psi} + \frac{2k\times\ell}{\ell^2+k^2r^2}(B_\phi - B_{\phi 1})\}, \quad (33)$$

and

$S(\psi) = 1$ if ψ is inside the plasma

$S(\psi) = 0$ if ψ is outside the plasma,

where $B_{\phi 1}$ is B_ϕ at the vacuum, and $B_{\phi 1} \cong B_{\phi o}$ if s is not large.

For convenience we are going to solve the equation inside $r \leqq a$ with the boundary condition

$$\psi(a_c) \equiv -\frac{kB_{\phi 1}}{2\ell} a_c^2 - \frac{\mu_o \ell}{2} \frac{k^2 a_c^2}{\sqrt{\ell^2+k^2 a_c^2}} K_\ell(ka_c) I_\ell'(ka_c)\cos\phi \times I_h. \quad (34)$$

The physical interpretation of this boundary condition is like this. The conducting shell is placed at $r = a_c$. The magnetic field due to the helical and toroidal coil currents penetrates fully the shell. But the stream function due to the plasma and the plasma current is constant on the shell because of its short life.

We may neglect the effect of the presence of the shell on the plasma for physical interpretation when we consider the deformation of the magnetic surface in the plasma, if the plasma radius is much shorter than the radius of the shell.

Equation (33) was solved by SOR method in (r, ϕ) coordinate, and at $r = 0$ we imposed the condition of $\left.\frac{d\psi}{dr}\right|_{r=o} = 0$. The mesh-width was reduced automatically by a factor 2 after the convergence at each mesh width. The final mesh points were 40×40. We define the plasma boundary in the following way. If the calculated separatrix is inside the inner radius of the limiter, $S(\psi) = 1$, if the ψ is inside the separatrix and $S(\psi) = 0$, if outside. In the case that there is no separatrix or the singular point on the separatrix is outside the limiter, then, $S(\psi) = 1$, if the ψ is inside the magnetic surface which just the limiter at $\theta = 0$.

4. Results and Discussion

As a check of the numerical scheme, the positions of the hyperbolic singular points calculated numerically by the method described above, are compared with the theoretical values. Figure 2 shows the good agreement between two values.

Figure 3 shows the plasma equilibria limited by the material limiter in the $\ell = 2$ stellarator. If the force-free plasma current is anti-parallel to the B_z magnetic field, the magnetic surface in the plasma tends to be more eccentric and vice versa. The calculated dependence of the ratio of the lengths of the major and minor axes of the magnetic surface of $\ell = 2$ plasma near the axis upon the value of $\frac{dB_\psi}{d\psi}$ is given in Fig.4. In this calculation, the inner radius of the limiter was at $\frac{1}{5}a_c$ in order to reduce the effect of the conducting shell placed at $r = a_c$. We can see that the iteration method till the first order of (δ/R) can give fairly accurate values. In Fig.5 the expansion of the plasma column by the force-free plasma current in the $\ell = 3$ stellarator is shown. If the plasma current is parallel to the B_z magnetic field, the plasma column expands, and vice versa. In this case, the plasma boundary is limited by the separatrix, because the separatrix is inside the limiter. The expansion and shrink of the separatrix of the $\ell = 3$ plasma by the force= free plasma current, can be well described by the equation (32), as we can see in Fig.6.

So far we have been dealing with the pressureless plasma with force-free current. The change of the shape of the magnetic surface due to the plasma pressure does not occur significantly till the β of more than 10%. So the effect of the plasma pressure upon the magnetic surface is quite weak compared with the case of the toroidal stellarator where the β larger than 1% or so can give the large effect on the magnetic surface.

So far we had the case of $\frac{dB_\psi}{d\psi} = s = \text{const}$, that is, the plasma current is quasi-homogeneous, in order to compare the numerical results with the theoretical prediction. We also have the convergence of the numerical scheme for the case of $\frac{dB_\psi}{d\psi} = \alpha\psi + \beta$ and $\alpha\psi^2 + \beta\psi + \gamma$, where the theoretical analysis is quite difficult.

The present numerical scheme suffers poor and no convergence at the particular region of the parameter s in the case of $\frac{dB_\psi}{d\psi} = s = \text{const}$, even when s is not large. The region of no convergence shrinks as we set the diameter of the limiter smaller if the plasma boundary are limited by the limiter. As the diameter of the limiter becomes larger, the region of no convergence grows wider in the parameter space and its number of the region increases. Even in the case of $\frac{dB_\psi}{d\psi} = \text{const}$, equation (1) is a nonlinear equation and it may have 2 or more solutions at the specific value of s. The reason of no convergence is now under study.

Conclusion

Diffuse and helical equilibrium of the plasma limited by the separatrix or the limiter was solved by a finite difference method as a boundary value problem at the conducting shell.

In the case of quasi-homogeneous force-free plasma current where the analytic solutions are known, the shift of position of the separatrix at $\ell = 3$ and the induced eccentricity of the magnetic surfaces of the $\ell = 2$ stellarator near the axis are in good agreement with the theoretical analysis. For the peaked current profile where the analytical solutions are unknown, the present numerical scheme shows convergence.

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Figure Captions

Fig.1 Schematics of the linear stellarator system for the present study.

Fig.2 The distance of the hyperbolic singular point on the separatrix to the axis, obtained by the finite difference method and the analytical one.

Fig.3 Magnetic surfaces of the $\ell = 2$ stellarator calculated by the finite difference method (a) the vacuum, (b) the force-free plasma current, $\frac{dB_\phi}{d\psi} = -1.25\ m^{-1}$. $B_{\phi 1} = 40$ kG, $I_h = 300$ kA, $k = 4\ m^{-1}$, $a_c = 30$ cm.

Fig.4 Dependence of the ratio of lengths of the major and minor axes near the axis upon $\frac{dB_\psi}{d\psi}$. The solid line is equation 27. The triangular point is by SOR. $B_{\phi 1} = 40$ kG, $I_h = 300$ kA, $k = 4\ m^{-1}$, $a_c = 30$ cm.

Fig.5 Magnetic surfaces of the $\ell = 3$ stellarator calculated by the finite-difference method. (a) the vacuum (b) the force-free plasma current $\frac{dB_\phi}{d\psi} = 2.5\ m^{-1}$. $B_{\phi 1} = 20$ kG, $I_h = 300$ kA, $k = 4\ m^{-1}$, $a_c = 30$ cm.

Fig.6 The distance from the singular point on the separatrix to the axis.
$B_{\phi 1} = 20$ kG, $I_h = 300$ kA, $k = 4\ m^{-1}$, $a_c = 30$ cm.
Solid line is equation 32.
The cross is the calculated value. The ambiguity is due to the mesh width.

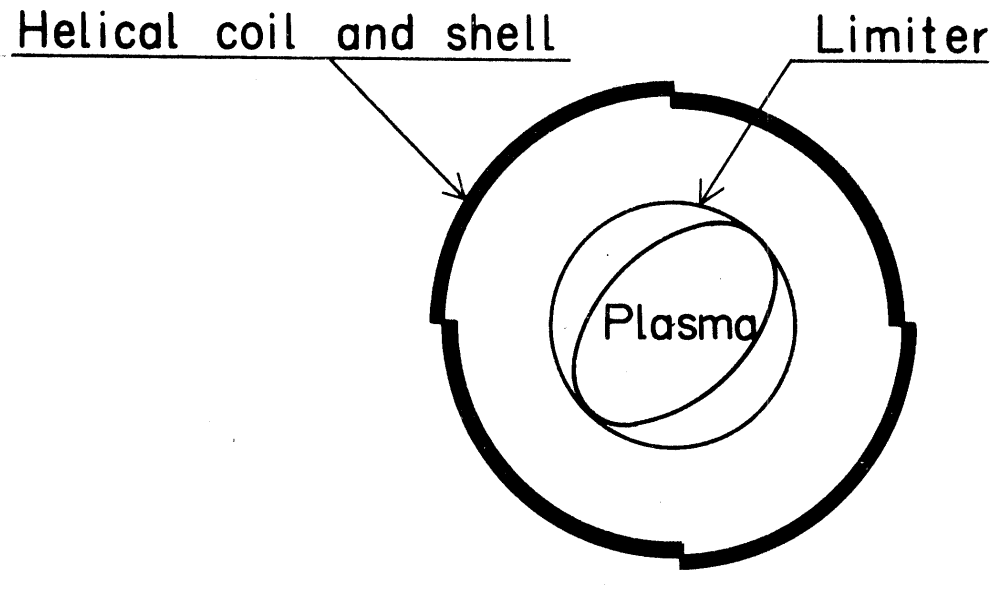


Fig.1

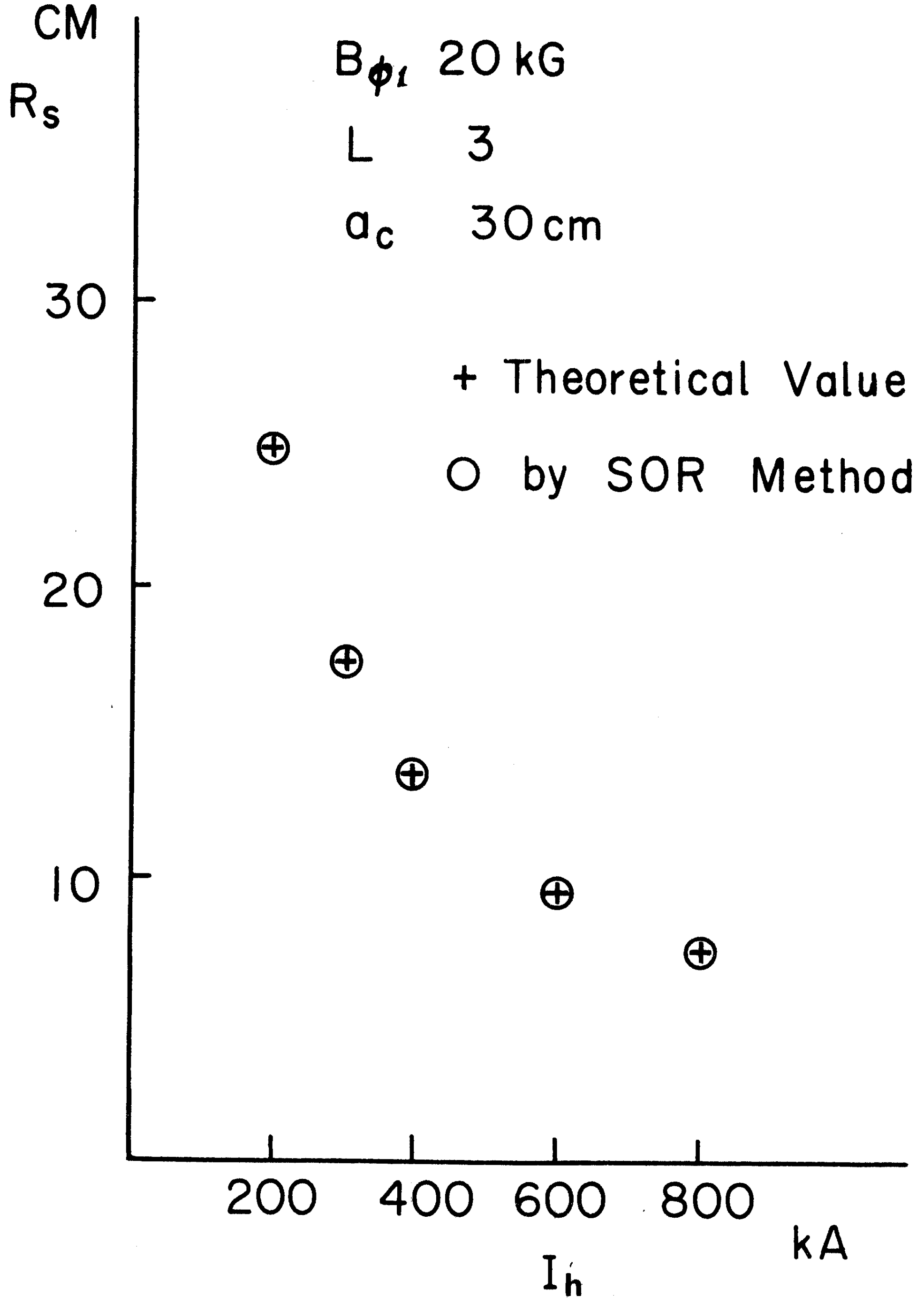
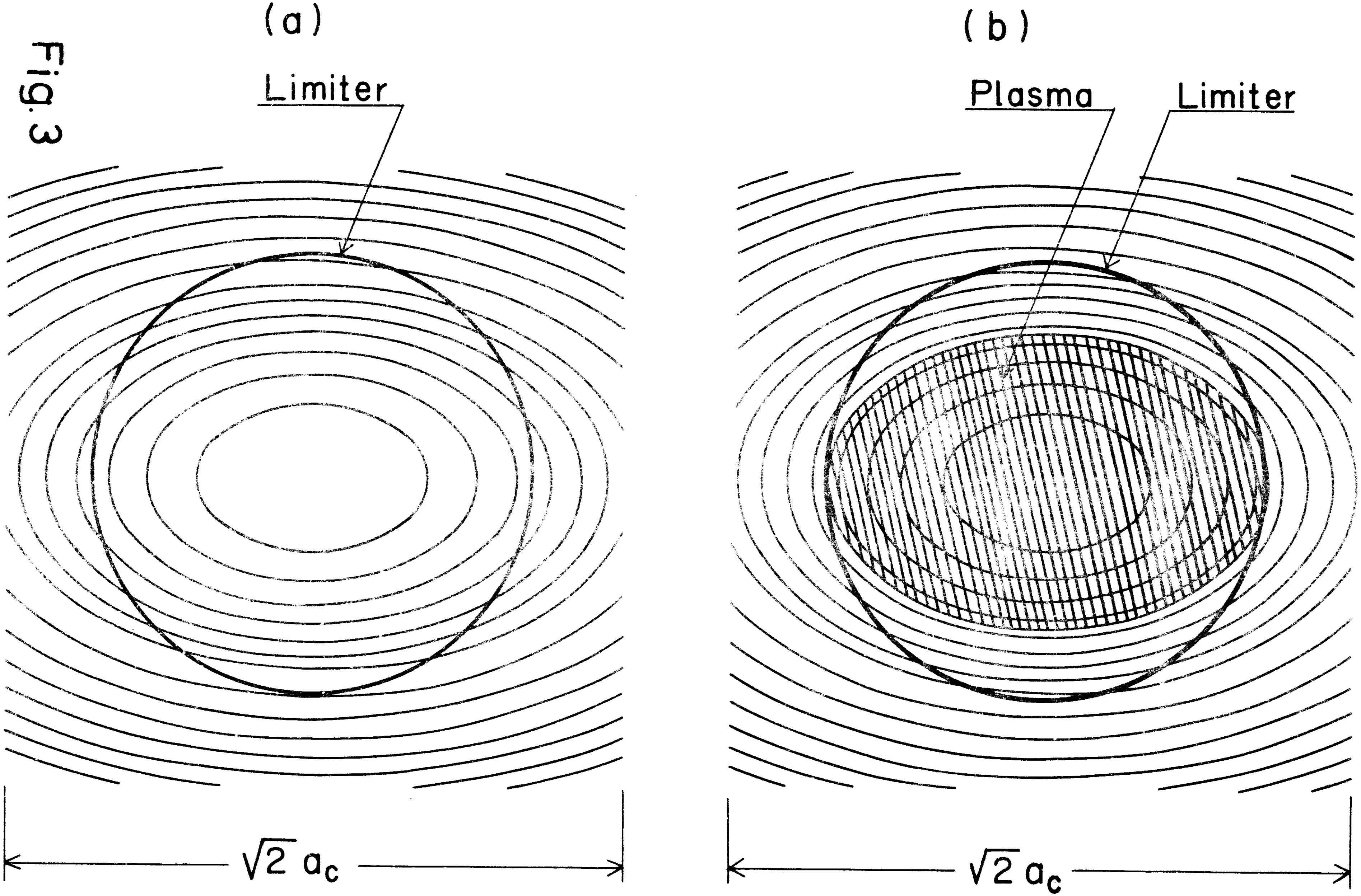

CM
R_s
$B_{\phi 1}$ 20 kG
L 3
a_c 30 cm
30
20
10
+ Theoretical Value
O by SOR Method
200
400
600
800
kA
I_h

Fig. 2

Fig. 3



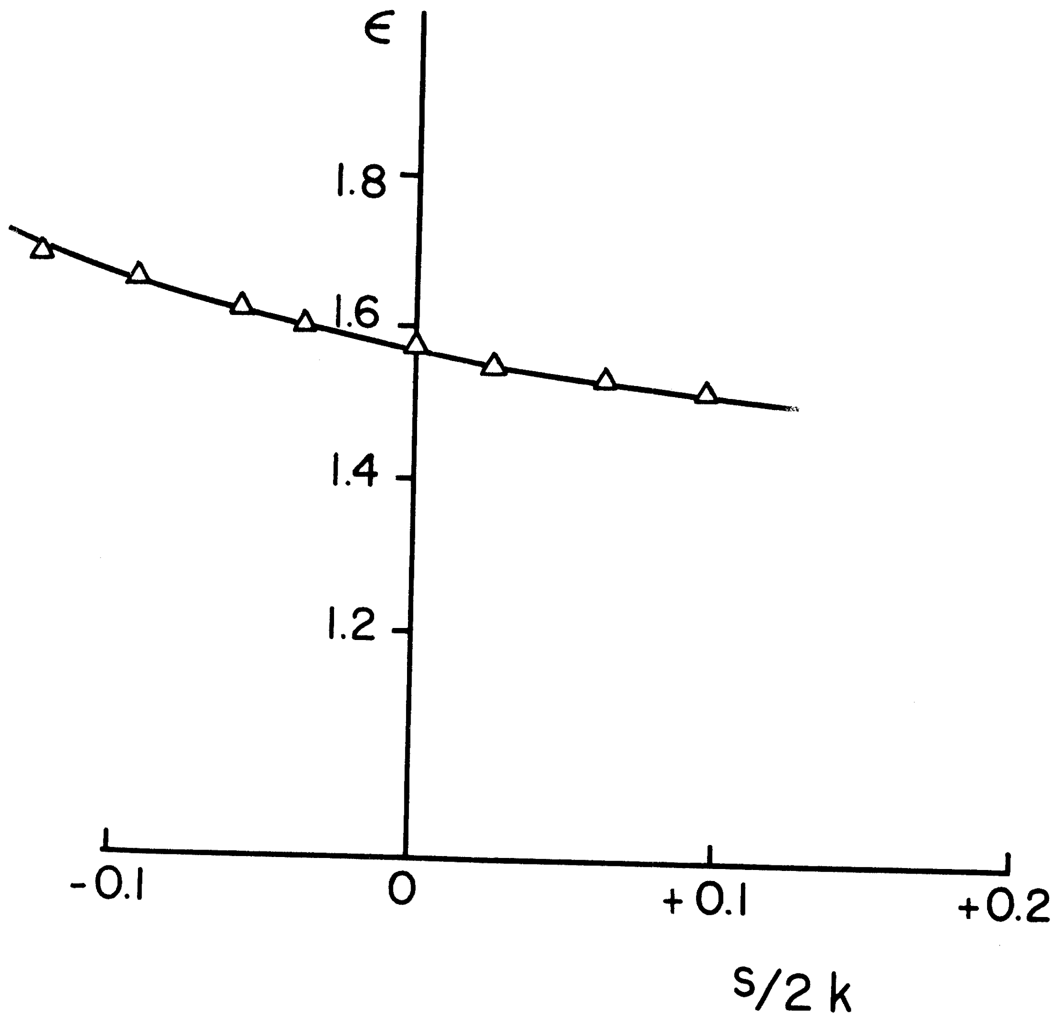


Fig. 4

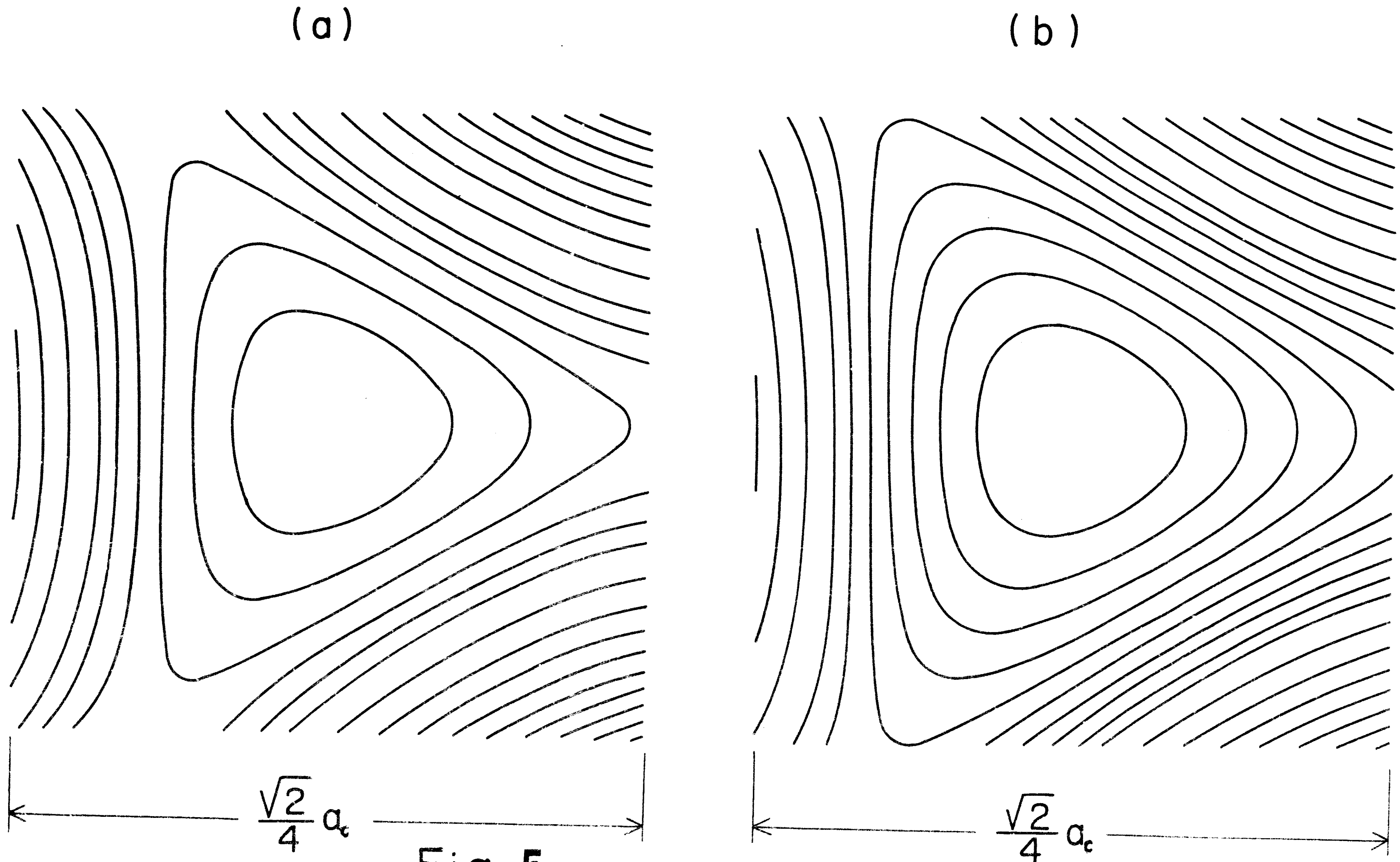
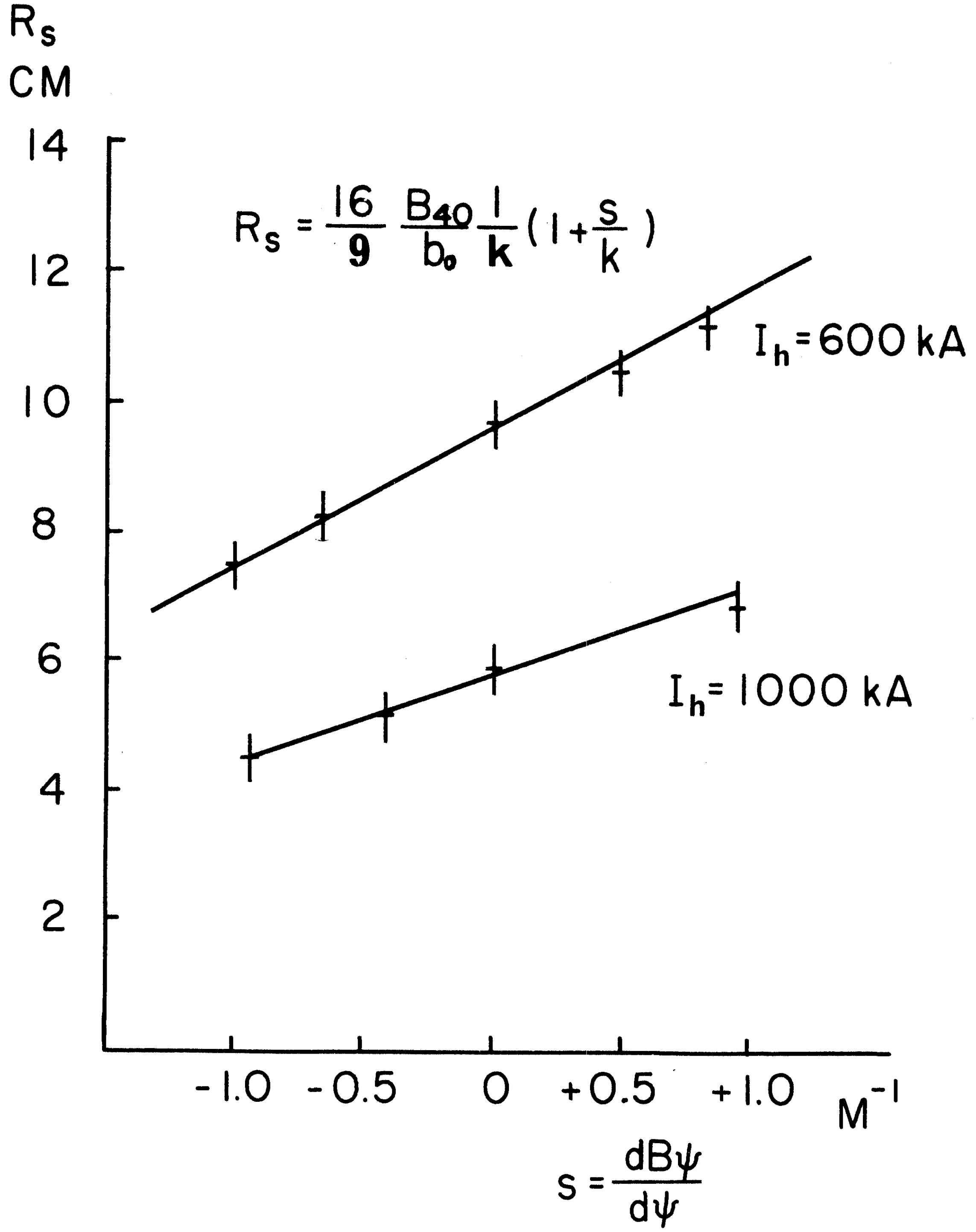


Fig. 5



R_s
CM
14
12
10
8
6
4
2
$R_s = \frac{16}{9} \frac{B_{40}}{b_0} \frac{1}{k} (1 + \frac{s}{k})$
$I_h = 600$ kA
$I_h = 1000$ kA
-1.0
-0.5
0
+0.5
+1.0
M^{-1}
$s = \frac{dB\psi}{d\psi}$

Fig. 6