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Contribution of the Second Order Terms to the Nonlinear Shallow Water Waves

Yoshi H. Ichikawa and T. Aoyama*

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* Department of Physics and Atomic Energy Research Institute, College of Science and Technology, Nihon University, Chiyoda-ku, Tokyo

Abstract

Contribution of the second order terms in the reductive perturbation theory has been investigated for the nonlinear shallow water waves. The fundamental equations are reduced to a coupled set of the Korteweg-de Vries equation for the first order horizontal velocity and a linear inhomogeneous equation for the second order arbitrary function. Structure of the coupled set of equations turns out to be the same as in the case of nonlinear ion acoustic wave. A steady state solution of the coupled set of equations has been examined in comparison with Laiton's analysis of the second order contribution of the Friedrich's expansion for the nonlinear shallow water waves.

§1. Introduction

Basing their analysis of large time asymptotic behaviour of the linear wave and of a nonlinear steady state solution, Gardner and Morikawa[1)] have presented a systematic derivation of the Korteweg-de Vries equation for the nonlinear shallow water wave in the long wave length limit as well as for the magnetohydrodynamic wave propagating perpendicular to an external magnetic field.

The Gardner-Morikawa transformation has been extended by Taniuti and Wei[2)] as the reductive perturbation theory for a system of nonlinear partial differential equations, which describes propagation of weakly dispersive wave. The reductive perturbation theory has been formulated also for strongly dispersive nonlinear systems[3)], giving rise to the nonlinear Schrödinger equation.

Concerning with the problem of shallow water wave, Keller[4)] has obtained steady solitary wave and cnoidal wave solutions as the first order approximation in the Friedrich's expansion[5)], which is based on the ordering in terms of the relative size of the water depth and wave length. Their solutions agree with the solutions of the Korteweg-de Vries equation. Later, Laiton[6)] have examined contribution of the second order terms in the Friedrich's expansion to the steady nonlinear shallow water wave propagation.

Now, we have investigated the higher order contribution of the reductive perturbation theory in a case of the weakly dispersive ion wave[7)]. Here, applying the same approach, we investigate contribution of the second order terms to the finite amplitude shallow water wave. In the next section, we derive a coupled set of the first order Korteweg-de Vries equation and the second order equation. In the third section, we examine the steady state solution in some details. We discuss the present results in comparison with Laiton's solution in the last section.

§2. The second order approximation to the nonlinear shallow water wave

Applying the reductive perturbation theory to a system illustrated in Fig.1, we investigate two dimensional irrotational motion of incompressible nonviscous fluid. We have the following set of fundamental equations,

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0 , \tag{1-a}$$

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = - \frac{1}{\rho} \frac{\partial}{\partial x} P , \tag{1-b}$$

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v = -g - \frac{1}{\rho} \frac{\partial}{\partial y} P , \tag{1-c}$$

$$\frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u = 0 . \tag{1-d}$$

As for the boundary conditions, we have

$$\frac{\partial}{\partial t} \eta + u(x,\eta,t) \frac{\partial}{\partial x} \eta - v(x,\eta,t) = 0 . \tag{1-e}$$

and

$$P(x, \eta, t) = 0 , \tag{1-f}$$

at the free surface $y=\eta(x,t)$, and we have

$$v(x,\ 0,\ t) = 0 \quad , \tag{1-g}$$

at the bottom y=0. In the above set of equations, ρ is the fluid density, g is the gravitational acceleration, u and v are the velocity component in the horizontal and the vertical direction, respectively. P is the static pressure.

As is well known, linearization of the above set of equations leads to the dispersion relation

$$\omega^2 = gk \tanh(hk) \tag{2}$$

which is reduced to

$$\omega = (gh)^{1/2}\, k\, \{1 - \frac{1}{6}(hk)^2 + \frac{19}{360}(hk)^4 - \cdots\cdots\} \tag{3}$$

in the long wave length limit hk<<1. Hence, ordering the wave number k to be small as the order of $\varepsilon^{1/2}$, we introduce stretched space-time variables $(\xi,\ y,\ \tau)$ defined as

$$\xi = \varepsilon^{1/2}(x-st), \quad y=y, \quad \tau = \varepsilon^{3/2} t \quad , \tag{4}$$

where the velocity s should be determined through perturbational analysis. The basic equations (1-a)~(1-g) are transformed into

$$\varepsilon^{1/2} \frac{\partial}{\partial\xi} u + \frac{\partial}{\partial y} v = 0 \quad , \tag{5-a}$$

$$\varepsilon^{3/2} \frac{\partial}{\partial\tau} u + \varepsilon^{1/2}(-s \frac{\partial}{\partial\xi} u + u \frac{\partial}{\partial\xi} u) + v \frac{\partial}{\partial y}u = -\varepsilon^{1/2}\frac{1}{\rho}\frac{\partial}{\partial\xi}P \quad , \tag{5-b}$$

$$\varepsilon^{3/2} \frac{\partial}{\partial\tau}v + \varepsilon^{1/2}(-s\frac{\partial}{\partial\xi}v + u\frac{\partial}{\partial\xi}v) + v\frac{\partial}{\partial y}v = -g - \frac{1}{\rho}\frac{\partial}{\partial y} P \quad , \tag{5-c}$$

$$\varepsilon^{1/2} \frac{\partial}{\partial \xi} v - \frac{\partial}{\partial y} u = 0 \quad , \tag{5-d}$$

$$\varepsilon^{3/2} \frac{\partial}{\partial \tau}\eta + \varepsilon^{1/2}(-s\frac{\partial}{\partial \xi}\eta + u(\xi,\eta,\tau)\frac{\partial}{\partial \xi}\eta) - v(\xi,\eta,\tau) = 0 \quad , \tag{5-e}$$

$$P(\xi,\eta,\tau) = 0 \tag{5-f}$$

$$v(\xi, 0, \tau) = 0 \tag{5-g}$$

Observing eqs.(5-a)~(5-d), we introduce the following perturbation expansions,

$$u(\xi, y, \tau) = \sum_{n\geq 1} \varepsilon^{n} u^{(n)}(\xi, y, \tau) \quad , \tag{6-a}$$

$$v(\xi, y, \tau) = \varepsilon^{1/2} \sum_{n\geq 1} \varepsilon^{n} v^{(n)}(\xi, y, \tau) \quad , \tag{6-b}$$

$$P(\xi, y, \tau) = P^{(0)}(y) + \sum_{n\geq 1} \varepsilon^{n} P^{(n)}(\xi, y, \tau) \quad , \tag{6-c}$$

$$\eta(\xi,\tau) = h + \sum_{n\geq 1} \varepsilon^{n}\eta^{(n)}(\xi,\tau) \quad . \tag{6-d}$$

With regards to quantities evaluated at the free surface boundary, we have to expand them as follows,

$$u(\xi,\eta,\tau) = \varepsilon u^{(1)}(\xi,h,\tau) + \varepsilon^{2}\{u^{(2)}(\xi,h,\tau) + u_{y}^{(1)}(\xi,h,\tau)\,\eta^{(1)}(\xi,\tau)\} + \cdots\cdots \quad , \tag{7-a}$$

$$v(\xi, \eta, \tau) = \varepsilon v^{(1)}(\xi, h, \tau) + \varepsilon^{2}\{v^{(2)}(\xi, h, \tau) + v_{y}^{(1)}(\xi, h, \tau)\eta^{(1)}(\xi,\tau)\} + \cdots\cdot \quad , \tag{7-b}$$

$$P(\xi, \eta, \tau) = P^{(0)}(h) + \varepsilon\{P^{(1)}(\xi, h,\tau) + P_{y}^{(0)}(h)\eta^{(1)}(\xi,\tau)\} + \varepsilon^{2}\{P^{(2)}(\xi, h, \tau) + P_{y}^{(1)}(\xi, h,\tau)\eta^{(1)}(\xi,\tau) + \frac{1}{2} P_{yy}^{(0)}(h)(\eta^{(1)}(\xi,\tau))^{2} + P_{y}^{(0)}(h)\eta^{(2)}(\xi,\tau)\} + \cdots\cdots \quad , \tag{7-c}$$

in which suffix y stands for the partial differential with respect to y.

Now, substituting the perturbation expansions of eqs.(6-a) ∿ (6-d) and (7-a) ∿ (7-c) into the transformed basic equations (5-a) ∿ (5-g), we obtain in the lowest order of ε,

$$\frac{\partial}{\partial \xi} P^{(0)} = 0 \quad , \tag{8-a}$$

$$- g - \frac{1}{\rho} \frac{\partial}{\partial y} P^{(0)} = 0 \quad , \tag{8-b}$$

$$P^{(0)}(\xi, h, \tau) = 0 \quad . \tag{8-c}$$

Eqs.(8-a) ∿ (8-c) determine the unperturbed static pressure as

$$P^{(0)}(\xi, y, \tau) = \rho g (h-y) \quad . \tag{9}$$

We have then, in the first order of ε,

$$\frac{\partial}{\partial \xi} u^{(1)} + \frac{\partial}{\partial y} v^{(1)} = 0 \quad , \tag{10-a}$$

$$s \frac{\partial}{\partial \xi} u^{(1)} = \frac{1}{\rho} \frac{\partial}{\partial \xi} P^{(1)} \quad , \tag{10-b}$$

$$\frac{\partial}{\partial y} P^{(1)} = 0 \quad , \tag{10-c}$$

$$\frac{\partial}{\partial y} u^{(1)} = 0 \quad , \tag{10-d}$$

$$v^{(1)}(\xi, h, \tau) + s \frac{\partial}{\partial \xi} \eta^{(1)}(\xi, \tau) = 0 \quad , \tag{10-e}$$

$$P^{(1)}(\xi, h, \tau) - \rho g \, \eta^{(1)}(\xi, \tau) = 0 \quad , \tag{10-f}$$

$$v^{(1)}(\xi, 0, \tau) = 0 \quad . \tag{10-g}$$

These equations determine the velocity s as

$$s = (gh)^{1/2} \quad , \tag{11}$$

which is in accord with the leading term of the linear dispersion relation (3). The first order perturbed quantities $v^{(1)}$, $p^{(1)}$ and $\eta^{(1)}$ are expressed in terms of the horizontal velocity component $u^{(1)}$ as

$$v^{(1)}(\xi, y, \tau) = -yu_{\xi}^{(1)}(\xi,\tau) \quad , \tag{12-a}$$

$$p^{(1)}(\xi, y, \tau) = \rho\, s\, u^{(1)}(\xi,\tau) \quad , \tag{12-b}$$

$$\eta^{(1)}(\xi,\tau) = (s/g)\, u^{(1)}(\xi,\tau) \quad , \tag{12-c}$$

where the suffix ξ expresses the partial differential with respect ξ. Explicit space-time variation of the first order horizontal velocity component $u^{(1)}(\xi,\tau)$ is not determined at this stage, but will be specified through the Korteweg-de Vries equation which is reduced as a compatibility condition of the second order equations.

In the second order in ε, we have

$$\frac{\partial}{\partial \xi} u^{(2)} + \frac{\partial}{\partial y} v^{(2)} = 0 \quad , \tag{13-a}$$

$$\frac{\partial}{\partial \tau}u^{(1)} - s\frac{\partial}{\partial \xi}u^{(2)} + u^{(1)}\frac{\partial}{\partial \xi}u^{(1)} = -\frac{1}{\rho}\frac{\partial}{\partial \xi} p^{(2)} \quad , \tag{13-b}$$

$$s\frac{\partial}{\partial \xi}v^{(1)} = \frac{1}{\rho}\frac{\partial}{\partial y} p^{(2)} \quad , \tag{13-c}$$

$$\frac{\partial}{\partial \xi}v^{(1)} - \frac{\partial}{\partial y} u^{(2)} = 0 \quad , \tag{13-d}$$

$$\frac{\partial}{\partial\tau}\eta^{(1)} - s\frac{\partial}{\partial\xi}\eta^{(2)} + u^{(1)}\frac{\partial}{\partial\xi}\eta^{(1)} - v^{(2)} + \eta^{(1)}\frac{\partial}{\partial\xi}u^{(1)} = 0, \qquad (13\text{-e})$$

$$p^{(2)}(\xi, h, \tau) - \rho g \eta^{(2)}(\xi,\tau) = 0 \quad , \qquad (13\text{-f})$$

$$v^{(2)}(\xi, 0, \tau) = 0 \quad . \qquad (13\text{-g})$$

We remark here that $u^{(1)}$ and $v^{(2)}$ of eq.(13-e) are evaluated at y=h, and that the last term of eq.(13-e) results from the term of $v_y^{(1)}(\xi, h,\tau)\,\eta^{(1)}$ of eq.(7-b). Integrating (13-d) with (12-a), we get

$$u^{(2)}(\xi, y,\tau) = -\frac{1}{2}y^2 u^{(1)}_{\xi\xi} + F(\xi,\tau) \quad , \qquad (14)$$

where $F(\xi,\tau)$ is an arbitrary function introduced through the integration with respect to y. Substitution of (14) into (13-a) yields

$$v^{(2)}(\xi, y, \tau) = \frac{1}{6}y^3 u^{(1)}_{\xi\xi\xi} - y F_\xi(\xi,\tau) \quad , \qquad (15)$$

in which the boundary condition (13-g) has been invoked to determine an integration constant with respect to y. Then, eq.(13-c) with (13-f) leads to

$$p^{(2)}(\xi, y, \tau) = \frac{1}{2}(h^2 - y^2)\rho s\, u^{(1)}_{\xi\xi} + \rho g\eta^{(2)}(\xi,\tau) \quad . \qquad (16)$$

Eq.(13-b) with (14) and (16) takes a form of

$$\frac{\partial}{\partial\tau}u^{(1)} + \frac{1}{2}sh^2\frac{\partial^3}{\partial\xi^3}u^{(1)} + u^{(1)}\frac{\partial}{\partial\xi}u^{(1)} = sF_\xi - g\,\eta^{(2)}_\xi \quad , \qquad (17)$$

while eq.(13-e) is reduced to

$$\frac{\partial}{\partial\tau}u^{(1)} - \frac{1}{6}sh^2\frac{\partial^3}{\partial\xi^3}u^{(1)} + 2u^{(1)}\frac{\partial}{\partial\xi}u^{(1)} = -sF_\xi + g\eta_\xi^{(2)} \quad . \qquad (18)$$

Therefore, we obtain the Korteweg-de Vries equation

$$\frac{\partial}{\partial \tau} u^{(1)} + \frac{1}{6} sh^2 \frac{\partial^3}{\partial \xi^3} u^{(1)} + \frac{3}{2} u^{(1)} \frac{\partial}{\partial \xi} u^{(1)} = 0 \quad , \tag{19}$$

as a compatibility condition of (17) and (18).

In order to determine the second order quantities $u^{(2)}$, $v^{(2)}$ and $\eta^{(2)}$, we have to determine the unknown function $F(\xi,\tau)$. Going up to the third order in ε, we get

$$\frac{\partial}{\partial \xi} u^{(3)} + \frac{\partial}{\partial y} v^{(3)} = 0 \quad , \tag{20-a}$$

$$\frac{\partial}{\partial \tau} u^{(2)} - s\frac{\partial}{\partial \xi} u^{(3)} + u^{(2)} \frac{\partial}{\partial \xi} u^{(1)} + u^{(1)} \frac{\partial}{\partial \xi} u^{(2)}$$
$$+ v^{(1)} \frac{\partial}{\partial \xi} u^{(2)} = - \frac{1}{\rho} \frac{\partial}{\partial \xi} P^{(3)} \quad , \tag{20-b}$$

$$\frac{\partial}{\partial \tau} v^{(1)} - s\frac{\partial}{\partial \xi} v^{(2)} + u^{(1)} \frac{\partial}{\partial \xi} v^{(1)} + v^{(1)} \frac{\partial}{\partial \xi} v^{(1)} = - \frac{1}{\rho} \frac{\partial}{\partial y} P^{(3)} \, , \tag{20-c}$$

$$\frac{\partial}{\partial \xi} v^{(2)} - \frac{\partial}{\partial y} u^{(3)} = 0 \quad . \tag{20-d}$$

Concerning with the free surface boundary condition, eq.(5-e) is reduced to

$$\frac{\partial}{\partial \tau} \eta^{(2)} - s\frac{\partial}{\partial \xi} \eta^{(3)} + u^{(1)} \frac{\partial}{\partial \xi} \eta^{(2)} + u^{(2)} \frac{\partial}{\partial \xi} \eta^{(1)}$$
$$- v^{(3)} - v_y^{(2)}(\xi, h, \tau)\, \eta^{(1)} - v_y^{(1)}(\xi, h, \tau)\, \eta^{(2)} = 0 \quad , \tag{20-e}$$

where $u^{(1)}$, $u^{(2)}$, $v^{(3)}$, $v_y^{(1)}$ and $v_y^{(2)}$ are evaluated at $y=h$.

Eq.(5-f) is reduced to

$$P^{(3)}(\xi, h, \tau) + P_y^{(2)}(\xi, h, \tau)\eta^{(1)}(\xi,\tau) + P_y^{(0)}(h)\eta^{(3)}(\xi,\tau) = 0 \quad . \tag{20-f}$$

At the bottom, we have

$$v^{(3)}(\xi,\ 0,\ \tau) = 0 \quad . \tag{20-g}$$

Integrating (20-d) with (15) with respect to y, we get

$$u^{(3)}(\xi,\ y,\ \tau) = \frac{1}{24}\, y^4\ u^{(1)}_{\xi\xi\xi} - \frac{1}{2}\, y^2 F_{\xi\xi} + G(\xi,\tau) \ , \tag{21-a}$$

where $G(\xi,\tau)$ is a third order arbitrary function. Substituting (21-a) into (20-a), we obtain

$$v^{(3)}(\xi,\ y,\tau) = -\frac{1}{120}\, y^5 \frac{\partial^5}{\partial\xi^5}\, u^{(1)} + \frac{1}{6}\, y^3 \frac{\partial^3}{\partial\xi^3}\, F - y\, \frac{\partial}{\partial\xi}\, G \ , \tag{21-b}$$

where the integration constant is fixed to be zero by the boundary condition (20.g). The third order perturbed pressure $P^{(3)}(\xi,\ y,\tau)$ is obtained by integration of (20-c) with (20.f) as follows,

$$P^{(3)}(\xi,\ y,\tau) = \frac{1}{2}\,\rho(y^2-h^2)u^{(1)}_{\xi\tau} + \rho\ s[\frac{1}{24}(y^4-h^4)\cdot$$

$$u^{(1)}_{\xi\xi\xi\xi} + \frac{1}{2}(h^2-y^2)F_{\xi\xi}] + \frac{1}{2}\,\rho(y^2-h^2)\ u^{(1)}u^{(1)}_{\xi}$$

$$+ \frac{1}{2}\rho(h^2-y^2)(u^{(1)}_{\xi})^2 + \rho sh\ \eta^{(1)}u^{(1)}_{\xi\xi} + \rho\, g\ \eta^{(3)} \ . \tag{21-c}$$

Then, eq.(20-b) with eqs.(14), (21-a) and (21-c) is reduced to

$$\frac{\partial}{\partial\tau}\, F + \frac{1}{2}\, sh^2\, \frac{\partial^3}{\partial\xi^3}\, F + \frac{\partial}{\partial\xi}(u^{(1)}F) = s\ G_{\xi} - g\ \eta_{\xi}^{(3)}$$

$$+ \frac{1}{24}sh^4\, \frac{\partial^5}{\partial\xi^5}\, u^{(1)} + \frac{1}{2}\, h^2\ u^{(1)}_{\xi\xi\tau} - \frac{1}{2}h^2(u^{(1)}_{\xi}{}^2)_{\xi}$$

$$- \frac{1}{2}\, h^2\, \frac{\partial}{\partial\xi}\, (u^{(1)}\ u^{(1)}_{\xi\xi}) \ , \tag{22-a}$$

where all of the terms depending on y are cancelled out. Eq.(20.e) gives rise to

$$\frac{\partial}{\partial\tau} F - \frac{1}{6} sh^2 \frac{\partial^3}{\partial\xi^3} F + 2 \frac{\partial}{\partial\xi}(u^{(1)}F) = - sG_{\xi} + g\eta_{\xi}^{(3)}$$
$$- \frac{1}{4s} \frac{\partial}{\partial\tau}(u^{(1)})^2 + \frac{1}{3}h^2 u^{(1)}_{\xi\xi\tau} - \frac{1}{120} sh^4 \frac{\partial^5}{\partial\xi^5} u^{(1)}$$
$$+ \frac{5}{6}h^2 u^{(1)} u^{(1)}_{\xi\xi\xi} + \frac{5}{6}h^2 u^{(1)}_{\xi} u^{(1)}_{\xi\xi} - \frac{3}{4s} u^{(1)^2} u^{(1)}_{\xi} . \quad (22\text{-b})$$

As the Korteweg-de Vries equation (19) has been derived by eliminating $(s F_{\xi} - g\eta_{\xi}^{(2)})$ from eqs.(17) and (18), elimination of $(s G_{\xi} - g \eta_{\xi}^{(3)})$ from eqs.(22-a) and (22-b) yields an equation for the unknown function $F(\xi,\tau)$ as

$$\frac{\partial}{\partial\tau} F + \frac{1}{6} sh^2 \frac{\partial^3}{\partial\xi^3} F + \frac{3}{2} \frac{\partial}{\partial\xi} (u^{(1)}F) = S(u^{(1)}), \quad (23\text{-a})$$

where the source term $S(u^{(1)})$ is defined as

$$S(u^{(1)}) = - \frac{19}{360} sh^4 \frac{\partial^5}{\partial\xi^5} u^{(1)} - \frac{53}{48} h^2 \frac{\partial}{\partial\xi}(\frac{\partial}{\partial\xi} u^{(1)})^2$$
$$- \frac{5}{12} h^2 u^{(1)} \frac{\partial^3}{\partial\xi^3} u^{(1)} . \quad (23\text{-b})$$

The first term of eq.(23-b) represents the third term of the linear dispersion term (3), while other two terms describe the wave-wave interaction between the nonlinear waves $u^{(1)}$, which is determined by the Korteweg-de Vries equation (19). Thus, the source term $S(u^{(1)})$ describes contributions of the higher order nonlinear effects in competition with the higher order dispersion effect. Solving eq.(23-a), we can determine the second order velocity components $u^{(2)}(\xi, y,\tau)$ and $v^{(2)}(\xi, y,\tau)$ through eqs. (14) and (15), and the second order water elevation $\eta^{(2)}(\xi,\tau)$ from eq.(17).

§3. A steady state solution

Although the coupled set of equations (19) and (23-a) with (23-b) has a full of variety of solution, we examine its steady state solution which depends on a translational variable

$$X = (sh^2)^{-1/2} (\xi - \lambda\tau) . \tag{24}$$

Here, λ is a parameter to be determined from the solution. Restricting our interest to the soliton solution with the vanishing derivatives at infinity, we can reduce the coupled set of eqs.(19) and (23-a) with (23-b) into the following set of equations,

$$\frac{d^2}{dx^2} u^{(1)} + \frac{3}{2} (3u^{(1)} - 4\lambda) u^{(1)} = 0 , \tag{25-a}$$

$$\frac{d^2}{dx^2} F + 3(3u^{(1)} - 2\lambda)F = - s^{-1}\{\frac{19}{60} \frac{d^4}{dx^4} u^{(1)} + \frac{5}{2}u^{(1)} \frac{d^2}{dx^2} u^{(1)} + \frac{43}{8} (\frac{d}{dx} u^{(1)})^2\} . \tag{25-b}$$

Now, as is well known, eq.(26-a) has a steady soliton solution

$$u^{(1)} = U \operatorname{sech}^2(DX) , \tag{26-a}$$

with

$$D = (3U/4)^{1/2} , \tag{26-b}$$

and

$$\lambda = \frac{1}{2} U . \tag{26-c}$$

Substitution of the one soliton solution (26-a)∿(26-c) into eq. (25-b) leads to a linear inhomogeneous second order differential equation for F(X).

Introducing a new variable

$$\mu = \tanh(DX) \quad , \tag{27}$$

we can transform eq.(25-b) as

$$\frac{d^2}{d\mu^2} F - 2 \frac{\mu}{1-\mu^2} \frac{d}{d\mu} F + \frac{12}{(1-\mu^2)^2} \left(\frac{2}{3} - \mu^2\right) F =$$

$$\frac{32}{9} \frac{D^4}{s} \left\{- \frac{19}{10} \frac{1}{1-\mu^2} - \frac{3}{2} + 4(1 - \mu^2)\right\} . \tag{28}$$

Since the associated homogeneous equation of (28) has two independent solutions,

$$P_3^2 (\mu) = 15 \mu(1 - \mu^2) \quad , \tag{29-a}$$

$$Q_3^2 (\mu) = \frac{15}{2} \mu(1-\mu^2) \log\frac{1+\mu}{1-\mu} + \frac{2}{1 - \mu^2} - 15(1-\mu^2)+5 \quad , \tag{29-b}$$

we can obtain the solution of (28) by a standard method of variation of undetermined coefficients setting as

$$F(\mu) = \phi_1(\mu) P_3^2 (\mu) + \phi_2(\mu) Q_3^2 (\mu) . \tag{30}$$

The coefficients $\phi_1(\mu)$ and $\phi_2(\mu)$ are easily calculated as

$$\phi_1(\mu) = - \frac{1}{60} \frac{U^2}{s} \left\{\frac{1}{4}\left[- \frac{17}{20} + 9\mu^2 - \frac{173}{4} \mu^4 + \frac{105}{2} \mu^6 - 15\mu^8\right]\log\frac{1+\mu}{1-\mu} + \left[- \frac{35}{8} \mu + \frac{115}{8} \mu^3 - \frac{95}{4}\mu^5 + \frac{15}{2}\mu^7 \right]\right\} \quad , \tag{31-a}$$

and

$$\phi_2(\mu) = \frac{1}{4} \frac{U^2}{s} \left\{ \frac{19}{40}(1-\mu^2)^2 + \frac{1}{4}(1 - \mu^2)^3 - \frac{1}{2}(1-\mu^2)^4\right\} . \tag{31-b}$$

Therefore, the final expression of $F(\mu)$ is obtained as

$$F(\mu) = \frac{1}{4}\frac{U^2}{s}(1-\mu^2)\ \{\frac{36}{5} - 14(1-\mu^2) + 5(1-\mu^2)^2$$
$$+ \mu\ DX[-\frac{6}{5} + 10(1-\mu^2) - 5(1-\mu^2)^2]\} \quad . \tag{32}$$

Hence, the second order horizontal velocity component $u^{(2)}$ is determined to be

$$u^{(2)} = \frac{3}{4}(\frac{y}{h})^2\frac{U^2}{s}\mathrm{sech}^2(DX)\ \{-2 + 3\ \mathrm{sech}^2(DX)\}$$
$$+ \frac{1}{4}\frac{U^2}{s}\mathrm{sech}^2(DX)\ \{\frac{36}{5} - 14\ \mathrm{sech}^2(DX) + 5\ \mathrm{sech}^4(DX)$$
$$+ DX\tanh(DX)[-\frac{6}{5} + 10\ \mathrm{sech}^2(DX) - 5\ \mathrm{sech}^4(DX)]\} \ . \tag{33}$$

The surface elevation $\eta(X)$ of the shallow water wave is determined correctly up to the second order as

$$\frac{\eta(X)}{h} = 1 + \frac{U}{s}\ \mathrm{sech}^2(DX) + (\frac{U}{s})^2\mathrm{sech}^2(DX)$$
$$\{\frac{4}{5} - \frac{7}{4}\ \mathrm{sech}^2(DX) + \frac{5}{4}\ \mathrm{sech}^4(DX) +$$
$$DX\tanh(DX)[-\frac{3}{10} + \frac{5}{2}\ \mathrm{sech}^2(DX) - \frac{5}{4}\ \mathrm{sech}^4(DX)]\} \ . \tag{34}$$

Fig.2 illustrates the present result (34) evaluated for a parameter of $U=0.12\ s$ and the second order contribution $\eta^{(2)}(X)$ expressed as the third term of (34). The dotted line is shape of the Korteweg-de Vries soliton for the same value of U.

Since experimental observation does not distinguish the first order and the second order contribution of the solitary wave, it would be appropriate to reexpress eq.(34) in terms of an actual height A defined as the water elevation at X=0. Taking X=0 in eq.(34), we have

$$U^2 + \frac{10}{3} s\, U - \frac{10}{3} g\, A = 0 \quad . \tag{35}$$

Eq.(35) determines U as

$$U = \{ \frac{A}{h} - \frac{3}{10}(\frac{A}{h})^2 + Q(\frac{A}{h})^3 \} s \quad . \tag{36}$$

Hence, the expression of water surface elevation takes the following form

$$\frac{\eta(X)}{h} = 1 + \frac{A}{h} \operatorname{sech}^2(DX) + \frac{1}{2}(\frac{A}{h})^2 \operatorname{sech}^2(DX) \; .$$

$$\{(1- \frac{5}{2}\operatorname{sech}^2(DX)(1 - \operatorname{sech}^2(DX)) + DX \tanh(DX) .$$

$$(- \frac{3}{5} + 5 \operatorname{sech}^2(DX) - \frac{5}{2} \operatorname{sech}^4(DX))\} , \tag{37}$$

in which DX is redefined as

$$DX = \frac{1}{h}(\frac{3A}{4h})^{1/2} (1 - \frac{3}{20}\frac{A}{h})(\xi - \lambda\tau) + O(\frac{A}{h})^{5/2} , \tag{38}$$

and

$$\frac{\lambda}{s} = \frac{1}{2}\frac{A}{h} - \frac{3}{20}(\frac{A}{h})^2 + O(\frac{A}{h})^3 \quad . \tag{39}$$

§4. Concluding discussions

Firstly, we examine the present result in comparison with the result given by Laiton. He has derived the following expression for the steady solitary wave solution,

$$\frac{\eta(X)}{h} = 1 + \frac{A}{h} \operatorname{sech}^2(LX) - \frac{3}{4}(\frac{A}{h})^2 \operatorname{sech}^2(LX)$$

$$(1 - \operatorname{sech}^2(LX)) + O(\frac{A}{h})^3 , \tag{40}$$

with

$$LX = \frac{1}{h}\left(\frac{3}{4}\frac{A}{h}\right)^{1/2}\left(1 - \frac{5}{8}\frac{A}{h}\right)X + O\left(\frac{A}{h}\right)^{5/2} . \tag{41}$$

The velocity of the solitary wave is given as

$$\frac{u(\infty)}{s} = 1 + \frac{1}{2}\frac{A}{h} - \frac{3}{20}\left(\frac{A}{h}\right)^2 + O\left(\frac{A}{h}\right)^3 . \tag{42}$$

Comparing the above expressions with eqs.(37), (38) and (39), we find that the soliton velocity turns out to be the same, although the shapes of the solitary wave are different in both theories.

It should be noticed that the essential nonlinear effect is fully accounted for by the third term of the Korteweg-de Vries equation (19), while the second order quantities are determined by a linear partial differential equation (23-a), of which inhomogeneous term is composed of the next order dispersion term and the terms representing the interaction between the fundamental nonlinear waves. Structure of the inhomogeneous term is essentially the same as our previous result obtained for the ion acoustic wave, except numerical coefficients of each terms. Thus, we can conclude that the reductive perturbation theory provides a systematic way to treat the self-interaction of nonlinear wave in the lowest order and then allows to treat the mode-mode interaction among the nonlinear waves as higher order perturbation.

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Captions of Figures

Fig. 1 The coordinate system of long-wave disturbance.

Fig.2 The steady solitary wave in the second order approximation for a value of U=0.12s. The broken line represents contribution of the second order terms. For the sake of comparison, the first order Korteweg-de Vries solitary wave is shown by the dotted line.

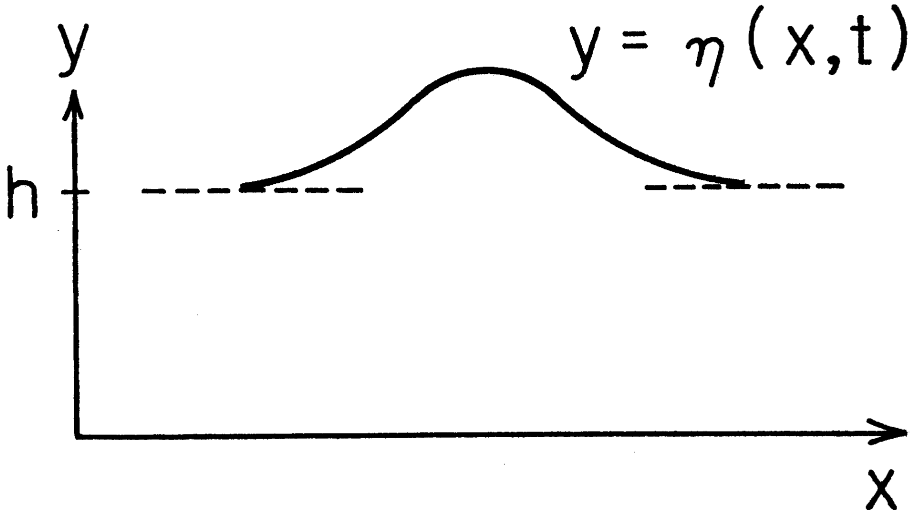

Fig. 1

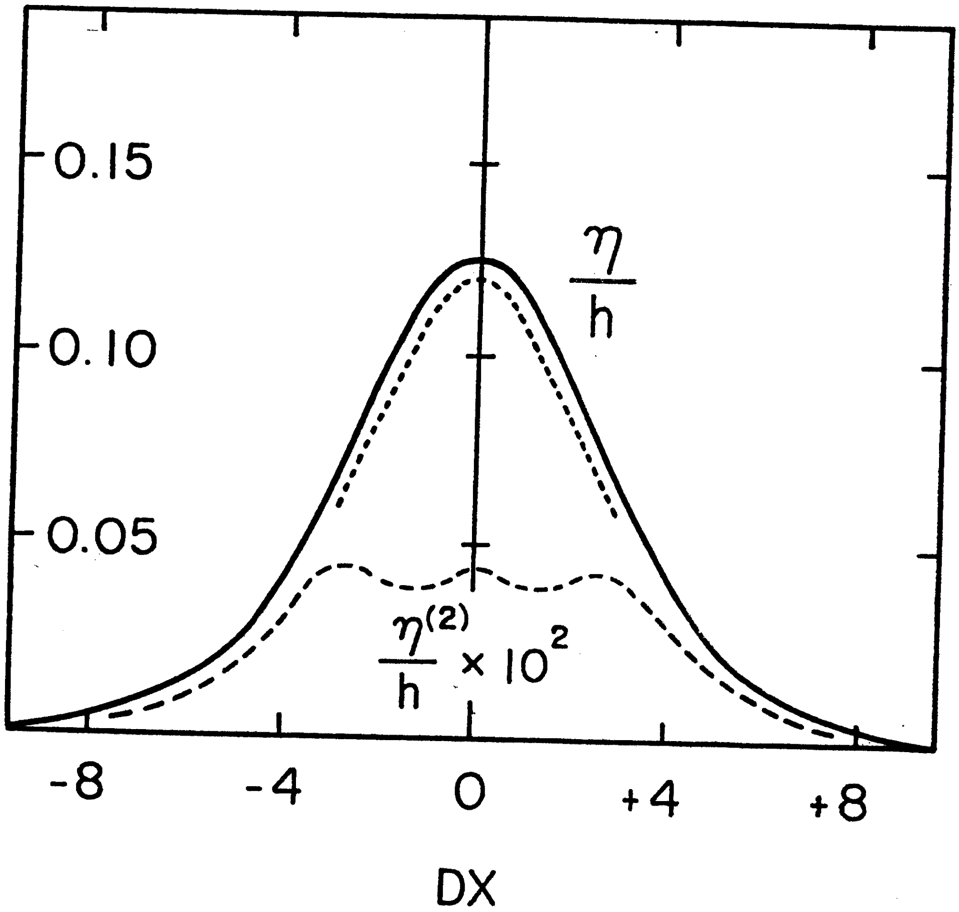

Fig. 2