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Asymptotic Expansion of Nonlinear Unstable Collisional Drift Wave

T. Hatori and Y. Terashima

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Synopsis

A nonlinear theory of the collisional drift instability is developed in a slab model based on the two fluid equations where the ion inertia, finite gyroradius and viscosity are included. A systematic expansion is introduced by taking $\varepsilon=|\kappa|\ell$ as a smallness parameter where κ is the degree of density gradient and ℓ is the linear scale of the slab along the density gradient. A set of the model equations is derived, which can describe the nonlinear evolution of the drift wave.

§1. Introduction

Nonlinear theory of unstable waves is one of the most important problems in the fundamental plasma physics, as well as in plasma confinement studies. It is generally believed that fluctuations due to instabilities play a significant role in the enhancement of plasma transport. Among various modes one of the most important modes with respect to the transport is the drift mode which occurs in the presence of the spatial gradient of density and/or of temperature of magnetically confined plasma. Recent experimental results of stellarator device[1) have shown that in the collisional regime the plasma is subject to the large amplitude oscillations of the collisional drift mode rather than to fluctuations, and usually a few unstable modes dominate. The observed particle losses are interpreted to be mainly due to the collisional drift instability. It is therefore necessary to treat nonlinear saturation near marginal stability and transition to the states with several unstable modes, and the consequent anomalous transport. The purpose of this paper is to present a method of asymptotic expansion for nonlinear unstable collisional drift wave in order to make these problems tractable.

Nonlinear analysis of the drift instability in the fully ionized collisional regime was carried out first by Stix[3)], and by Hinton and Horton[4)]. Subsequently, Monticello and Simon[5)] performed a self-consistent calculation which includes the effect of both the zero-frequency harmonic and the radial derivatives. They have found that the modification of the background density by the zero-frequency harmonic is the primary mechanism for saturation of the isntability, which was omitted hitherto. Their treatment, however, is mostly concerned with a steadily oscillating drift mode above a critical magnetic field. Recently, Sato[6] has presented a numerical work for this instability. Although the wide range of physical parameters is treated in his work, the ion inertia terms and stress tensor is neglected.

We here have interests in the following parameter range of a low beta plasma; $k_\perp^2 a_i^2 << 1$, $\Omega_\alpha >> \nu_\alpha >> \omega$, $k_\parallel v_{T\alpha} (\alpha=e,i)$ where ω is the frequency of a drift wave, $k_\perp$ and $k_\parallel$ are the components of its wave vector perpendicular to and parallel to the magnetic field, respectively, a_i is the mean ion gyroradius, and Ω_α the cyclotron frequency for the respective species, ν_α the collision frequency, and $v_{T\alpha} = \sqrt{2T_\alpha/m_\alpha}$. And also

$$\omega_* \gtrsim \omega >> \gamma_L \simeq \omega_* (\omega_* - \omega) \nu_{ei}/k_\parallel^2 v_{Te}^2 \quad , \tag{1.1}$$

where ω_* is the electron diamagnetic frequency and γ_L is the linear growth rate. Therefore, we use the two fluid description and consider the electrostatic field $E=-\nabla\phi$ and the uniform magnetic field B.

§2. Basic Equations

We start from the following set of the equations for two component isothermal fluid[7]

$$0 = -\vec{\nabla} p_e - en_e(\vec{E} + \frac{1}{c}\vec{v}_e \times \vec{B}) + \vec{R} \quad , \tag{2.1}$$

$$m_i n_i(\frac{\partial}{\partial t} + \vec{v}_i \cdot \vec{\nabla})\vec{v}_i = -\vec{\nabla} p_i - \vec{\nabla}\Pi + e\, n_i(\vec{E} + \frac{1}{c}\vec{v}_i \times \vec{B}) - \vec{R}, \tag{2.2}$$

$$\vec{R} = -m_e n_e \nu_{ei}(0.51\, \vec{u}_{\parallel} + \vec{u}_{\perp}), \quad \vec{u} = \vec{v}_e - \vec{v}_i \quad , \tag{2.3}$$

$$\frac{\partial n_e}{\partial t} + \mathrm{div}(n_e \vec{v}_e) = Q_e \quad , \tag{2.4}$$

$$\frac{\partial n_i}{\partial t} + \mathrm{div}(n_i \vec{v}_i) = Q_i \quad , \tag{2.5}$$

where the electron inertia and stress tensor are neglected, the ion charge is assumed to be Z=1, the $\vec{\nabla}\Pi$ term is the ion stress tensor term which includes the finite ion gyroradius and the ion viscosity contributions, the source terms $Q_{e,i}$ are introduced to sustain the steady unperturbed state, and the other symbols have the usual meanings.

We henceforth assume the quasi-neutrality condition $n_e \simeq n_i = n$ under the condition that $k_\perp^2 \lambda_D^2 << 1$, and λ_D is the Debye length. We solve eq.(2.1) for $\vec{v}_{e\parallel}$ and $\vec{v}_{e\perp}$,

$$\vec{v}_{e\parallel} = D_{c\parallel} \vec{\nabla}_{\parallel}(e\phi/T_e - \log n) \quad , \tag{2.6}$$

$$\vec{v}_{e\perp} = \vec{u}_E + \vec{u}_{de} - D_{c\perp} \vec{\nabla}_\perp \log n \quad , \tag{2.7}$$

where

$$\vec{u}_E = (c/B^2)(\vec{B}\times\vec{\nabla}\phi), \quad \vec{u}_{de} = -(T_e/m_e\Omega_e)(\vec{B}\times\vec{\nabla}n)/Bn \quad , \tag{2.8}$$

and $D_{c\parallel,\perp}$ are the classical diffusion coefficients defined, respectively, by

$$D_{c\parallel} = 1.96\, T_e/m_e \nu_{ei}, \quad D_{c\perp} = (T_e + T_i)\nu_{ei}/m_e\Omega_e^{\,2}. \tag{2.9}$$

In the derivation of (2.6) the ion parallel motion is neglected, and in the derivation of (2.7) by omitting the ion inertia and viscosity terms the following approximation is made,

$$\vec{R}_\perp \cong (\nu_{ei}/\Omega_e)(T_e + T_i)(\vec{B}/B) \times \vec{\nabla} n \quad .$$

Substituting (2.6) and (2.7) into (2.4), one obtains

$$\frac{\partial n}{\partial t} + \vec{u}_E\cdot\vec{\nabla} n + nD_{c\parallel}\ \nabla_\parallel^{\,2}\left(\frac{e\phi}{T_e} - \log n\right) - \vec{\nabla}_\perp\cdot(D_{c\perp}\vec{\nabla}_\perp n) = Q_e . \tag{2.10}$$

For the ion component, we shall solve eqs. (2.2) and (2.5) to the second order in $k_\perp a_i$, and $\vec{v}_{i\perp}$ is deduced by iteration from (2.2) as

$$\vec{v}_{i\perp} = \vec{u}_E + \vec{u}_{di} + \frac{1}{\Omega_i}\,\frac{\vec{B}}{B}\times\left\{\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\vec{v} + \frac{1}{m_i n}\vec{\nabla}\Pi\right\}_{\vec{v}=\vec{u}_E+\vec{u}_{di}}$$

$$- D_{c\perp}\vec{\nabla}_\perp \log n \quad ,$$

where

$$\vec{u}_{di} = (T_i/m_i\Omega_i)(\vec{B}\times\vec{\nabla} n)/(Bn). \tag{2.11}$$

Substituting this relation into (2.5) and neglecting the ion parallel motion, we obtain

$$\begin{aligned}
&\frac{\partial n}{\partial t} + \vec{u}_E\cdot\vec{\nabla} n - \vec{\nabla}_\perp(D_{c\perp}\vec{\nabla}_\perp n) \\
&\quad + \mathrm{div}\left\{\frac{n}{\Omega_i}\,\frac{\vec{B}}{B}\times\left[\frac{\partial}{\partial t} + (\vec{u}_E+\vec{u}_{di})\cdot\vec{\nabla}\right]\vec{u}_E\right\} \\
&\quad + \mathrm{div}\left\{\frac{n}{\Omega_i}\,\frac{\vec{B}}{B}\times\left[\left(\frac{\partial}{\partial t} + \vec{u}_E\cdot\vec{\nabla}\right)\vec{u}_{di} + \frac{1}{m_i}\,\vec{\nabla}_\perp\Pi_{FLR}(\vec{u}_\perp=\vec{u}_E)\right]\right\} \\
&\quad + \mathrm{div}\left\{\frac{1}{m_i\Omega_i}\,\frac{\vec{B}}{B}\times\vec{\nabla}\Pi_s(\vec{u}_\perp=\vec{u}_E + \vec{u}_{di})\right\} = Q_i ,
\end{aligned} \tag{2.12}$$

where $\vec{\nabla}\Pi_s$ is the ion viscosity part of $\vec{\nabla}\Pi$ and $\vec{\nabla}\Pi_{FLR}$ the finite ion gyroradius one, namely,

$$\vec{\nabla}\Pi_{FLR}(\vec{u}_\perp) = - \frac{T_i}{2\Omega_i} \{[n\Delta_\perp + \vec{\nabla}_\perp n\cdot\vec{\nabla}_\perp](\vec{u}_\perp \times \frac{\vec{B}}{B}) + (\frac{\vec{B}}{B} \times \vec{\nabla} n)\cdot\vec{\nabla}_\perp \vec{u}_\perp\} \quad . \tag{2.13}$$

In the derivation of (2.12), the following identity is used

$$\mathrm{div}\{\frac{\vec{B}}{B} \times [n(\vec{u}_{di}\cdot\vec{\nabla})\vec{u}_{di} + \frac{1}{m_i} \vec{\nabla} \Pi_{FLR}(\vec{u}_\perp = \vec{u}_{di})]\} = 0 \quad . \tag{2.14}$$

We thus have two equations, the eqs. (2.10) and (2.12), to determine the variables n and ϕ.

§3. Nonlinear Analysis

For equilibrium state, we adopt a slab model with the density profile

$$N(x) = N_0(1 + \kappa x) \quad , \quad 0 \leq x \leq \ell. \tag{3.1}$$

The magnetic field is taken to be in the z-direction. The electrostatic potential in equilibrium is set to be zero for simplicity.

We assume $\varepsilon \equiv |\kappa|\ell << 1$ and introduce the following ordering scheme;

$$\begin{aligned} &\tilde{n}/N \sim e\tilde{\phi}/T_e \sim \omega/\Omega_i \sim \omega_*/\Omega_i \sim O(\varepsilon) \quad , \\ &k_y\ell \sim O(\varepsilon^0) \quad , \quad \nu_{ii}/\Omega_i \sim O(\varepsilon^2) \quad , \\ &k_\parallel{}^2 D_{c\parallel}/\omega_* \sim O(\varepsilon^{-1}) \quad , \quad k_y{}^2 D_{c\perp}/\omega_* \sim O(\varepsilon) \quad , \end{aligned} \tag{3.2}$$

where $\tilde{n}$ and $\tilde{\phi}$ are the perturbed density and potential, ω and $(k_y, k_\parallel = k_z)$ are their frequency and wave numbers and ω_* is defined as $\omega_* = -k_y \kappa T_e / m_e \Omega_e$. This ordering differs from those adopted by Hinton and Horton, and by Monticello and Simon.

When we substitute $n = N + \tilde{n}$ and $\phi = \tilde{\phi}$ into the eqs(2.10) and (2.12) and separate nonlinear terms from linear ones, these equations can be expressed in a matrix form as

$$LU = S \ , \qquad U = \begin{pmatrix} \rho \equiv \tilde{n}/N \ , \\ \psi \equiv e\tilde{\phi}/T_e \end{pmatrix} , \tag{3.3}$$

where L is the linear operator and S is the nonlinear term. The expressions of L and S are given in the Appendix. By the ordering in (3.2), we may expand

$$L = L^{(0)} + \varepsilon L^{(1)} + \varepsilon^2 L^{(2)} + \cdots \ , \ U = \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots \ ,$$

$$S = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \cdots \ . \tag{3.4}$$

Also

$$\frac{\partial}{\partial t} = \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \left(\frac{\partial}{\partial t_2} - v_g \frac{\partial}{\partial y_1} \right) + \cdots \ ,$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y_0} + \varepsilon \frac{\partial}{\partial y_1} + \cdots \ , \tag{3.5}$$

where v_g is the group velocity to be determined in the course of calculation. The reason why the time derivative begins in the first order in ε is due to that ω is thought to be first order quantity in (3.2).

[1] 1st order calculation.

From (3.3), we have

$$L^{(0)} U^{(1)} = \begin{pmatrix} -D_{C\parallel} \dfrac{\partial^2}{\partial z^2} \ , & D_{C\parallel} \dfrac{\partial^2}{\partial z^2} \\ 0 \ , & 0 \end{pmatrix} U^{(1)} = 0 \ . \tag{3.6}$$

Here and hereafter $D_{c\parallel}$ means its value for $n=N_0$. Therefore, we may write

$$\rho^{(1)} = \psi^{(1)} \equiv f_{(1)} = h(x,y_1,t_2=\varepsilon^2 t) + \sum_{k_\parallel}\{g(x,y_1,t_2)e^{2ik_\parallel z} + c.c.\}$$

$$+ \sum_{k_\parallel, k_y} \{[f_+(x,y_1,t_2)e^{ik_\parallel z} + f_-(x,y_1,t_2)e^{-ik_\parallel z}]e^{-i\omega t_1 + ik_y y_0} + c.c.\} \tag{3.7}$$

where h and g denote the modulation of the background density, and f_+ and f_- denote the amplitude of drift wave. We have also assumed that the zero-frequency part of the potential perturbation is equal to the background density modulation. It should be noted that ω is taken to be real in the present representation.

[2] 2nd order calculation.

We have

$$L^{(0)}U^{(2)} + L^{(1)}U^{(1)} = S^{(2)}.$$

The components of this equation are

$$-D_{c\parallel}\frac{\partial^2}{\partial z^2}(\rho^{(2)}-\psi^{(2)}) + (\frac{\partial}{\partial t_1} + V_* \frac{\partial}{\partial y_0})f_{(1)} = -D_{c\parallel}\frac{\partial^2}{\partial z^2}(\frac{1}{2}f^2_{(1)}) , \tag{3.8}$$

$$\{\frac{\partial}{\partial t_1}[1+(1+\lambda)\hat{b}] + V_* \frac{\partial}{\partial y_0}\}f_{(1)} = \lambda(\lambda+1)\frac{a_i^2}{4}\Omega_i$$

$$\vec{\nabla}_{\perp 0}\cdot(-\frac{\partial f_{(1)}}{\partial y_0}\frac{\partial}{\partial x} + \frac{\partial f_{(1)}}{\partial x}\frac{\partial}{\partial y_0})\vec{\nabla}_{\perp 0} f_{(1)} , \tag{3.9}$$

where $V_* = -\kappa T_e/m_e\Omega_e$, $\hat{b} = -(1/2)a_i^2\Delta_\perp$, $\vec{\nabla}_{\perp 0} = (\partial/\partial x, \partial/\partial y_0)$, and $\lambda = T_e/T_i$.

From (3.8), $(\rho^{(2)} - \psi^{(2)})$ is found to be

$$\rho^{(2)} - \psi^{(2)} = \frac{1}{2}f^2_{(1)} + \sum (-i)\delta_k\{(f_+ e^{ik_\parallel z} + f_- e^{-ik_\parallel z})$$

$$\times e^{-i\omega t_1 + ik_y y_0} + c.c.\} , \tag{3.10}$$

where

$$\delta_k = \frac{\omega_* - \omega}{k_\parallel{}^2 D_{C\parallel}}$$

is the phase shift between $\tilde{n}$ and $\tilde{\phi}$ which is proportional to the linear growth rate. From (3.9) with use of (3.7), we obtain

$$\{(-i\omega)(1+(1+\lambda)\hat{b})+ik_yV_* +\lambda(\lambda+1)\frac{a_i{}^4}{4}\Omega_i(\frac{\partial^3 h}{\partial x^3} - \frac{\partial h}{\partial x}\Delta_\perp)(ik_y)\}\begin{pmatrix} f_+ \\ , \\ f_- \end{pmatrix}$$

$$+\lambda(\lambda+1)\frac{a_i{}^4}{4}\Omega_i\{\frac{\partial^3}{\partial x^3}\begin{pmatrix} g \\ , \\ g^* \end{pmatrix} - \frac{\partial}{\partial x}\begin{pmatrix} g \\ , \\ g^* \end{pmatrix}\Delta_\perp\}(ik_y)\begin{pmatrix} f_- \\ , \\ f_+ \end{pmatrix} = 0, \quad (3.11)$$

$\Delta_\perp = -k_y{}^2 + \partial^2/\partial x^2$, and g^* is the complex conjugate of g. The equation gives the dispersion for ω(real) which includes nonlinear frequency shift. If one neglects the small terms in (3.11), the linear dispersion relation will be recovered

$$\omega = \omega_*/[1 + (1+\lambda)b] \quad , \qquad (3.12)$$

where $b = (1/2)\ k_y{}^2 a_i{}^2$.

In the above, we have assumed that $f_{(1)}$ in (3.7) contains the terms up to the first harmonic. But the righthand side of (3.9) generates the second harmonic. It is easily seen that the second harmonic is smaller by the factor b than the first harmonic, and can be neglected.

[3] 3rd order calculation.

The third order equation of (3.3) is

$$L^{(0)}U^{(3)} + L^{(1)}U^{(2)} + L^{(2)}U^{(1)} = S^{(3)} \quad , \qquad (3.13)$$

of which the ion part is given by

$$\{[1+(1+\lambda)\hat{b}]\frac{\partial}{\partial t_2} + (-v_g[1+(1+\lambda)\hat{b}]+ V_* -(1+\lambda)a_i^2\frac{\partial^2}{\partial t_1 \partial y_0})\frac{\partial}{\partial y_1}$$

$$-[\kappa x V_* \frac{\partial}{\partial y_0} + (1+\lambda)\frac{\kappa a_i^2}{2}\frac{\partial^2}{\partial t_1 \partial x}] + \frac{3}{10}(\lambda+1)\nu_{ii}\hat{b}^2\} \quad f_{(1)}$$

$$= \frac{\partial}{\partial t_1}(1+\hat{b})[\psi^{(2)} - \rho^{(2)}] + S_i^{(3)} \quad , \qquad (3.14)$$

in which the expression of $S_i^{(3)}$ is given in the Appendix and $\rho^{(2)} - \psi^{(2)}$ is given by the (3.10). If we substitute (3.7) into (3.14), we shall obtain the equations for h, g, f_+ and f_-. They are

$$\frac{\partial h}{\partial t_2} = \hat{O}h - 2\Sigma \frac{\omega_* \delta_k}{\kappa}\frac{\partial}{\partial x}[|f_+|^2 + |f_-|^2] \quad , \qquad (3.15)$$

$$\frac{\partial g}{\partial t_2} = \hat{O}g - 2\Sigma \frac{\omega_* \delta_k}{\kappa}\frac{\partial}{\partial x}(f_+ f_-^*) \quad , \qquad (3.16)$$

$$(\frac{\partial}{\partial t_2} - \Delta v_g \frac{\partial}{\partial y_1})f_+ = [\gamma_k(1+\frac{1}{\kappa}\frac{\partial h}{\partial x}) + \hat{O} + i\delta\omega]f_+$$

$$+ \frac{1}{\kappa}\omega_* \delta_k \frac{\partial g}{\partial x} f_- \quad , \qquad (3.17)$$

$$(\frac{\partial}{\partial t_2} - \Delta v_g \frac{\partial}{\partial y_1})f_- = [\gamma_k(1+\frac{1}{\kappa}\frac{\partial h}{\partial x})] + \hat{O} + i\delta\omega']f_-$$

$$+ \frac{1}{\kappa}\omega_* \delta_k \frac{\partial g^*}{\partial x} f_+ \quad , \qquad (3.18)$$

where the summation Σ is taken over k_y and $k_{\parallel}$, and

$$\hat{O} \equiv D_{c\perp}(n=N_0)\Delta_\perp - \frac{3}{10}(\lambda+1)\nu_{ii}(\frac{1}{2}a_i^2\Delta_\perp)^2, \quad \Delta_\perp = \frac{\partial^2}{\partial x^2} - k_y^2 \quad ,$$

$$\gamma_k = \omega_* \delta_k \quad , \quad \Delta v_g = (1+\lambda)V_* k_y^2 a_i^2 .$$

In (3-15) and (3-16) we have chosen $v_g = v_{g0} \equiv V_*[1+(1+\lambda)\hat{b}]^{-1}$ so as to drop the derivative with respect to y_1, and in (3-17) and (3-18) we have set v_g equal to $v_{g0} + \Delta v_g$. On the right-hand side of (3.17) and (3.18), $\delta\omega$ and $\delta\omega'$ denote the nonlinear frequency shift,

$$\delta\omega = \kappa x \omega_* \{1 + O(b)\} ,$$

$$\delta\omega' = \kappa x \omega_* \{1 + O(b)\} .$$

The γ_k corresponds to the linear growth rate γ_L in (1.1) and the factor of $(1+\kappa^{-1}\ \partial h/\partial x)$ means the reduction of the growth rate due to the change in the gradient of the background density. Precisely speaking, δ_k must be determined by solving eq. (3.11) under appropriate boundary conditions. In another way, by inspection of (3.11) one may use throughout eqs. (3.15) ∿ (3.18) the relation

$$\delta_k \begin{pmatrix} f_+ \\ , \\ f_- \end{pmatrix} = \frac{(\lambda+1)\omega_*}{k_{\parallel}{}^2 D_{c\parallel}} \hat{b} \begin{pmatrix} f_+ \\ , \\ f_- \end{pmatrix} + \lambda(\lambda+1)\frac{\Omega_i}{k_{\parallel}{}^2 D_{c\parallel}} \frac{a_i{}^4 k_y}{4}\left(\frac{\partial h}{\partial x}\Delta_\perp - \frac{\partial^3 h}{\partial x^3}\right) \begin{pmatrix} f_+ \\ , \\ f_- \end{pmatrix}$$

$$+ \lambda(\lambda+1)\frac{\Omega_i}{k_{\parallel}{}^2 D_{c\parallel}} \frac{a_i{}^4 k_y}{4} \left\{\frac{\partial}{\partial x}\begin{pmatrix} g \\ , \\ g^* \end{pmatrix}\Delta_\perp - \frac{\partial^3}{\partial x^3}\begin{pmatrix} g \\ , \\ g^* \end{pmatrix}\right\} \begin{pmatrix} f_- \\ , \\ f_+ \end{pmatrix} .$$

Note that eqs. (3.15) ∿ (3.18) are of quasi-linear form. One sees that the most important nonlinear effect is the quasi-linear one, which is rigorously demonstrated under the present ordering.

We add a remark on the electron part of the third order equation of (3.13). That is

$$-D_{c\parallel}\frac{\partial^2}{\partial z^2}(\rho^{(3)} - \psi^{(3)}) + \left(\frac{\partial}{\partial t_1} + V_* \frac{\partial}{\partial y_0}\right)\rho^{(2)}$$

$$+ \left(\frac{\partial}{\partial t_1} - D_{c\perp}\Delta_\perp - \kappa x V_* \frac{\partial}{\partial y_0}\right) f_{(1)} + V_* \frac{\partial}{\partial y_0} \Sigma\ i[\delta_k\ f_{(1)}]$$

$$+ D_{c\parallel}\ \kappa x \frac{\partial^2}{\partial z^2}\left(-\frac{1}{2} f_{(1)}^2 + \Sigma\ i[\delta_k\ f_{(1)}]\right)$$

$$= \frac{T_e}{m_e \Omega_e}\left(\frac{\partial f_{(1)}}{\partial x}\frac{\partial}{\partial y_0} - \frac{\partial f_{(1)}}{\partial y_0}\frac{\partial}{\partial x}\right)\Sigma\ i[\delta_k\ f_{(1)}] + D_{c\parallel}\frac{\partial^2}{\partial z^2}\left(-f_{(1)}\rho^{(2)} + \frac{1}{3} f_{(1)}^3\right) , \qquad (3.19)$$

where we have used the relation (3.10) to eliminate $\psi^{(2)}$ and $-i\ \Sigma[\delta_k f_{(1)}]$ means the second term on the righthand side of (3.10). When we compare the zero and first harmonic components of this equation with eqs. (3.15) ∿ (3.18), we shall find the consistency conditions which include the plausible requirement for $\rho^{(3)} - \psi^{(3)}$. We may take the zero and first harmonic of $\rho^{(2)}$ be equal to zero, and also assume consistently the second harmonic of $\rho^{(3)} - \psi^{(3)}$ equal to zero. Therefore from the second harmonic component of the eq.(3.19), we obtain the equation for $\rho^{(2)}$

$$\left(\frac{\partial}{\partial t_1} + V_* \frac{\partial}{\partial y_0} + D_{C\parallel} \frac{\partial^2}{\partial z^2} Y\right)\rho^{(2)}$$

$$= D_{C\parallel}\ \kappa x \frac{\partial^2}{\partial z^2}\left(\frac{1}{2} f^2_{(1)}\right)_{2nd} + \frac{T_e}{m_e\Omega_e}\left[\left(\frac{\partial f_{(1)}}{\partial x}\frac{\partial}{\partial y_0} - \frac{\partial f_{(1)}}{\partial y_0}\frac{\partial}{\partial x}\right)\right.$$

$$\left.\times\ \Sigma\ i[\delta_k\ f_{(1)}]\right]_{2nd} + D_{C\parallel}\frac{\partial^2}{\partial z^2} Y(f^2_{(1)})_{2nd}\ , \qquad (3.20)$$

where Y is the zero harmonic part of $f_{(1)}$ and the subscript "2nd" on the righthand side means to take only the second harmonic parts. Hence, remembering eq.(3.10), one will be able to determine $\rho^{(2)}$ and $\psi^{(2)}$. Exactly speaking, the expressions of the frequency shift $\delta\omega$ and $\delta\omega'$ which appear on the righthand sides of eqs. (3. 17) and (3.18), include $\rho^{(2)}$ and $\psi^{(2)}$. We can now express them in terms of $f_{(1)}$.

§4. Model Equations

Nonlinear behaviour of the system under consideration may be described substantially by the following set of the "model" equations for h and f (f stands for $f_{\pm}$)

$$\frac{\partial h}{\partial t_2} = D_{c\perp}\frac{\partial^2 h}{\partial x^2} - (\frac{3}{40}(\lambda+1)\nu_{ii}\ a_i{}^4)\frac{\partial^4}{\partial x^4}h + 2\frac{\gamma_L}{|\kappa|k_y{}^2}\frac{\partial}{\partial x}(f\Delta_\perp f) \quad , \tag{4.1}$$

$$\frac{\partial f}{\partial t_2} = \frac{\gamma_L}{k_y{}^2}(1 - \frac{1}{|\kappa|}\frac{\partial h}{\partial x})(-\Delta_\perp f) - \frac{3}{40}(\lambda+1)\nu_{ii}\ a_i{}^4\ \Delta_\perp{}^2\ f \ , \tag{4.2}$$

where $\Delta_\perp = \partial^2/\partial x^2 - k_y{}^2$, and

$$\gamma_L = \omega_*(1+\lambda)(1/2)(k_y{}^2 a_i{}^2)/k_\parallel{}^2 D_{c\parallel} = (1/4)\kappa^2(k_y{}^2/k_\parallel{}^2)k_y{}^2 a_i{}^2 D_{c\perp} \ .$$

In the above, we have retained only the important nonlinear interaction terms which appear in (3.15) $\sim$ (3.18).

We rewrite the eqs.(4.1) and (4.2) in dimensionless form as

$$\frac{\partial H}{\partial \tau} = (\delta\frac{\partial^2}{\partial \xi^2} - \eta\frac{\partial^4}{\partial \xi^4})H + 2\ \frac{\partial}{\partial \xi}\ (F\Delta F)\ ,\ \Delta \equiv \frac{\partial^2}{\partial \xi^2} - k^2 \ , \tag{4.3}$$

$$\frac{\partial F}{\partial \tau} = -\ (1 - \frac{\partial H}{\partial \xi})\Delta F - \ \eta\ \Delta^2 F \ , \tag{4.4}$$

where

$$H = (\pi/|\kappa|\ell)h, \quad F = (\pi/|\kappa|\ell)f \ , \quad k = k_y\ell/\pi$$

$$\tau = \gamma_L\ t_2(\pi/\ell k_y)^2 \ , \quad \xi = (\pi/\ell)x \equiv k_x x \ ,$$

$$\delta = \frac{k_y^2\ D_{c\perp}}{\gamma_L} = 4\ (\frac{k_\parallel{}^2 k_x{}^2}{\kappa^2 k_y{}^2})(\frac{T_i}{T_e})\ \frac{1}{k_x{}^2 a_i{}^2} \ ,$$

$$\eta = \frac{3}{40}(\lambda+1)\nu_{ii}(k_x a_i)^4\ \frac{k_y{}^2}{\gamma_L k_x{}^2} = \frac{3\sqrt{2}}{10}(\frac{k_\parallel{}^2 k_x{}^2}{\kappa^2 k_y{}^2})(\frac{T_e}{T_i}\ \frac{m_i}{m_e})^{1/2}$$

By the coupled eqs(4.3) and (4.4), we can treat such problems as one-mode saturation near marginal stability, transition to higher instabilities and particle transport, which will be reported elsewhere.

References

1) K. Kawahata and M. Fujiwara, J. Phys. Soc. Japan 14 (1976) 1150.

2) There has been the extensive literature for linear analysis of this instability. See, for example, A.A. Rukhadze and V.P. Silin, Soviet Phys. Uspekhi, 11 (1969) 659.

3) T.H. Stix, Phys. Fluids 12 (1969) 627.

4) F.L. Hinton and C.W. Horton, Jr., Phys. Fluids 14 (1971) 116.

5) D.A. Monticello and A. Simon, Phys. Fluids 17 (1974) 791.

6) T. Sato, to be published in J. Phys. Soc. Japan 41,No.1(1976).

7) S.I. Braginskii, Review of Plasma Physics, Vol 1. (Consultants Bureau, New York, 1965) p205.

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Appendix

i) Expressions of L and S.

$$L = \begin{bmatrix} \frac{\partial}{\partial t} - D_{c\parallel}(x)\,\frac{\partial^2}{\partial z^2} - D_{c\perp}(x)\frac{1}{N^2}\,\Delta_\perp (N^2 \quad , & V_{*e}\,\frac{\partial}{\partial y} + D_{c\parallel}(x)\,\frac{\partial^2}{\partial z^2} \\ \frac{\partial}{\partial t} - \frac{a_i^{\,2}}{2}(\kappa\frac{\partial}{\partial x} + \Delta_\perp)\frac{\partial}{\partial t} + d_\eta - D_{c\perp}(x)\,\frac{1}{N^2}\,\Delta_\perp (N^2 \quad , & V_{*e}\,\frac{\partial}{\partial y} - \frac{\lambda a_i^{\,2}}{2}(\kappa\frac{\partial}{\partial x} + \Delta_\perp)\frac{\partial}{\partial t} + \frac{a_i^{\,2}}{2}\,\kappa\, V_{*e}\frac{\partial^2}{\partial x\partial y} + \lambda\, d_\eta \end{bmatrix} \qquad \text{(A-1)}$$

where $D_{c\parallel}(x)$ and $D_{c\perp}(x)$ depend on x through $n=N(x)\equiv N_0(1+\kappa x)$, $\lambda=T_e/T_i$, and

$$V_{*e} = \frac{V_*}{1+\kappa x} = -\,\frac{T_e}{m_e\Omega_e}\,\frac{\kappa}{1+\kappa x}\ , \qquad d_\eta = \frac{3}{10}\,\nu_{ii}(\frac{1}{2}\,a_i^{\,2}\,\Delta_\perp)^2\ . \qquad \text{(A-2)}$$

The term which includes $\nabla\Pi_s$ (see eq.(2.12)) is approximated as

$$\mathrm{div}\,\{\frac{1}{m_i\Omega_i}\;\frac{\vec{B}}{B}\times\vec{\nabla}\,\Pi_s\} \;\cong\; N\, d_\eta(\lambda\,\frac{e\tilde{\phi}}{T_e} + \frac{\tilde{n}}{N})\ .$$

Also

$$S = \begin{bmatrix} S_e\,, \\ \\ S_i \end{bmatrix}\ . \qquad \text{(A-3)}$$

With use of $\vec{b} = \vec{B}/B$, $\rho=\tilde{n}/N$ and $\psi=e\tilde{\phi}/T_e$,

$$S_e = -\,\rho\, V_{*e}\,\frac{\partial\psi}{\partial y} - \frac{T_e}{m_e\Omega_e}\,\vec{b}\times\vec{\nabla}\psi\cdot\vec{\nabla}\rho + D_{c\parallel}(x)\frac{\partial^2}{\partial z^2}[\log(1+\rho)-\rho]$$
$$+\, D_{c\perp}(x)\,\frac{1}{2N^2}\,\Delta_\perp(N^2\rho^2)\quad , \qquad \text{(A-4)}$$

$$S_i = -\rho V_{*e}\frac{\partial\psi}{\partial y} - \frac{T_e}{m_e\Omega_e}\vec{b}\times\vec{\nabla}\psi\cdot\vec{\nabla}\rho + D_{c\perp}(x)\frac{1}{2N^2}\Delta_\perp(N^2\rho^2)$$

$$+ \frac{1}{2N}\lambda a_i^2(\vec{\nabla}_\perp n\cdot\vec{\nabla}_\perp + n\Delta_\perp)\frac{\partial\psi}{\partial t}$$

$$+ \frac{1}{2N}\lambda a_i^2(\frac{T_e}{m_e\Omega_e})\,\mathrm{div}[n(\vec{b}\times\vec{\nabla}\psi)\cdot\vec{\nabla}_\perp\psi]$$

$$+ \frac{1}{2N} a_i^2(\frac{T_e}{m_e\Omega_e})\{\vec{b}\times\vec{\nabla}n\cdot\vec{\nabla}(\Delta_\perp\psi) + \vec{\nabla}_\perp(\frac{\partial\psi}{\partial y})\cdot\vec{\nabla}_\perp(\frac{\partial n}{\partial x}) - \vec{\nabla}_\perp(\frac{\partial\psi}{\partial x})\cdot\vec{\nabla}_\perp(\frac{\partial n}{\partial y})\}$$

$$+ \frac{1}{2N} a_i^2\{\vec{\nabla}_\perp n\cdot\frac{\partial}{\partial t}\vec{\nabla}_\perp \log n + n\frac{\partial}{\partial t}\Delta_\perp \log n\}$$

$$+ \frac{1}{2N} a_i^2(\frac{T_e}{m_e\Omega_e})\,\mathrm{div}[n(\vec{b}\times\vec{\nabla}\psi\cdot\vec{\nabla})\vec{\nabla}_\perp \log n]$$

$$+ \frac{1}{4N} a_i^2(\frac{Te}{m_e\Omega_e})\,\mathrm{div}[(n\,\Delta_\perp + \vec{\nabla}_\perp n\cdot\vec{\nabla}_\perp)\vec{b}\times\vec{\nabla}\psi - \vec{b}\times\vec{\nabla}n\cdot\vec{\nabla}\,\vec{\nabla}_\perp\psi], \qquad (A-5)$$

where $n=N(1+\rho)$ is to be substituted.

ii) Expression of $S_i^{(3)}$.

$$S_i^{(3)} = \frac{T_e}{m_e\Omega_e}\{\kappa f\frac{\partial f}{\partial y_0} - (\vec{b}\times\vec{\nabla}f\cdot\vec{\nabla})(\rho^{(2)} - \psi^{(2)})\}$$

$$+ \frac{1}{2}\lambda a_i^2[f\,\Delta_\perp\frac{\partial f}{\partial t_1} + \vec{\nabla}_\perp f\cdot\vec{\nabla}_\perp\frac{\partial f}{\partial t_1}] - \frac{1}{2}a_i^2\frac{\partial f}{\partial t_1}\Delta_\perp f$$

$$+ \frac{a_i^2}{2}\frac{T_e}{m_e\Omega_e}\{(\lambda+1)\kappa(\vec{b}\times\vec{\nabla}f\cdot\vec{\nabla})\frac{\partial f}{\partial x}$$

$$+(\lambda+1)[(\vec{b}\times\vec{\nabla}f\cdot\vec{\nabla})\Delta_\perp\rho^{(2)} + (\vec{b}\times\vec{\nabla}\rho^{(2)}\cdot\vec{\nabla})\Delta_\perp f$$

$$-(\lambda+2)f(\vec{b}\times\vec{\nabla}f\cdot\vec{\nabla})\Delta_\perp f - \lambda(\vec{\nabla}_\perp f)\cdot[(\vec{b}\times\vec{\nabla}f\cdot\vec{\nabla})\vec{\nabla}_\perp f]$$

$$+ (\lambda+1)[\vec{b}\times\vec{\nabla}(\psi^{(2)} - \rho^{(2)} + \frac{1}{2}f^2)\cdot\vec{\nabla}]\,\Delta_\perp f$$

$$+ \lambda(\vec{b}\times\vec{\nabla}f\cdot\vec{\nabla})\Delta_\perp(\psi^{(2)} - \rho^{(2)} + \frac{1}{2}f^2)$$

$$+ [(\vec{\nabla}_\perp\frac{\partial f}{\partial y_0})\cdot\vec{\nabla}_\perp\frac{\partial}{\partial x} - (\vec{\nabla}_\perp\frac{\partial f}{\partial x})\cdot\vec{\nabla}_\perp\frac{\partial}{\partial y_0}](\rho^{(2)} - \psi^{(2)} - \frac{1}{2}f^2)\}. \qquad (A-6)$$

In the above, the abbreviation $f\equiv f_{(1)}$ is used.