INSTITUTE OF PLASMA PHYSICS

NAGOYA UNIVERSITY

RESEARCH REPORT

NAGOYA, JAPAN

Stochastic Acceleration by an Electrostatic Wave near Ion Cyclotron Harmonics

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IPPJ-259 September 1976

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ABSTRACT

A nonlinear effect of a large amplitude electrostatic wave propagating perpendicularly to a static magnetic field on the motion of an ion is studied. We consider the case where the frequency of the wave ω is sufficiently close to an multiple of the ion cyclotron frequency Ω_i. If the trapping frequency $\omega_t << \Omega_i$, most of hot ions ($v_\perp > \omega/\kappa_\perp$) are trapped in cells of a phase space separated by separatrices. However it is found that the trapping motion is influenced by periodic forces. These forces lead to randomization of the trapping motion near the separatrix and ions are expected to be stochastically accelerated. In order to confirm the results, numerical calculations are carried out.

§1 Introduction

The formation of a high energy tail in a velosity distribution of ions has often been observed in experiments on lower hybrid resonance heating.[1) In order to explain the tail formation, we consider a nonlinear motion of an ion affected by a coherent electrostatic wave propagating perpendicularly to a uniform magnetic field. In the case of an oblique propagation, Smith and Kaufman[2) have shown the occurrence of stochastic acceleration along a magnetic field subjected to stochastic perpendicular acceleration. However the threshold of the wave amplitude is so large, when $k_{\parallel} v_{\mathrm{thermal}} \ll \Omega_i$, that their results are not applicable to the lower hybrid wave. In this paper, we show that destruction of adiabaticity of the magnetic moment can appear even if $k_{\parallel} = 0$ and no spatial inhomogeneity exists; this inhomogeneity has been essential in many analogous studies on cyclotron heating in a mirror field.[3-6)

In the next section, by introducing appropriate canonical variables, we obtain an expression of the Hamiltonian of a test ion as a sum of a time-independent part H_0 and a time-dependent one H_1. The motion of a phase point derived from H_0 is studied in §3, where we find that the majority of hot ions ($v_{\perp} > \omega/k_{\perp}$) are trapped in cells of a phase space separated each other by separatrices, when the wave frequency is sufficiently close to an ion cyclotron harmonics. The analysis of a periodic force derived from H_1 shows in §4 that it resonates with the periodic motion determined by H_0 and forms islands on a phase trajectory. The overlapping of islands near a separatrix causes stochastic instability[7]; then the motion of an ion becomes stochastic in this region. In the last section, we shall discuss the stochastic acceleration of a high energy ion as well as the result of numerical calculations.

§2 Basic equations

We consider the motion of an ion, mass m and charge e, in a uniform magnetic field $\vec{B}_0 = B_0\vec{z}$ in the presence of a monochromatic electrostatic wave $\phi\cdot\cos(kx - \omega t)$ propagating perpendicularly to the magnetic field. The Hamiltonian of a test ion takes a form,

$$H(x,y,P_x,P_y,t) = \{P_x^2+(P_y-eB_0x)^2\}/2m + e\phi\cos(kx-\omega t), \tag{1}$$

where P_x and P_y are the canonical momentum conjugate to x and y, respectively ($P_x = mv_x$, $P_y = mv_y + eB_0x$). Since the Hamiltonian is independent of y, the momentum P_y is a constant of motion. Defining the magnetic moment μ and the cyclotron phase θ by the relations,

$$\mu \equiv \frac{1}{2m\Omega_i}\{P_x^2 + (P_y - m\Omega_i x)^2\} = \frac{mv_\perp^2}{2\Omega_i}, \tag{2}$$

$$\theta \equiv \arcsin\left(\frac{x - P_y/m\Omega_i}{\sqrt{2\mu/m\Omega_i}}\right) = \arctan\left(-\frac{v_y}{v_x}\right), \tag{3}$$

where the ion cyclotron frequency is denoted by $\Omega_i = eB_0/m$, we may rewrite the Hamiltonian after the canonical transformation from (x, P_x) to (θ, μ),

$$H(\theta,\mu,t) = \mu\Omega_i + e\phi \cos(k\sqrt{2\mu/m\Omega_i}\,\sin\theta - \omega t) \tag{4}$$

$$= \mu\Omega_i + e\phi \sum_{n=-\infty}^{\infty} J_n(k\sqrt{2\mu/m\Omega_i}) \cos(n\theta - \omega t) , \tag{5}$$

here a series expansion of Bessel functions J_n is applied.

By the use of an integer n_0 nearest to ω/Ω_i, we again perform the canonical transformation from (θ, μ) to (ξ, M) defined by the generating function,

$$S(\theta,M,t) = (n_0\theta - \omega t)M , \tag{6}$$

which contains the time t explicitly. New variables and the Hamiltonian are given by

$$\xi \equiv n_0\theta - \omega t , \tag{7}$$

$$M \equiv \mu/n_0 , \tag{8}$$

$$H(\xi,M,t) = (n_0\Omega_i - \omega)M$$

$$+ e\phi \sum_n J_n(k\sqrt{2n_0M/m\Omega_i}) \cos\{n\xi/n_0 + (n-n_0)\omega t/n_0\} \tag{9}$$

This Hamiltonian is expressed by a sum of a time-independent part $H_0(\xi,M)$ and a time-dependent one $H_1(\xi,M,t)$;

$$H_0(\xi,M) \quad = \delta\omega\ M + e\phi\ J_{n0}(k\rho(M))\ \cos\xi\ , \tag{10}$$

$$H_1(\xi,M,t) = e\phi \sum_{n\neq n_0} J_n(k\rho(M))\ \cos\{n\xi/n_0 + (n-n_0)\omega t/n_0\}, \tag{11}$$

here we have introduced, for simplicity,

$$\delta\omega \equiv n_0\Omega_i - \omega\ , \tag{12}$$

$$\rho(M) \equiv \sqrt{2n_0M/m\Omega_i} = v_\perp/\Omega_i\ , \tag{13}$$

§3 Motion derived from H_0

If the motion derived from H_0 is characterized by sufficiently small $d\xi/dt$ and dM/dt, the time average of $H_1(\xi,M,t)$ approaches zero and its contribution on an ion motion may be neglected; therefore we first consider the motion derived from H_0 alone in this section. The equations of motion obtained from eq. (10) are

$$\frac{dk\rho}{dt} = -\frac{kn_0}{m\Omega_i\rho}\frac{\partial H_0}{\partial\xi} = \frac{kn_0}{m\Omega_i\rho}\, e\phi\ J_{n0}(k\rho)\ \sin\xi\ , \tag{14}$$

$$\frac{d\xi}{dt} = \frac{\partial H_0}{\partial M} = \delta\omega + \frac{kn_0}{m\Omega_i\rho}\, e\phi\ J'_{n0}(k\rho)\ \cos\xi\ , \tag{15}$$

and the qualitative phase trajectories are illustrated in Fig. 1. Trapped regions enclosed by separatrices can exist where $k\rho \gtrsim n_0$ if $\delta\omega$ is smaller than $(kn_0/m\Omega_i\rho)\ e\phi\ \max\{|J_{n0}(k\rho)|\}$. With $\delta\omega$ approaching to zero, however, untrapped regions vanish and all ions are trapped in rectangular cells.

When $k\rho \gg n_0$, we can use the asymptotic expansion of J_{n0}:

$$J_{n0} \sim \sqrt{2/\pi k\rho}\,\cos(k\rho - (2n_0+1)\pi/4) \simeq \sigma\sqrt{2/\pi k\rho_0}\,\cos k\tilde{\rho}, \qquad (16)$$

where ρ_0 satisfies $J'_{n0}(k\rho_0) = 0$ and it has been assumed that $\tilde{\rho} \equiv \rho - \rho_0 \ll \rho_0$. The quantity σ takes the value ± 1 depending on $k\rho_0$. Making use of the bounce frequency at the center of the trapped region,

$$\omega_t \equiv \frac{kn_0}{m\Omega_i\rho_0}\, e\phi\, J_{n0}(k\rho_0) \simeq \frac{\omega_B^2}{\Omega_i}\,\frac{n_0}{(k\rho_0)^{3/2}}\,\frac{2^{1/2}}{\pi^{1/2}}, \qquad (17)$$

we obtain the equations of motion,

$$dk\tilde{\rho}/dt = \sigma\,\omega_t\,\cos k\tilde{\rho}\,\sin\xi\ , \qquad (18)$$

$$d\xi/dt = \delta\omega - \sigma\,\omega_t\,\sin k\tilde{\rho}\,\cos\xi\ , \qquad (19)$$

where we have denoted the bounce frequency without a magnetic field by $\omega_B \equiv (k^2 e\phi/m)^{1/2}$.

We assume that $\delta\omega \ll \omega_t$ in the following, in order to make quantitative analyses possible. On this assumption, the equation of motion in the cell takes the form,

$$dk\tilde{\rho}/dt = \sigma\,\omega_t\,(q^2 - \sin^2 k\tilde{\rho})^{1/2}, \qquad (20)$$

The quantity q defined by

$$q^2 \equiv 1 - \left(\frac{kn_0}{m\Omega_i\rho_0}\,\frac{H_0}{\omega_t}\right)^2, \qquad (21)$$

takes the value zero at the center of the cell and unity at the separatrix; and this quantity q will be used as the parameter of elliptic functions and elliptic integrals introduced below. Using Jacobian elliptic functions, we can express the solutions to eq.(20) as follows,

$$\sin k\tilde{\rho} = q\ \mathrm{sn}\{\omega_t(t-t_0),q\}\ , \tag{22}$$

$$\cos k\tilde{\rho} = \sigma_1\ \mathrm{dn}\{\omega_t(t-t_0),q\}\ , \tag{23}$$

$$\sin\xi = q\ \mathrm{cn}\{\omega_t(t-t_0),q\}\ /\ \mathrm{dn}\{\omega_t(t-t_0),q\}\ , \tag{24}$$

$$\cos\xi = \sigma_2\ \sqrt{1-q^2}\ /\ \mathrm{dn}\{\omega_t(t-t_0),q\}\ . \tag{25}$$

The quantity σ_1 and σ_2 which take the value ± 1, and the time t_0 depend on the initial value; however the initial conditions are not so important in the analysis that we assume $\sigma_1 = \sigma_2 = 1$ and $t_0 = 0$ for brevity.

The Jacobian elliptic functions are periodic with a period $4F_0(q)$; here complete and incomplete elliptic integrals of the first kind are denoted by $F_0(q) = F(\pi/2,q)$ and $F(\psi,q)$, respectively. Hence the motion derived from H_0 has a period $T = 4F_0(q)/\omega_t$ and we can express it by the form

$$\mathrm{d}I/\mathrm{d}t = 0\ , \tag{26}$$

$$\mathrm{d}\eta/\mathrm{d}t = \Omega_t(I) = (\pi/2F_0(q))\ \omega_t\ , \tag{27}$$

making use of the action-angle variables (I,η) defined by

$$I(H_0) = \frac{1}{2\pi}\oint M\,\mathrm{d}\xi = \frac{2}{\pi}\frac{m\Omega_i\rho_0}{kn_0}\int_0^q \frac{q}{\sqrt{1-q^2}}\,\mathrm{F}_0(q)\,\mathrm{d}q \;, \tag{28}$$

$$\eta(\xi,H_0) = \frac{\partial}{\partial I}\int^{\xi} M\,\mathrm{d}\xi = \frac{\pi}{2}\,\frac{\mathrm{F}\{\arcsin(\sin\xi/q),q\}}{\mathrm{F}_0(q)} \;. \tag{29}$$

The relation between variables (ξ,M) and (I,η) are illustrated in Fig. 2; it is easy to see that the phase space area enclosed by a trajectory equals to the action $2\pi I$ and the angle η means the phase on the trajectory, $I \doteq$ *constant*.

§4 Effect of periodic force derived from H_1

Since the time-dependent part of the Hamiltonian, H_1, has an explicit periodicity, the force derived from H_1 is able to resonate with the periodic motion determined by H_0 and to form islands on the resonant trajectory.

When we use the asymptotic expansion of J_n as well as eq.(16) on the assumption that $k\rho \gg n_0$ and take account of only the terms $n = n_0 \pm 1$ which are dominant for the island formation, the time-dependent part H_1 becomes

$$H_1 = -\,2e\phi\sqrt{2/\pi k\rho_0}\,\sin k\tilde{\rho}\,\sin\xi\,\sin\{(\xi+\omega t)/n_0\} \;. \tag{30}$$

Considering the existence of H_1, we find that $\mathrm{d}I/\mathrm{d}t$ does not vanishes, *i.e.*,

$$\frac{dI}{dt} = \frac{dI}{dH_0} \left(- \frac{\partial H_0}{\partial M} \frac{\partial H_1}{\partial \xi} + \frac{\partial H_0}{\partial \xi} \frac{\partial H_1}{\partial M} \right) \tag{31}$$

$$= \frac{m\Omega_i \rho_0}{kn_0} \frac{2\omega_t^2}{\Omega_t} (\cos^2 k\tilde{\rho} \sin^2 \xi - \sin^2 k\tilde{\rho} \cos^2 \xi) \sin(\frac{\xi + \omega t}{n_0}) \tag{32}$$

$$= \frac{m\Omega_i \rho_0}{kn_0} \omega_t \frac{8\pi}{F_0} \sum_{s=0}^{\infty} \frac{(2s+1)\, r^{2s+1}}{1 - r^{2(2s+1)}}$$

$$\times \left(\sin\{\frac{\omega t + \xi}{n_0} + 2(2s+1)\Omega_t t\} + \sin\{\frac{\omega t + \xi}{n_0} - 2(2s+1)\Omega_t t\} \right). \tag{33}$$

Here we have used the series expansions [8] of Jacobian elliptic functions in terms of the nome: $r \equiv \exp\{-\pi F_0(\sqrt{1-q^2})/F_0(q)\}$.

In a case of our interest, $n_0 \gg 1$, the time variation of $\dot{\xi}/n_0 \sim O(\omega_t/n_0)$ can be neglected; therefore the resonance condition is

$$\Omega_t(I_r) = \frac{1}{2(2s+1)} \frac{\omega}{n_0} \sim \frac{1}{2(2s+1)} \Omega_i \qquad (s;\ integer). \tag{34}$$

Because of the I dependence of $\Omega_t(I)$, this resonance leads to the island formation on the resonant trajectory. Near the resonance $I = I_r$, the equations of motion are approximated by the expression,

$$d\Delta I/dt = \Gamma_s \sin\Delta\eta \ , \tag{35}$$

$$d\Delta\eta/dt = 2(2s+1)\{d\Omega_t(I_r)/dI\}\Delta I \ , \tag{36}$$

here the variables ΔI and $\Delta\eta$ are defined by

$$\Delta I \equiv I - I_r \ , \tag{37}$$

$$\Delta\eta \equiv 2(2s+1)\{\Omega_t(I) - \Omega_t(I_r)\}t \ , \qquad (38)$$

and the nonresonant terms in eq. (33) have been neglected. The first integral of eqs. (35) and (36),

$$G = 2(2s+1)\{d\Omega_t(I_r)/dI\}\{(\Delta I)^2/2\} + \Gamma_s \cos\Delta\eta, \qquad (39)$$

gives the width of the island,

$$\Delta I_{max} = 2\left(\frac{|\Gamma_s|}{2(2s+1)|d\Omega_t(I_r)/dI|}\right)^{1/2} ; \qquad (40)$$

therefore the frequency width of the island is

$$\Delta\Omega_{t\ max} = 2\left(\frac{|d\Omega_t(I_r)/dI|\ |\Gamma_s|}{2(2s+1)}\right)^{1/2} . \qquad (41)$$

On the other hand, the frequency spacing of adjacent resonances is obtained from eq. (34) as follows,

$$\delta\Omega_t \equiv \Omega_t^{(s-1)} - \Omega_t^{(s)} = \left\{\frac{1}{2(2s-1)} - \frac{1}{2(2s+1)}\right\}\frac{\omega}{n_0} \sim \frac{4\Omega_t^2}{\Omega_i} . \qquad (42)$$

The explanation of the meaning of $\Delta\Omega_{t\ max}$ and $\delta\Omega_t$ is shown in Fig. 3. It is known [7] that if adjacent islands overlap, then the motion of a phase point becomes stochastic. This condition, called a stochastic condition, is given by the following expression,

$$K \equiv (2\Delta\Omega_{t\ max}/\delta\Omega_t)^2$$

$$= \frac{\Omega_i^2}{\omega_t^2} \frac{1}{\sinh\{\Omega_i F_0(\sqrt{1-q^2})/\omega_t\}} \frac{8}{\pi} \frac{E_0 - (1-q^2)F_0}{\sqrt{1-q^2}\ \ q^2} \gtrsim 1 , \tag{43}$$

here E_0 denotes the complete elliptic integral of the second kind $E_0(q)$. It should be noted that if q approaches to unity, the quantity K diverges provided that $\omega_t/\Omega_i \neq 0$.

§5 Discussion of the results

The motion of an ion is considered to be stochastic in the region where $K \gtrsim 1$. The ratio R between the phase space area of the stochastic layer in the cell and the total phase space area of the cell $2\pi I(q=1)$,

$$R(\omega_t/\Omega_i) = 1 - I(q;K=1)/I(q=1) , \tag{44}$$

is exhibited in Fig. 4. It is found that the ratio R begins to increase fairly abruptly when ω_t/Ω_i exceeds 0.2, which is ragarded as a threshold for an applied electric field to bring stochastic behavior of an ion motion. The saturation of R can be seen where $\omega_t/\Omega_i \gtrsim 0.5$; however, if we take account of the higher order resonances, $n = n_0 \pm j$ $(j \geqq 2)$, it is infered that the ratio R keeps increasing to attain unity.

Since the stochastic layer near a separatrix contacts with the stochastic one in the adjacent cell, it should be expected that the ion contained in this region stochastically moves into the adjacent cell and that the diffusion in the ion perpendicular velocity distribution takes place.

In order to confirm the above theoretical consideration, numerical calculations are performed according to the equations of motion deduced from eq. (4). Some examples are illustrated in Fig. 5 and 6; in each case ω/Ω_i is equal to 5 and the phase points are plotted at intervals of $\Delta t = 2\pi/\Omega_i$; theoretically predicted separatrix and the center of cell are indicated in the figures. Two cases for different wave amplitudes are shown in Fig. 5; the theoretical bounce frequencies at the center are calculated: a) $\omega_t/\Omega_i = 0.13$ and b) $\omega_t/\Omega_i = 0.26$. In the case of b), islands at the resonance of $\Omega_t(I)/\Omega_i = 1/6$ are observed and the motions are randomized outside of the trajectory which starts from $(k\rho,\xi) = (8.15,0.0)$. Figure 6 shows the phase points initially at $(k\rho,\xi) = (7.4,0.0)$ when the wave amplitude is as same as Fig. 5 b). It can be seen that the phase point randomly wonder from a cell to an adjacent cell.

We conclude that a monochromatic electrostatic wave propagating across a magnetic field can generate a high energy tail in a perpendiculr velocity distribution if $\delta\omega << \omega_t$. The stochastic formation of this high energy tail is restricted by the condition $v_\perp > \omega/k_\perp$ and

$$\frac{\omega_t}{\Omega_i} \simeq \frac{e\phi}{m} \frac{k^{1/2}\,\omega}{\Omega_i^{3/2}\, v_\perp^{3/2}} \gtrsim 0.2 , \qquad (45)$$

in the case of $\omega/\Omega_i \gg 1$. A high energy limit of this tail may be given by the latter condition.

Acknowledgements

The authors wish to thank Drs. R. Sugihara and T. Takizuka for their stimulating suggestions and useful discussions on numerical calculations.

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Figure Captions

Fig. 1 Phase trajectory obtained from eq. (10) for the case a) $\delta\omega \neq 0$ and b) $\delta\omega = 0$.

Fig. 2 Illustration of the relation between the action-angle variables (I,η) and (ξ,M).

Fig. 3 A sketch of the frequency width of islands $\Delta\Omega_t$ and the frequency spacing of adjacent resonances $\delta\Omega_t$.

Fig. 4 The ratio between the phase space area of the stochastic region in the cell and the total phase space area of the cell $2\pi I(q=1)$.

Fig. 5 Motion of phase point calculated from eq. (4). The value ω/Ω_i is equal to 5 and $(\omega_B/\Omega_i)^2$ equals to a) 1.0 and b) 2.0. Points are plotted at intervals of $\Delta t = 2\pi/\Omega_i$. A circle indicates the initial point.

Fig. 6 Motion of a phase point. Parameters are the same as Fig. 5 b). The initial point (indicated by a circle) differs from the previous case.

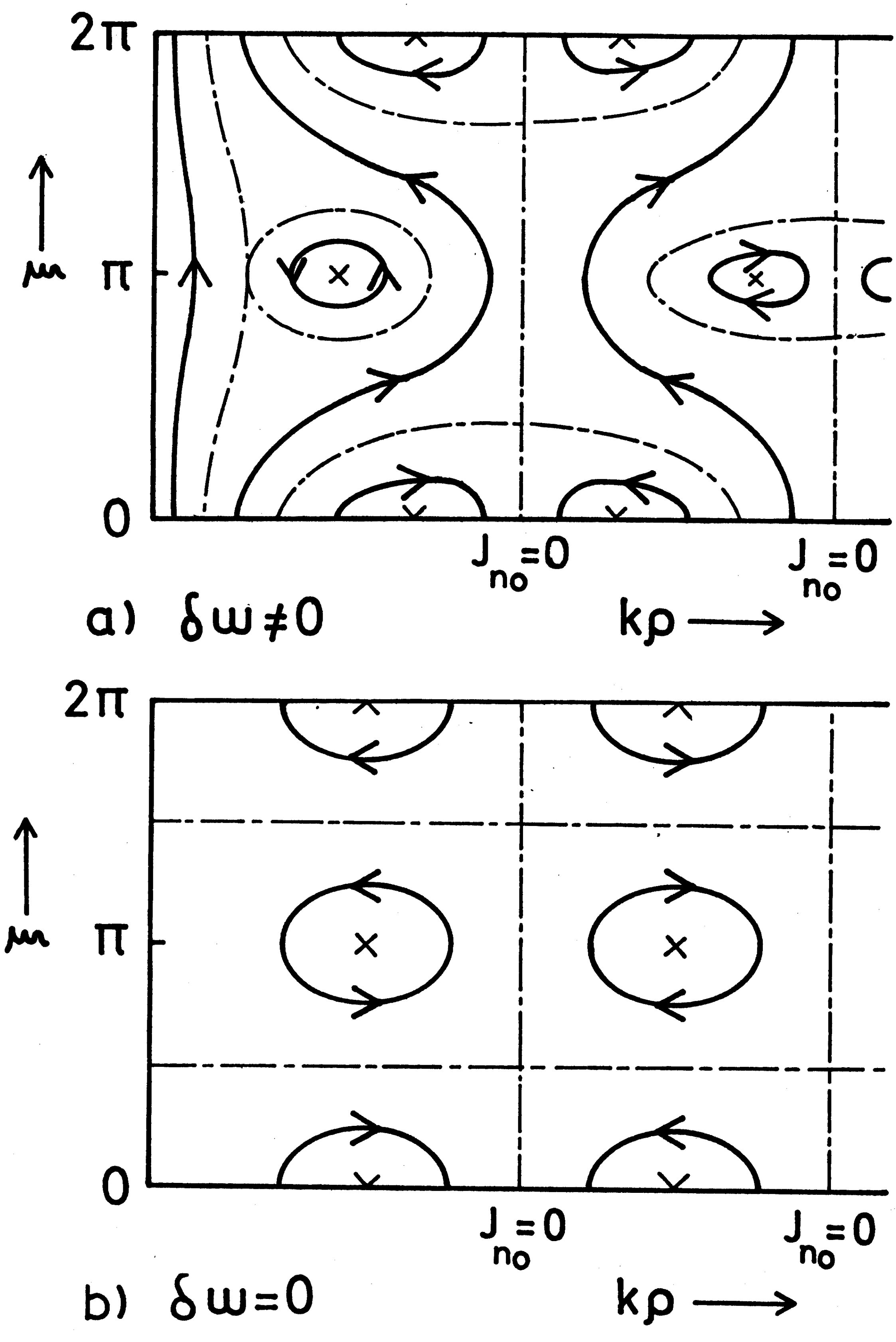
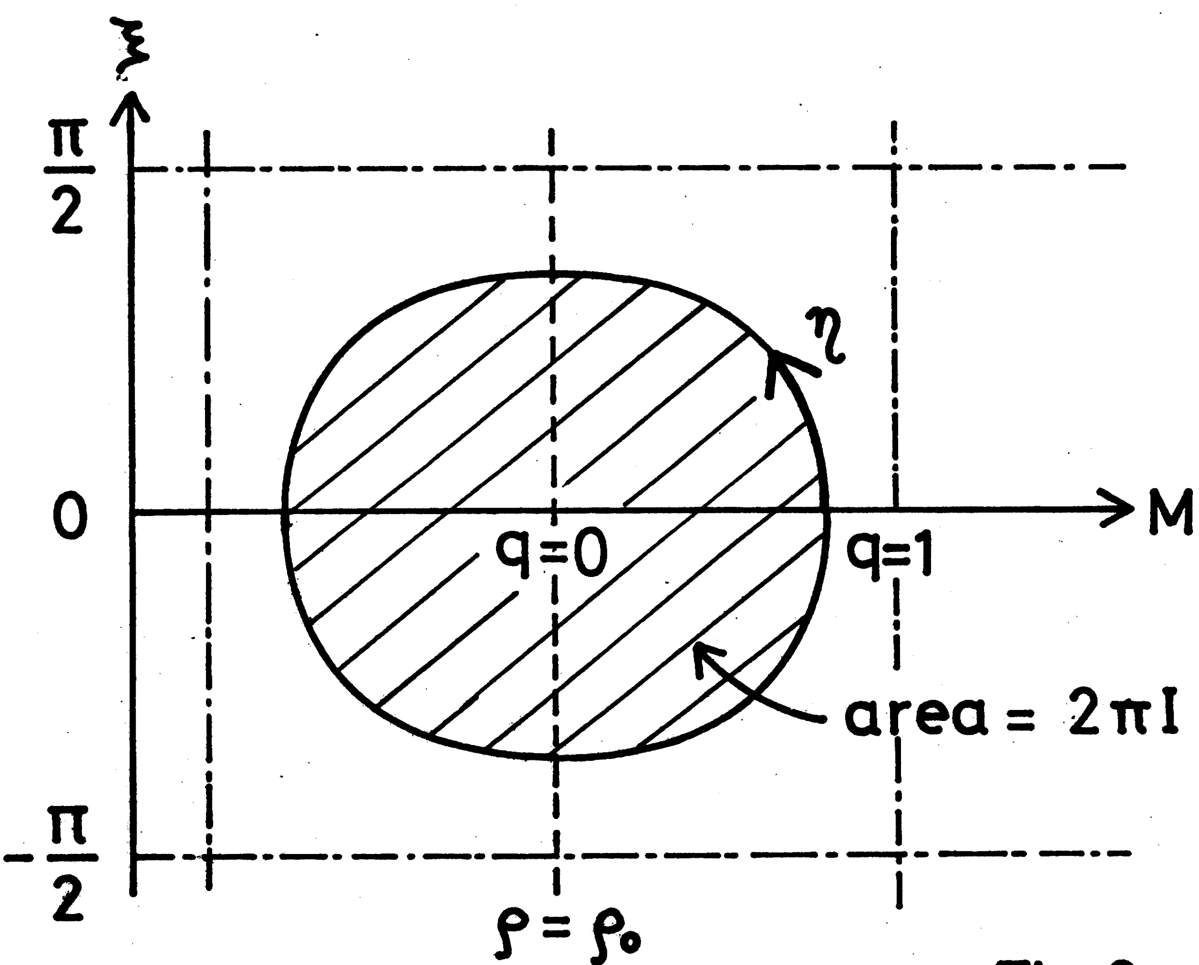
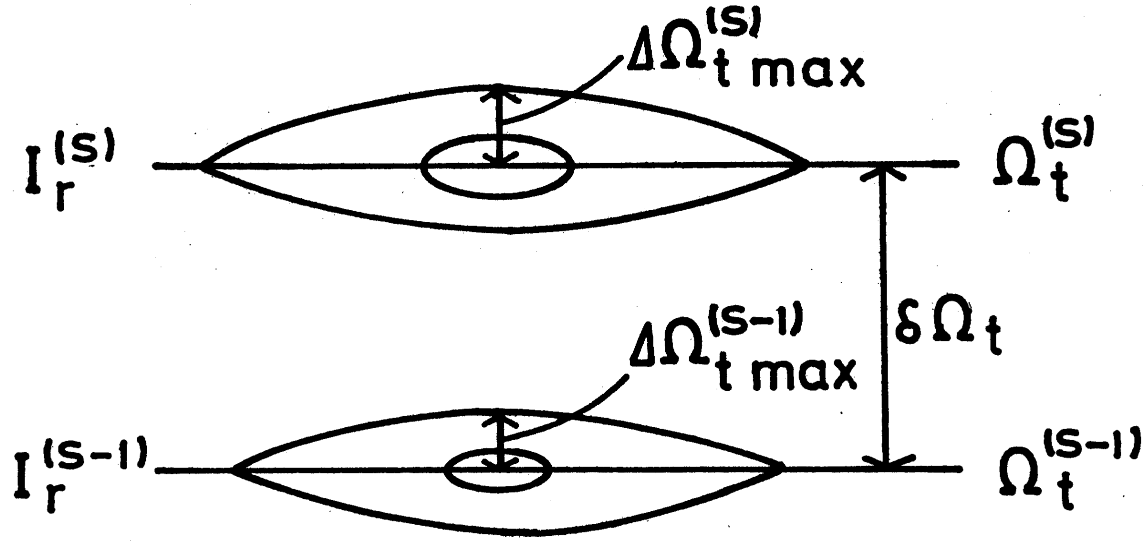


Fig. 1



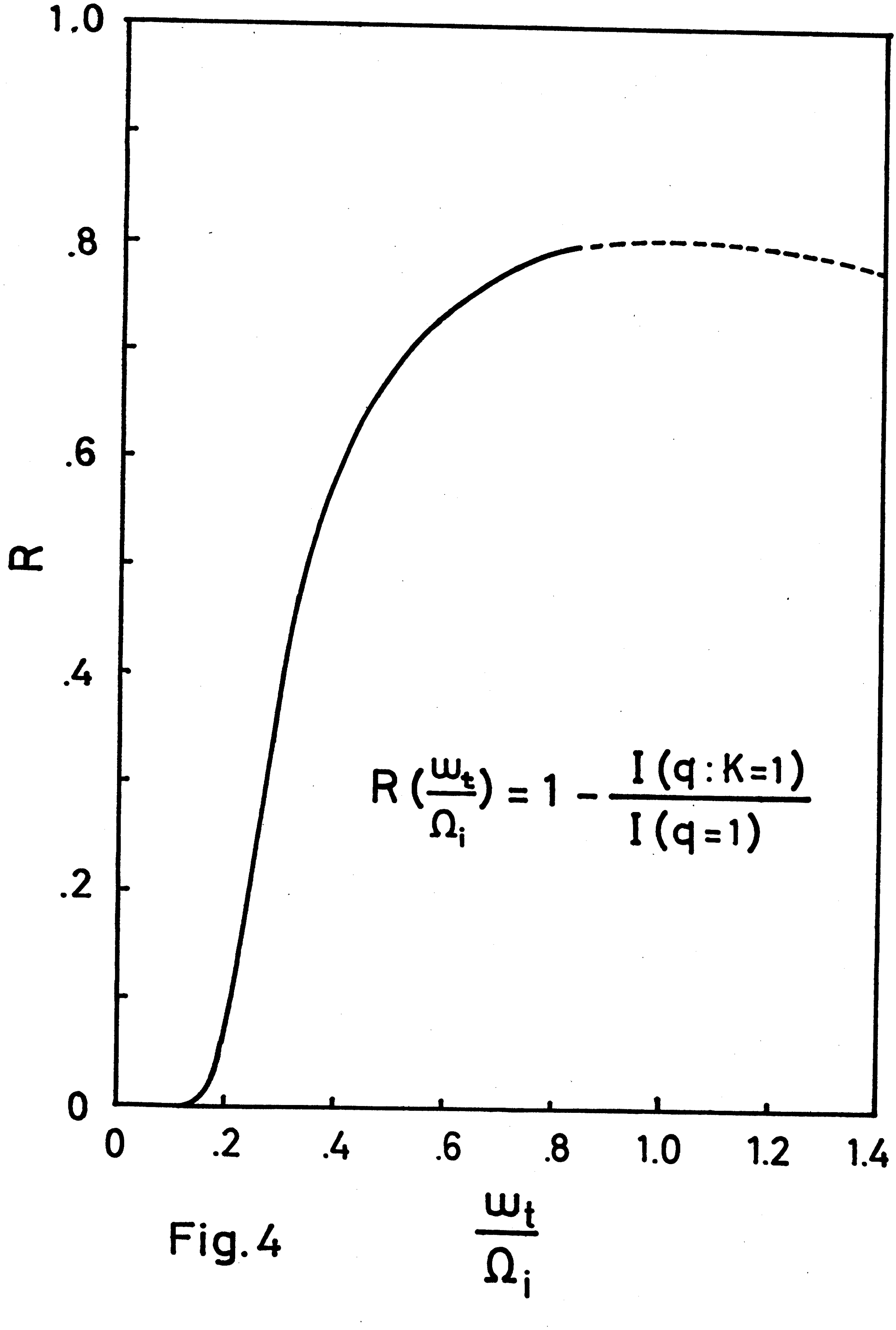
ξ
π/2
0
−π/2
η
M
q=0
q=1
area = 2πI
ρ = ρ₀

Fig. 2



ΔΩ(S) t max
I(S) r
Ω(S) t
δΩt
ΔΩ(S−1) t max
I(S−1) r
Ω(S−1) t

Fig. 3


1.0
.8
.6
.4
.2
0
R
$R\left(\frac{\omega_t}{\Omega_i}\right) = 1 - \frac{I\,(q : K=1)}{I\,(q=1)}$
0
.2
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1.2
1.4
$\frac{\omega_t}{\Omega_i}$

Fig. 4

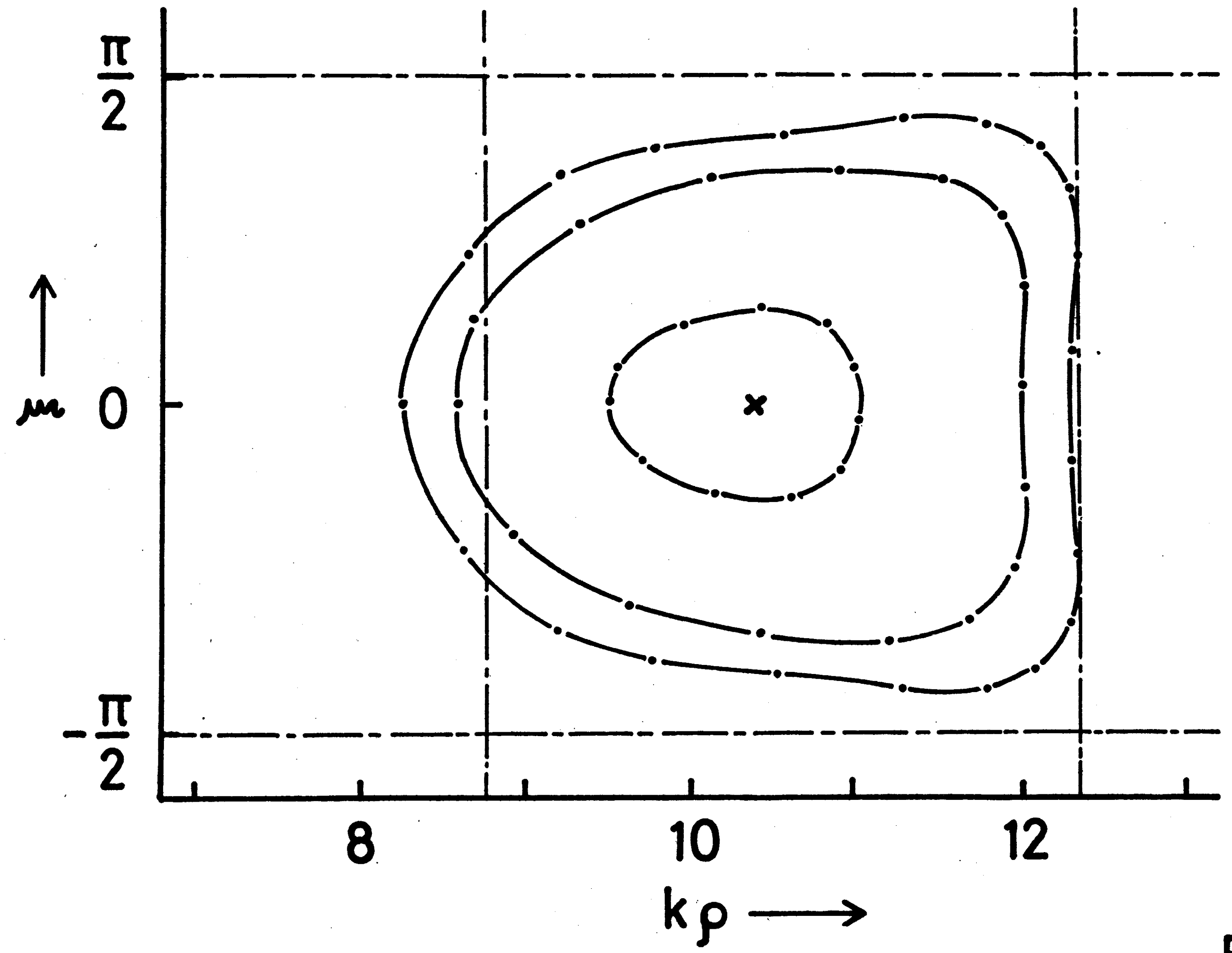

Fig. 5 a

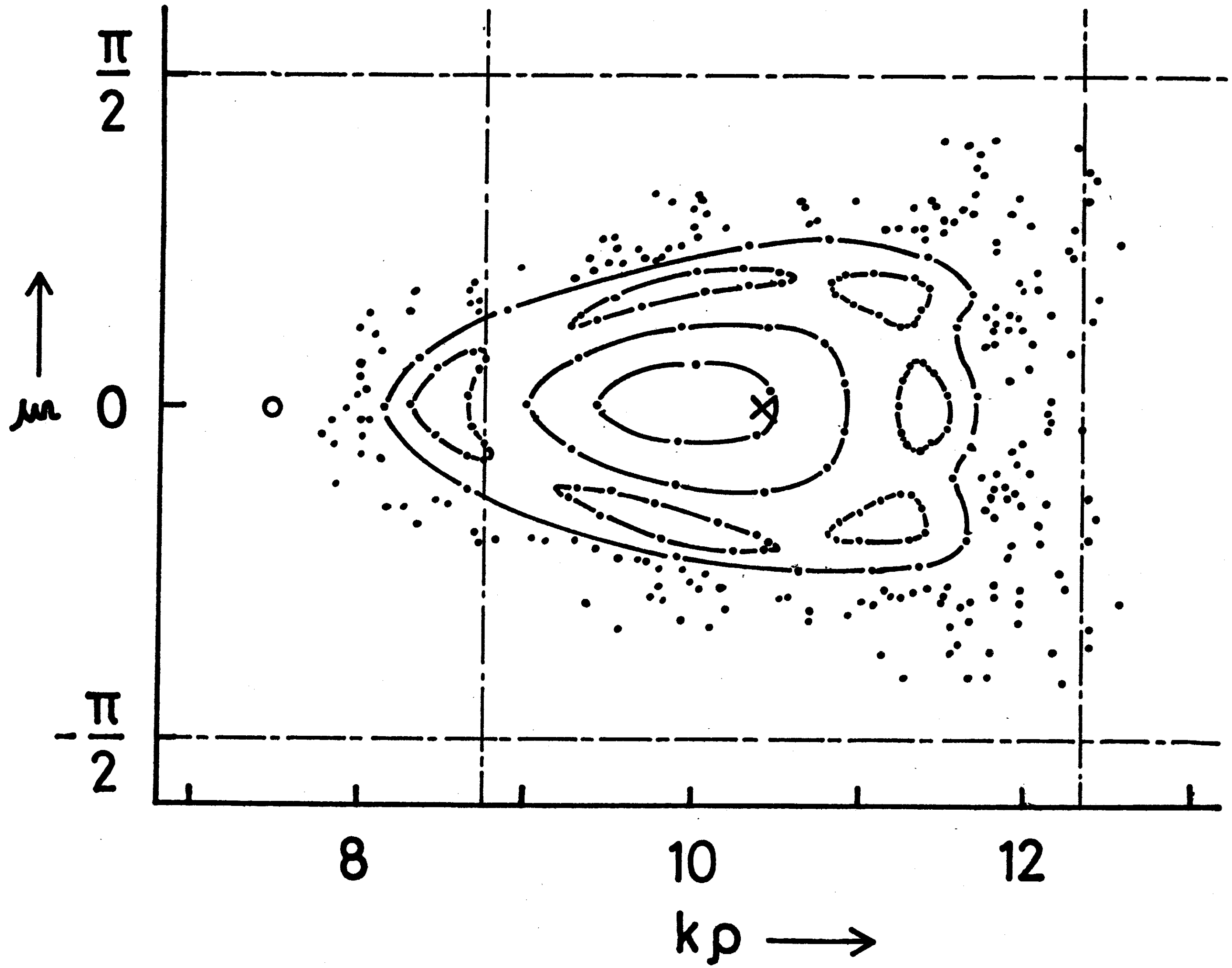

Fig. 5 b

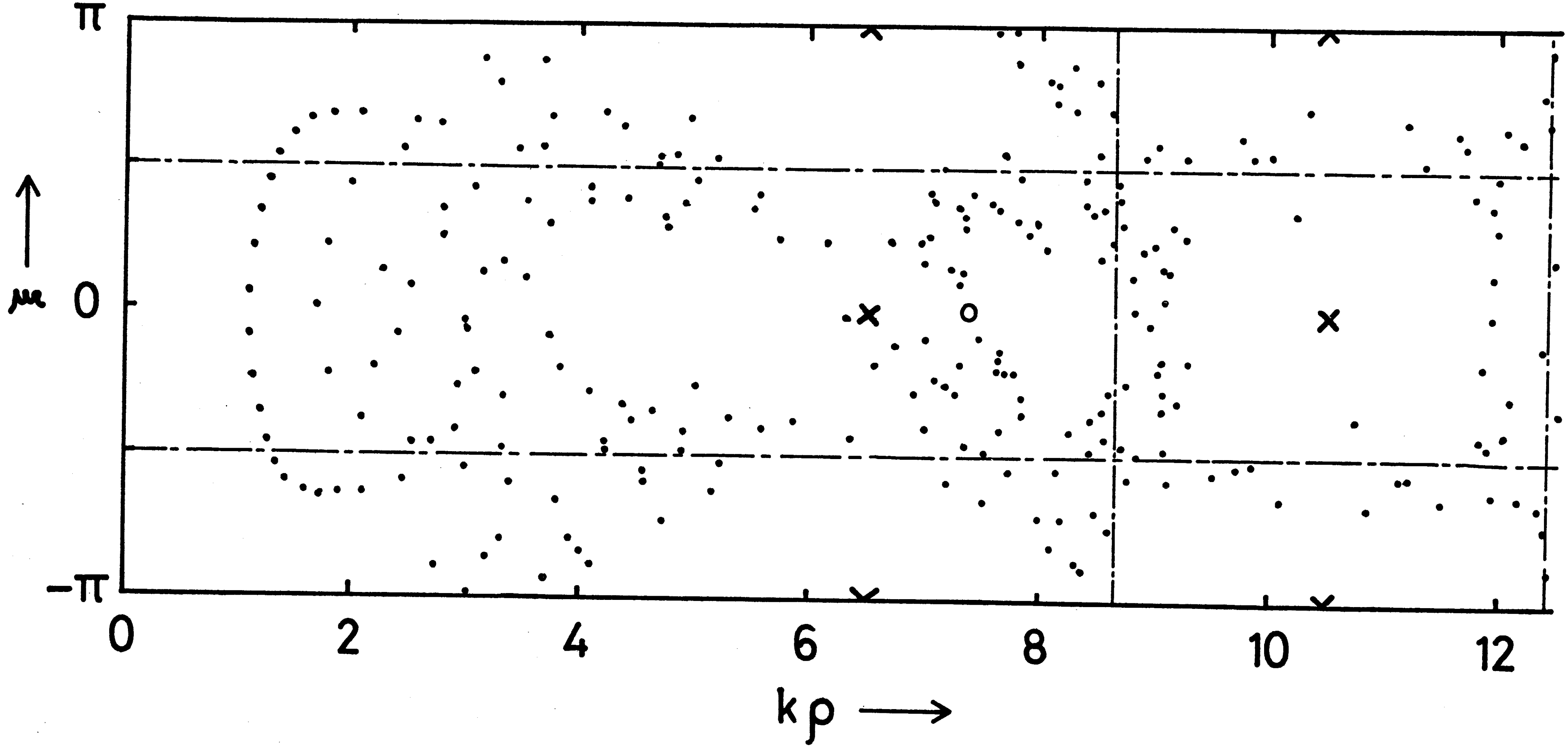

Fig. 6