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Abstract

Kinetic theory of the self-modulation of a quasi-monochromatic ordinary wave propagating across an external magnetic field is presented. Explicit expressions of the dispersion and the nonlinear coupling coefficients are given for a Maxwellian plasma. In the limit of zero-temperature, a Karpman-Kruskal linear stability of the O-mode envelop is discussed.

§1. Introduction

The linear stability of electromagnetic waves propagating across an external magnetic field $\vec{B}_o$ has been extensively studied by a number of authors (Dnestrovskii et al 1961, Hamasaki 1968, Davidson and Wu 1970, Gaffey et al 1972, 73, 75). They found that instabilities may exist only in high-β plasmas with a large temperature anisotropy. In their recent work, Freund and Wu (1976) have shown that low-β plasmas can also be unstable if an anisotropic population of hot relativistic electrons is added to a Maxwellian electron distribution.

With regards the nonlinear propagation of finite-amplitude linear waves in dispersive media, much attention has been paid during a last decade on the modulational instability of electromagnetic waves in plasmas. Self-modulation of the ion acoustic waves is governed either by a nonlinear Schrödinger (NLS) equation or by a Korteweg-de Vries (KdV) equation according as their wave-length is short or long (Shimizu and Ichikawa 1972). Both of them have already been experimentally confirmed (Watanabe 1976, Ikezi 1973). Finite-amplitude electron plasma waves are known to be described by the NLS equation (Zakharov 1972, Ichikawa et al 1972) which includes a nonlinear nonlocal term representing a wave-particle resonant interaction at the group velocity (Ichikawa and Taniuti 1973, Ichikawa et al 1973). Nonlinear propagation of whistlers (Hasegawa 1972) has also been shown to obey the NLS equation, which involves the nonlinear nonlocal term when the effect of nonlinear Landau damping is taken into account (Suzuki and

Ichikawa 1973, Kako et al 1973). As for electromagnetic ordinary and extraordinary (O and X) modes propagating across $\vec{B}_o$, only the hydrodynamic theories are available (Kako 1972, Furutani 1975).

The aim of this paper is then two-fold: (1) to obtain explicitly the dispersion and the nonlinear coupling coefficients of a NLS equation for the O-mode envelop and thereby (2) to discuss its modulational instability. The paper is organized thus: within the framework of the reductive perturbation technique, explicit evaluation of the dispersion coefficient is given in §2 and that of the nonlinear coupling coefficient in §3. With the aid of their limiting expressions (cold electron plasma), a linear stability based on the Karpman-Kruskal criterion is briefly discussed in §4. In the concluding remark, we elucidate a structural difference of the nonlinear coupling coefficient between the whistler (cyclotron wave) and the O-mode.

§2. Reductive Perturbation Method for Ordinary Waves

A quasi-monochromatic electromagnetic wave is assumed to propagate across an external magnetic field $\vec{B}_0$, directed to the z-axis, in a collision-free two-component plasma. When all quantities vary in the x-direction, the Vlasov-Maxwell equations are reduced to

$$[\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} - \frac{e_\alpha}{m_\alpha}(\frac{\partial}{\partial x}\phi + \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\cdot\frac{\partial}{\partial \vec{v}}) + \frac{e_\alpha}{m_\alpha c}(\vec{v}_T\cdot\frac{\partial}{\partial x}\vec{A}_T\frac{\partial}{\partial v_x} - v_x\frac{\partial}{\partial x}\vec{A}_T\cdot\frac{\partial}{\partial \vec{v}_T}) - \omega_{c\alpha}\frac{\partial}{\partial \psi}]\mathcal{F}_\alpha(\vec{x},\vec{v},t) = 0 \qquad (1)$$

$$(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\begin{pmatrix}\vec{A}\\ \phi\end{pmatrix} = -\frac{4\pi}{c}\sum_\alpha e_\alpha\int d\vec{v}\begin{pmatrix}\vec{v}\\ 1\end{pmatrix}\mathcal{F}_\alpha \qquad (2)$$

where ψ is the azimuthal angle in velocity space and $\omega_{c\alpha} = \frac{e_\alpha B_0}{m_\alpha c}$ is the cyclotron frequency of the α-species. The suffix T denotes a transverse component relative to the x-axis. $\mathcal{F}_\alpha$, $\vec{A}$ and ϕ are a velocity distribution function, a vector potential and a scalar potential, respectively. Furthermore $\vec{A}$ and ϕ are subject to the Lorentz condition

$$\frac{1}{c}\frac{\partial}{\partial t}\phi + \frac{\partial}{\partial x}A_x = 0 \qquad (3)$$

In view of applying the reductive perturbation method to the set (1) ~ (3), we introduce the stretched variables

$$\sigma = \varepsilon t, \quad \eta = \varepsilon x \text{ and } \zeta = \varepsilon^2 x$$

and expand the distribution function and the electromagnetic potentials into the following double series

$$\mathcal{F}_\alpha = F_\alpha(\vec{v}) + \sum_{n=1}^{\infty} \varepsilon^n \sum_{\ell=-\infty}^{\infty} f_{\alpha,\ell}^{(n)} (\vec{v},\sigma,\eta,\zeta;k,\omega) \times \exp[i\ell(kx - \omega t)] \quad (4)$$

$$X = \sum_{n=1}^{\infty} \varepsilon^n \sum_{\ell=-\infty}^{\infty} X_\ell^{(n)}(\sigma,\eta,\zeta;k,\omega) \times \exp[i\ell(kx - \omega t)] \quad (5)$$

where X stands for A_x, A_y, A_z and ϕ. The unperturbed plasma is characterized by $F_\alpha(\vec{v})$ which, axially symmetric about the z-axis, is assumed to be an even function of v_z. The reality condition requires $f_{\alpha,-\ell}^{(n)} = f_{\alpha,\ell}^{(n)*}$ and $X_{-\ell}^{(n)} = X_\ell^{(n)*}$, where the asterisk denotes the complex conjugate.

After having expanded and collected terms of the n-th order in ε, we can obtain a set of coupled equations for a given pair (n,l). Since we do not find it interesting to write them down in a general form, we content ourselves to present a step-by-step analysis which, though analogous to the previous kinetic treatment of whistlers (Kako et al 1973), makes it clear that for the O-mode (as well as the X-mode) the second-order second-harmonic component does not vanish.

2-1 Linear Dispersion relation $n=|\ell|=1$

Taking account of (3), we obtain from (1)

$$d_1(\psi) f_{\alpha,1}^{(1)} = i \frac{\omega}{B_0} \vec{A}_1 \cdot \vec{L} F_\alpha, \quad d_\ell(\psi) \equiv \frac{\partial}{\partial \psi} + i \frac{\ell}{\omega_{c\alpha}} (\omega - kv_z) \quad (6)$$

where the vector $\hat{\vec{A}}$ and the vectorial operator $\vec{L}$ are defined as

$$\hat{A}_x = (1 - N^2)A_x, \quad \hat{A}_y = A_y \text{ and } \quad \hat{A}_z = A_z \tag{7a}$$

$$L_x = \frac{\partial}{\partial v_x}, \quad L_y = \frac{\partial}{\partial v_y} - \frac{k}{\omega}\frac{\partial}{\partial \psi} \text{ and } L_x = \frac{1}{\omega}\{(\omega - kv_x)\frac{\partial}{\partial v_z} + kv_z \frac{\partial}{\partial v_x}\} \tag{7b}$$

Now introducing $G_{\ell k}(\psi, \psi') = G_{\ell k}(\psi)\, G_{\ell k}^{*}(\psi')$ with $G_{\ell k}(\psi) = \exp[i\ell(kv_{\perp}\sin\psi - \omega\psi)/\omega_{c\alpha}]$, we obtain

$$f^{(1)}_{\alpha,1} = i\,\frac{\omega}{B_0}\,\hat{\vec{A}}^{(1)}_1 \cdot \vec{P}_\alpha(1) \tag{8}$$

where $\vec{P}_\alpha(\ell) \equiv \vec{P}_\alpha(\ell k, \vec{v}) = \int^{\psi} d\psi'\; G_{\ell k}(\psi, \psi')\vec{L}\, F_\alpha$. Unless specified, $\vec{P}_\alpha$ denotes $\vec{P}_\alpha(1)$ in the sequel. Substitution of (8) into the first-order version of (2) and (3) yields

$$\overleftrightarrow{D}(k, \omega) \cdot \hat{\vec{A}}^{(1)}_1 = 0 \tag{9}$$

where the propagation tensor $\overleftrightarrow{D}(k,\omega)$ is defined by

$$\overleftrightarrow{D}(k,\omega) = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} - N^2 & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} - N^2 \end{pmatrix} \tag{10}$$

in which

$$\varepsilon_{\nu\mu} = \delta_{\nu\mu} + \chi_{\nu\mu} \tag{11a}$$

$$\chi_{\nu\mu} = i \sum_{\alpha} \frac{\omega^2_{p\alpha}}{\omega\, \omega_{c\alpha}} \int d\vec{v} \quad v_\nu\, P_{\alpha,\mu} \tag{11b}$$

and N is the refractive index. Since $F_{\alpha}(\vec{v})$ is assumed to be an even function of v_z, $P_{\alpha,z}$ is then odd and we have $\varepsilon_{xz}=\varepsilon_{zx}=\varepsilon_{yz}=\varepsilon_{zy}=0$. The determinant of $\overleftrightarrow{D}(k,\omega)$ gives rise to two dispersion relations

$$\det \overleftrightarrow{D}_X(k,\omega) \equiv \varepsilon_{xx}(\varepsilon_{yy} - N^2) - \varepsilon_{xy}\varepsilon_{yx} = 0 \qquad \text{X-mode} \quad (12)$$

and

$$D_O(k,\omega) \equiv D_{zz} = \varepsilon_{zz} - N^2 = 0 \qquad \text{O-mode} \quad (13)$$

Since we are concerned with the 0-mode in the present article, $\det \overleftrightarrow{D}_X(k,\omega) \neq 0$, leading to $A^{(1)}_{x,1} = A^{(1)}_{y,1} = 0$. With this in mind, (8) is reduced to

$$f^{(1)}_{\alpha,1} = \frac{e_\alpha}{m_\alpha c} A^{(1)}_{z,1} \left(\frac{\partial}{\partial v_z} F_\alpha + i \frac{k v_z}{\omega_{c\alpha}} P_{\alpha,x}\right) \qquad (14)$$

The dispersion relation turns out to be

$$D_O(k,\omega) \equiv 1 - N^2 - \sum_\alpha \frac{\omega^2_{p\alpha}}{\omega^2} + \Delta\chi = 0 \qquad (15)$$

where $\Delta_{zz} = i \sum_\alpha \frac{\omega^2_{p\alpha}}{\omega^2} \frac{k}{\omega_{c\alpha}} \int d\vec{v} \; v_z^2 \; P_{\alpha,x}$ is a portion of the permittivity which accounts for the thermal effect of constituent particles. The dispersion characteristics of the 0-mode in a hot magnetoplasma have already been completely studied (see, for instance, Dnestrovskii et al 1961, Clemmow & Dougherty 1969) and explicit expressions for the components of the dielectric tensor are also well known for a Maxwellian plasma (e.g., Furutani & Kalman 1965), which we do not reproduce here.

2-2 Second-order "slow" mode n=2, ℓ=0

Setting n=2 and ℓ=0 in (1), $f^{(2)}_{\alpha,0}$ is found to obey the equation

$$d_0(\psi) f_{\alpha,0}^{(2)} = -i \frac{\omega}{B_0} A_{z,1}^{(1)*} L_z f_{\alpha,1}^{(1)} + \text{c.c.} \tag{16}$$

which is integrated to give

$$f_{\alpha,0}^{(2)} = H + K \tag{17}$$

where

$$H = \frac{\omega_{c\alpha}}{B_0{}^2} |A_{z,1}^{(1)}|^2 \{ i \frac{\partial}{\partial v_z} (k v_z P_{\alpha,x}) + \frac{(k v_z)^2}{\omega_{c\alpha}} \int^{\psi} d\psi' \frac{\partial}{\partial v'_x} P_{\alpha,x} \}$$

and (17.a)

$$K = \frac{\omega_{c\alpha}^2}{B_0{}^2} |A_{z,1}^{(1)}|^2 [\frac{\partial^2}{\partial v_z^2} F_\alpha - i \frac{k v_z}{\omega_{c\alpha}} \frac{v_z}{v_\perp} \frac{\partial}{\partial v_\perp} \frac{\partial}{\partial \omega} \{ \omega \int_0^{2\pi} \frac{d\psi}{2\pi} P_{\alpha,x} \}]$$

(17.b)

and c.c. denotes the complex conjugate. Detailed derivation of K is given in Appendix A. Inserting now $f_{\alpha,0}^{(2)}$ into (2) with n=2 and ℓ=0, we can show that it automatically satisfies $\vec{J}_0^{(2)} = \rho_0^{(2)} = 0$.

2-3 Compatibility Condition n=2, ℓ=1

Turning to the components with ℓ=1, we obtain from (1)

$$d_1(\psi) f_{\alpha,1}^{(2)} = \frac{1}{B_0} [i\omega \vec{A}_1^{(2)} \cdot \vec{L} F_\alpha + \frac{1}{B_0} (\frac{\partial}{\partial \sigma} \frac{\partial}{\partial \omega} - \frac{\partial}{\partial \eta} \frac{\partial}{\partial k}) \{ \omega A_{z,1}^{(1)} L_z F_\alpha \}]$$

$$+ \frac{1}{\omega_{c\alpha}} (\frac{\partial}{\partial \sigma} + v_x \frac{\partial}{\partial \eta}) f_{\alpha,1}^{(1)} \tag{18}$$

To solve (18) which, though similar to (5), now contains the source terms (second and third terms of the right-hand side) depending on ψ through v_z and $\partial/\partial v_x$, the following identities are useful

$$\int^{\psi} d\psi' \; G_k(\psi,\psi') P_{\alpha,x}(k,\psi') = i \, \omega_{c\alpha} \frac{\partial}{\partial \omega} P_{\alpha,x} \tag{19}$$

$$\int^{\psi} d\psi' \; v'_x \, G_k(\psi,\psi') P_{\alpha,x}(k,\psi') = -i \, \omega_{c\alpha} \frac{\partial}{\partial k} P_{\alpha,x}$$

Substitution into (18) of $f_{\alpha,1}^{(1)}$ given by (8) yields

$$f_{\alpha,1}^{(2)} = \frac{1}{B_0}[i\omega \vec{A}_1^{(2)} \cdot \vec{P}_\alpha(k,\vec{v}) - v_z(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})\{kA_{z,1}^{(1)}P_{\alpha,x}(k,\vec{v})\}] \tag{20}$$

which then gives, in view of (16), the current density associated with $f_{\alpha,1}^{(2)}$

$$\frac{4\pi c}{\omega^2}\vec{J}_1^{(2)} = \overleftrightarrow{\chi}\cdot\hat{A}_1^{(2)} + i\hat{z}\frac{1}{\omega^2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})\{\omega^2(\Delta\chi_{zz})A_{z,1}^{(1)}\} \tag{21}$$

where $\hat{z}$ is a unit vector directed to the positive z-axis. Correspondingly an equation obtained from (2) becomes

$$(1-N^2)\vec{A}_1^{(2)} + \frac{i}{\omega^2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})\{\omega^2(1-N^2)\vec{A}_1^{(1)}\} + \frac{4\pi c}{\omega^2}\vec{J}_1^{(2)} = 0 \tag{22}$$

In the light of (21), the x- and y- components of (22) gives

$$\overleftrightarrow{D}(k,\omega)\cdot\vec{A}_{\perp,1}^{(2)} = 0 \quad ,$$

which gives rise to the relations $A_{x,1}^{(2)} = A_{y,1}^{(2)} = 0$, where the subscript $\perp$ stands for the components normal to the z-axis. Now the z-component of (22) yields

$$(\varepsilon_{zz} - N^2)A_{z,1}^{(2)} + \frac{i}{\omega^2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})[\omega^2(1+\Delta\chi_{zz} - N^2)A_{z,1}^{(1)}] = 0 \tag{23}$$

We then see that the coefficient of $A_{z,1}^{(2)}$ drops out by virture of the dispersion relation and that those of $\partial A_{z,1}^{(1)}/\partial\sigma$ and $\partial A_{z,1}^{(1)}/\partial\eta$ can be easily identified as $i\,\partial D_0/\partial\omega$ and $-i\,\partial D_0/\partial k$, respectively, by direct differentiation of (15). From this it follows

$$\left(\frac{\partial}{\partial \sigma} + v_g \frac{\partial}{\partial \eta} \right) A_{z,1}^{(1)} = 0 \tag{24}$$

where $v_g = -(\partial D_0/\partial k)/(\partial D_0/\partial \omega)$ is the group velocity of an envelop of linear carriers. The above relation, called compatibility condition, imposes a (η,σ) dependence of $A_{z,1}^{(1)}$ which is determined as a general solution to (24)

$$A_{z,1}^{(1)}(\sigma,\eta,\zeta) = A_{z,1}^{(1)}(\eta - v_g\sigma,\ \zeta) \tag{25}$$

2-4 Second harmonic of the second order $n=2$, $\ell=2$

The second harmonic ($|\ell|=2$) of the second order is governed by

$$d_2(\psi) f_{\alpha,2}^{(2)} = i \frac{\omega}{B_0} \left(2\, \hat{\vec{A}}_2 \cdot \vec{L}\, F_\alpha + A_{z,1}^{(1)} L_z f_{\alpha,1}^{(1)} \right) \tag{26}$$

which formally gives

$$f_{\alpha,2}^{(2)} = i \frac{\omega}{B_0} [2\hat{\vec{A}}_2^{(2)} \cdot \vec{P}_\alpha(2) + i\frac{\omega}{B_0} A_{z,1}^{(1)^2} \int^{\psi} d\psi' G_{2k}(\psi,\psi') L_z P_{\alpha,z}(1)] \tag{27}$$

Now, setting $n=\ell=2$ in (2), we obtain the equation

$$4(1 - N^2)\vec{A}_2^{(2)} + \frac{4\pi c}{\omega^2} \vec{J}_2^{(2)} = 0$$

which is separable into two parts. The one is

$$D_0(2k,2\omega) A_{z,2}^{(2)} = 0 \tag{28a}$$

from which $A_{z,2}^{(2)} = 0$ by virtue of the fact that $D_0(2k,2\omega) \neq 0$, when $D_0(k,\omega) = 0$. The other is

$$\overleftrightarrow{D}_x(2k,2\omega) \cdot \hat{\vec{A}}_{\perp,2}^{(2)} = \frac{k}{B_0} A_{z,1}^{(1)^2} \vec{C}(k,\omega) \tag{28b}$$

where the j component of C is defined by

$$C_j = \frac{1}{4k} \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega_{c\alpha}} \int d\vec{v}\, v_j \int^{\psi} d\psi' G_{2k}(\psi,\psi') L_z P_{\alpha,z}(1) \qquad (29)$$

It is now obvious that the second harmonic is polarized in the x-y plane and, as (28b) suggests, can be written as

$$\hat{\vec{A}}^{(2)}_{\perp,2} = \frac{k}{B_0} A_{z,1}^{(1)^2}\, \vec{\Gamma}\,(k,\omega) \qquad (30)$$

Explicit expressions of $\vec{\Gamma}$ are given in appendix B.

2-5 Fundamental Mode of the third order n=3, ℓ=1

To this order we only need to consider the fundamental mode. Setting n=3 and ℓ=1 in (1), we obtain

$$d_1(\psi) f_{\alpha,1}^{(3)} = S_L + S_{NL} \qquad (31)$$

in which the linear source term S_L is given by

$$S_L = i \frac{\omega}{B_0} \hat{\vec{A}}_1^{(3)} \cdot \vec{L}\, F_\alpha - \frac{1}{B_0}[\{\frac{\partial}{\partial\sigma} A_{z,1}^{(2)} + v_x(\frac{\partial}{\partial\eta} A_{x,1}^{(2)} +$$

$$\frac{\partial}{\partial\zeta} A_{x,1}^{(1)})\}\frac{\partial}{\partial v_z} F_\alpha - v_z(\frac{\partial}{\partial\eta} A_{z,1}^{(2)} + \frac{\partial}{\partial\zeta} A_{z,1}^{(1)})\frac{\partial}{\partial v_z} F_\alpha]$$

$$+ \frac{1}{\omega_{c\alpha}}[\frac{\partial}{\partial\sigma} f_{\alpha,1}^{(2)} + v_x(\frac{\partial}{\partial\eta} f_{\alpha,1}^{(2)} + \frac{\partial}{\partial\zeta} f_{\alpha,1}^{(1)})] \qquad (32)$$

and the nonlinear term S_{NL} by

$$S_{NL} = i \frac{\omega}{B_0} \{2\, \hat{\vec{A}}_2^{(2)} \cdot \vec{L}\, f_{\alpha,1}^{(1)*} - A_{z,1}^{(1)*} L_z\, f_{\alpha,2}^{(2)}$$

$$+ A_{z,1}^{(1)}\, L_z\, f_{\alpha,0}^{(2)}\} \qquad (33)$$

Therefore a solution to (31) should be a superposition of two terms

$$f_{\alpha,1}^{(3)} = f_{\alpha,1}^{(3)L} + f_{\alpha,1}^{(3)NL}$$

where $f_{\alpha,1}^{(3)L}$ and $f_{\alpha,1}^{(3)NL}$ correspond to S_L and S_{NL}, respectively. Consider first the linear term. Upon substituting (14) and (20) into (31), the linear part of (31) is readily integrated to give

$$f_{\alpha,1}^{(3)L} = \frac{1}{B_O}[i\omega\hat{\vec{A}}_1^{(3)}\cdot\vec{P}_\alpha - v_z\{(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})(A_{z,1}^{(2)}kP_{\alpha,x})$$
$$- \frac{\partial}{\partial\zeta}A_{z,1}^{(1)}\frac{\partial}{\partial k}(k\,P_{\alpha,x}) + \frac{i}{2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})^2(A_{z,1}^{(1)}kP_{\alpha,x})\}] \tag{34}$$

where we have made use of the identities

$$\int^\psi d\psi'\ G_k(\psi,\psi')\frac{\partial}{\partial\omega}P_{\alpha,x} = i\ \frac{\omega_{c\alpha}}{2}\ \frac{\partial^2}{\partial\omega^2}\ P_{\alpha,x} \tag{35a}$$

$$\int^\psi d\psi'\ G_k(\psi,\psi')(\frac{\partial}{\partial k} - v_x'\ \frac{\partial}{\partial\omega})P_{\alpha,x} = i\omega_{c\alpha}\ \frac{\partial^2}{\partial\omega\partial k}\ P_{\alpha,x} \tag{35b}$$

and

$$\int^\psi d\psi'\ G_k(\psi,\psi')v_x'\ \frac{\partial}{\partial k}\ P_{\alpha,x} = -i\ \frac{\omega_{c\alpha}}{2}\ \frac{\partial^2}{\partial k^2}\ P_{\alpha,x} \tag{35c}$$

The linear part of the current density $\underset{\sim}{J}_1^{(3)L}$ is then given by

$$\frac{4\pi c}{\omega^2}J_{1,z}^{(3)L} = \chi_{zz}\ A_{z,1}^{(3)} + \frac{i}{\omega^2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})(\omega^2\Delta\chi_{zz}A_{z,1}^{(2)})$$
$$- i\ \frac{\partial}{\partial k}\Delta\chi_{zz}\frac{\partial}{\partial\zeta}\ A_{z,1}^{(1)} - \frac{1}{2\omega^2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})^2(\omega\Delta\chi_{zz}A_{z,1}^{(1)}) \tag{36}$$

Substitution of this expression into the corresponding equation (2) yields

$$(\varepsilon_{zz} - N^2)A_{z,1}^{(3)} + \frac{i}{\omega^2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial})\{\omega^2(1+\Delta\chi_{zz})A_{z,1}^{(2)}\} + (\frac{N^2}{R^2}\frac{\partial^2}{\partial\eta^2} - \frac{1}{\omega^2}\frac{\partial^2}{\partial\sigma^2})A_{z,1}^{(1)}$$
$$+ i\ \frac{\partial}{\partial k}(N^2 - \Delta\chi_{zz})\frac{\partial}{\partial\zeta}A_{z,1}^{(1)} - \frac{1}{2\omega^2}(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k})^2(\omega^2\Delta\chi_{zz}A_{z,1}^{(1)})$$
$$= -\frac{4\pi c}{\omega^2}\ \vec{J}_{z,1}^{(3)NL} \tag{37}$$

where the nonlinear current $J_{z,1}^{(3)NL}$ has no compact expression as (36). Remark in (37) that the coefficient of $A_{z,1}^{(3)}$ vanishes due to the linear dispersion. That of $A_{z,1}^{(2)}$ has been shown previously to vanish by virtue of the compatibility condition. Finally the coefficient of $\partial A_{z,1}^{(1)}/\partial\zeta$ can be shown to be equal to $-\partial D_O/\partial k$. From these considerations, (37) is reduced to

$$i \frac{\partial}{\partial k} D_O \frac{\partial}{\partial\zeta} A_{z,1}^{(1)} + \frac{1}{2}\left(\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\omega} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial k}\right)^2 D_O A_{z,1}^{(1)} = \frac{4\pi c}{\omega^2} J_{z,1}^{(3)NL} \tag{38}$$

This equation can be simplified farther by introducing a set of new variables $\xi=\eta - v_g\sigma$ and $\tau=\zeta/v_g$. The result is

$$\left(i\frac{\partial}{\partial\tau} + \frac{1}{2} v_g' \frac{\partial^2}{\partial\xi^2}\right) A_{z,1}^{(1)} = \frac{4\pi c}{\omega^2}\left(\frac{\partial}{\partial\omega}D_O\right)^{-1} J_{z,1}^{(3)NL} \tag{39}$$

where use has been made of the identity

$$- v_g' \frac{\partial}{\partial\omega} D_O = \left(v_g^2 \frac{\partial^2}{\partial\omega^2} + 2v_g \frac{\partial^2}{\partial\omega\partial k} + \frac{\partial^2}{\partial k^2}\right) D_O \tag{40}$$

§3. Calculation of the Nonlinear Current Density $J_{z,1}^{(3)NL}$

In order to evaluate $J_{z,1}^{(3)NL}$, we substitute into (33) the expressions of $f_{\alpha,1}^{(1)}$, $f_{\alpha,0}^{(2)}$, $f_{\alpha,2}^{(2)}$ and $\hat{A}_{\perp,2}^{(2)}$ given, respectively, by (14), (17), (27) and (30), and obtain

$$f_{\alpha,1}^{(3)NL} = i \frac{\omega}{B_O} \int^{\psi} d\psi' \, G_k(\psi,\psi') [2\hat{\vec{A}}_2^{(2)} \cdot \vec{L} \, f_{\alpha,1}^{(1)*} - i\frac{\omega}{B_O} A_{z,1}^{(1)*} L_z$$
$$\cdot \{2\vec{A}_2^{(2)} \cdot \vec{P}_\alpha(2) + i \frac{\omega}{B_O} A_{z,1}^{(1)^2} \int^{\psi'} d\psi'' \, G_{2k}(\psi',\psi'') L_z P_{\alpha,z}\}$$
$$+ A_{z,1}^{(1)} \, L_z \, f_{\alpha,0}^{(2)}] \tag{41}$$

Denote by S_σ (σ=A,B,C &D) the part of $(4\pi c/\omega^2)J^{(3)NL}_{z,1}$ corresponding to f_σ, where f_A, f_B, f_c and f_D represent, respectively, the first, second, third and fourth terms in the square bracket of (41). Then, after simple algebra, we get

$$S_A = i\frac{2}{B_O} A^{(1)*}_{z,1} \sum_\alpha \frac{\omega^2_{p\alpha}}{n_\alpha \omega^2} \int d\vec{v}[-\omega\{\hat{A}^{(2)}_{x,2} P_{\alpha,x}(1) + \hat{A}_{y,2}P_{\alpha,y}(1)\}$$
$$- i\omega \frac{k\ v_z^2}{\omega_{c\alpha}} \int^\psi d\psi'\ G_k(\psi,\psi')\{\hat{A}^{(2)}_{x,2} \frac{\partial}{\partial v'_x} + A^{(2)}_{y,2} \frac{\partial}{\partial v'_y}\}P^*_{\alpha,x}(1)$$
$$+ i \frac{(k\ v_z)^2}{\omega_{c\alpha}} A^{(2)}_{y,2} \int^\psi d\psi'\ G_k(\psi,\psi') \frac{\partial}{\partial \psi'} P^*_{\alpha,x}(1)] \qquad (42.a)$$

$$S_B = \frac{2}{B_O} A^{(1)*}_{z,1} \sum_\alpha \frac{\omega^2_{p\alpha}}{n_\alpha \omega^2} \int d\vec{v}[-i\{\hat{A}^{(2)}_{x,2}(P_{\alpha,x}(2) - P_{\alpha,x}(1)) +$$
$$A^{(2)}_{y,2}(P_{\alpha,y}(2) - P_{\alpha,y}(1))\} + \frac{kv_z^{\ 2}}{\omega_{c\alpha}} \int^\psi d\psi' G_k(\psi,\psi') \frac{\partial}{\partial v_x'} \{\hat{A}^{(2)}_{x,2}P_{\alpha,x}(2)$$
$$+ A^{(2)}_{y,2}\ P_{\alpha,y}\ (2)\}] \qquad (42.b)$$

$$S_C = -\frac{k}{B_O^{\ 2}}|A^{(1)}_{z,1}|^2 A^{(1)}_{z,1} \sum_\alpha \frac{\omega^2_{p\alpha}}{n_\alpha \omega^2} \int d\vec{v}[i\omega_{c\alpha}\{2P_{\alpha,x}(1) - P_{\alpha,x}(2)\}$$
$$+ k\ v_z^{\ 2} \int^\psi d\psi'\ G_k(\psi,\ \psi')\frac{\partial}{\partial v_x'}\{3P_{\alpha,x}(1) + P_{\alpha,x}(2)\}$$
$$- kv_z^{\ 2} \int^\psi d\psi'\ G_{2k}(\psi,\psi')\frac{\partial}{\partial v_x'} P_{\alpha,x}(1) - i\frac{(kv_z)^2}{\omega_{c\alpha}} \int d\psi'$$
$$\times\ G_k(\psi,\psi')\frac{\partial}{\partial v_x'} \int^{\psi'} d\psi''\ G_{2k}(\psi',\psi'')\frac{\partial}{\partial v_x''} P_{\alpha,x}(1)] \qquad (42c)$$

and

$$S_D = \frac{k}{B_O^{\ 2}}|A^{(1)}_{z,1}|^2 A^{(1)}_{z,1} \sum_\alpha \frac{\omega^2_{p\alpha}}{n_\alpha \omega^2} \int d\vec{v}[2i\omega_{c\alpha}P_{\alpha,x}(1) +$$
$$3kv_z^{\ 2} \int^\psi d\psi' G_k(\psi,\ \psi')\frac{\partial}{\partial v_x'} P_{\alpha,x}(1) - kv_z^2 \int^\psi d\psi' G_k(\psi,\psi')\frac{\partial}{\partial v_x'} P^*_{\alpha,x}(1)$$
$$+ i\frac{(kv_z^{\ 2})^2}{\omega_{c\alpha}} \int^\psi d\psi' G_k(\psi,\psi')\frac{\partial}{\partial v_x'} \int^{\psi'} d\psi''\ \frac{\partial}{\partial v_x''}\{P_{\alpha,x}(1) + P^*_{\alpha,x}(1)\}$$
$$+\ (kv_z^2)^2 \int^\psi\ d\psi' G_k(\psi,\psi')\frac{\partial}{\partial v_x'}\ \frac{\partial}{\partial \omega}\ \{\frac{\omega}{kv_\perp}\ \frac{\partial}{\partial v_\perp}\ \langle P_{\alpha,x}(1)\rangle\} \qquad (42d)$$

where the bracket <P> means an angular average

$$< P >= \frac{1}{2\pi} \int_0^{2\pi} P(\psi)\, d\psi$$

By virtue of (30), we can now put $(4\pi c/\omega^2) J_{z,1}^{(3)NL}$ into the form

$$\frac{4\pi c}{\omega^2} J_{z,1}^{(3)NL} = q^{NL} \, |A_{z,1}^{(1)}|^2 A_{z,1}^{(1)} \tag{43}$$

where q^{NL} is the nonlinear coupling coefficient which can be written as

$$q^{NL} = q^{I} + q^{II}$$

Obviously q^I comes from $S_A + S_B$ which involves the polarization vector $\vec{\Gamma}$ and represents the coupling between the second-order second-harmonic $\vec{A}_{\perp,2}^{(2)}$ and the fundamental. With regards q^{II}, it represents all other couplings between the slow mode and the fundamental. Their explicit expressions are

$$\begin{aligned} q^{I} = {} & 2 \frac{k}{B_O{}^2} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha \omega^2} \int d\vec{v} \int -i\omega\{\Gamma_x P_{\alpha,x}(2) + \Gamma_y P_{\alpha,y}(2)\} \\ & + k v_z{}^2 \frac{\omega}{\omega_{c\alpha}} \int^{\psi} d\psi' G_k(\psi,\psi') \{\Gamma_x \frac{\partial}{\partial v'_x}(P^*_{\alpha,x}(1) + P_{\alpha,x}(2)) \\ & + \Gamma_y \left(\frac{\partial}{\partial v'_y} P^*_{\alpha,x}(1) + \frac{\partial}{\partial v'_x} P_{\alpha,x}(2)\right)\} \\ & - \frac{(k\, v_z)^2}{\omega_{c\alpha}} \Gamma_y \int^{\psi} d\psi' \, G_k(\psi,\psi') \frac{\partial}{\partial \psi'} P^*_{\alpha,x}(1)] \end{aligned} \tag{44a}$$

and

$$\begin{aligned} q^{II} = {} & \frac{k^2}{B_O{}^2} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha \omega^2} \int d\vec{v} [i \frac{\omega_{c\alpha}}{k} P_{\alpha,x}(2) + v_z{}^2 \int^{\psi} d\psi' G_{2k}(\psi,\psi') \frac{\partial}{\partial v'_x} P_{\alpha,x}(1) \\ & - v_z{}^2 \int^{\psi} d\psi' G_k(\psi,\psi') \frac{\partial}{\partial v'_x} \{P^*_{\alpha,x}(1) + P_{\alpha,x}(2)\} \\ & + i \frac{k\, v_z{}^4}{\omega_{c\alpha}} \int^{\psi} d\psi' \, G_k(\psi,\psi') \frac{\partial}{\partial v'_x} \int^{\psi'} d\psi'' \{G_{2k}(\psi',\psi'') \frac{\partial}{\partial v''_x} P_{\alpha,x}(1) \\ & + \frac{\partial}{\partial v''_x}(P_{\alpha,x}(1) + P^*_{\alpha,x}(1))\} + k v_z{}^4 \int^{\psi} d\psi' G_k(\psi,\psi') \frac{\partial}{\partial v'_x} \frac{\partial}{\partial \omega}\{\frac{\omega}{k v_\perp} \frac{\partial}{\partial v_\perp} < P_{\alpha,x}(1) >\}] \end{aligned} \tag{44b}$$

Finally we obtain from (39) and (43) the NLS equation which reads

$$(i \frac{\partial}{\partial \tau} + \frac{1}{2}v_g' \frac{\partial^2}{\partial \xi^2})A_{z,1}^{(1)} - (\partial D_O/\partial \omega)^{-1} q^{NL}|A_{z,1}^{(1)}|^2 A_{z,1}^{(1)} = 0 \tag{45}$$

A solution to (45) has been extensively studied by a number of authors in the past (e.g., Zakharov and Shabat 1971) and may be subject to the modulational instability whenever the product of two coefficients, $v_g'/2$ and $-(\partial D_O/\partial \omega)^{-1} q^{NL}$, becomes negative.

§4. Stability of the O-mode envelop in a cold electron plasma

In order to determine a frequency range over which the O-mode envelop can be stable, explicit expressions for the dispersion coefficient and the nonlinear coupling coefficient are required. Their derivation involving thermal correction up to first order in λ_α, with $\lambda_\alpha = (ka_\alpha/\omega_{c\alpha})^2$, is given in some detail in Appendices B and C.

(a) Nonlinear coupling coefficient   The results are

$$q^I = 2\frac{k^2}{B_O^2} \sum_\alpha \frac{R_\alpha}{\nu_\alpha(4\nu_\alpha^2-1)} [2\Gamma_x\{1+\lambda_\alpha \frac{3(2\nu_\alpha^2-1)}{(\nu_\alpha^2-1)^2}\} + \frac{i}{\nu_\alpha}\Gamma_y \times\{1+\lambda_\alpha \frac{13\nu_\alpha^2-7}{(\nu_\alpha^2-1)^2}\}] \tag{46a}$$

and

$$q^{II} = -2\frac{k^2}{B_O^2} \sum_\alpha \frac{R_\alpha}{\nu_\alpha^2(4\nu_\alpha^2-1)} \{1 + \frac{1}{2}\lambda_\alpha \frac{6\nu_\alpha^2-1}{(\nu_\alpha^2-1)^2}\} \tag{46b}$$

with $R_\alpha = \frac{\omega^2_{p\alpha}}{\omega^2_{c\alpha}}$ and $\nu_\alpha = \omega/\omega_{c\alpha}$. Now, since high-frequency O-mode can not be affected by the ion dynamics, we content ourselves to consider only the electron contribution to the polarization vector $\vec{\Gamma}$, the two components of which are given as

$$\Gamma_x = -\frac{1}{D}\frac{3R^2}{8\nu^3(4\nu^2-1)}\{1+\frac{2}{3}\frac{\lambda}{(4\nu^2-1)(\nu^2-1)^2}(66\nu^4-62\nu^2+11)\} \tag{47a}$$

$$\Gamma_y = \frac{i}{D}\frac{R}{4\nu^2(4\nu^2-1)}[1+\frac{\lambda}{(4\nu^2-1)(\nu^2-1)^2}\{-(5\nu^2-2)R + 2(7\nu^2-4)(4\nu^2-1)\}] \tag{47b}$$

where

$$D=-\frac{R}{4\nu^2(4\nu^2-1)}[3R-4(3\nu^2-1)+\frac{12\lambda}{\nu^2-1}\{2R-(5\nu^2-1)\}] \tag{47c}$$

Next, although thermal correction may be important at the immediate vicinity of a pseudo-resonance at $\nu^2=1$ (in any way, there is no cyclotron resonance when $\vec{k} \perp \vec{B}_0$) and at $\nu^2 = 1/4$ (resonance at the subharmonic), it is legitimate to neglect these resonant effects over nearly all frequency ranges whenever $\lambda_\alpha \ll 1$. Substitution of (47) into (46) then yields

$$q^{NL} = -\frac{1}{2}\frac{k^2}{B_0^2}\frac{R}{\nu^2(\nu^2-\nu_0^2)} \tag{48}$$

where $\nu_0^2 = \frac{1}{3}+\frac{R}{4}$. On the other hand, since $kv_g'/v_g = 1-(v_g/c)^2$ with $v_g=(\omega/k)N^2$, we have

$$v_g' = \frac{\omega}{k^2}N^2(1-N^2) \tag{49}$$

which is non-negative. Finally, by virtue of (48) and (49), the NLS equation (45) can be put into the form

$$\left(i \frac{\partial}{\partial u} + p \frac{\partial^2}{\partial v^2}\right) a^{(1)} + q|a^{(1)}|^2 \quad a^{(1)} = 0 \tag{50}$$

where the new variables $u = \omega\tau$ and $v = k\xi$ are introduced in (45) and $a^{(1)}=kA_{z,1}^{(1)}/B_O$. Also have we defined $p=N^2(1-N^2)/2$ and and $q=\frac{1}{4}\frac{R}{\nu^2(\nu_0{}^2-\nu^2)}$. We can then see that the stability of the O-mode envelop depends solely on the sign of q, because of the positiveness of p. Fig.(a), (b), (c) and (d) show the qualitative behaviours of p and q as a function of ν^2. The p curve starts from zero at $\nu^2=R(p>0)$, takes a maximum value 1/8 at $\nu^2 = 2R$, irrespective of R, and tends to 0 as $\nu^2 \to \infty$. The q curve has a minimum value $R/\nu_0{}^4$ at $\nu^2=\nu_0{}^2/2$ and presents two asymptotes $\nu^2=0$ and ν_0^2. Consequently the following four cases are envisaged:

a) $\nu_0{}^2 < R$. In this case p and q are of the opposite sign and the envelop is modulationally unstable.

b) $R<\nu_0{}^2<2R$. Qualitatively there exists a frequency range $R\leq\nu^2<\nu_0{}^2$, over which p and q are of the same sign and thus the envelop is stable. Owing to the fact that q is much larger than p, however, we can hardly expect to observe a stable formation of the wave packet, in that the steepening effect (nonlinearity) will overcome the broadening effect (dispersion) of the profile.

c) $R<\nu_0{}^2/2<2R$ and d) $2R<\nu_0{}^2/2$ In these cases, too, p can not be of the same order as q, though they are of the same sign. This is because we have no real solution ν^2 to the equation p=q, since

$$\nu^2 - R = [-\{2(R - \nu_0{}^2) + 1\}\pm\{(2R - 2\nu_0{}^2+1)^2 - 8R\}^{1/2}]/4 ,$$

where ν^2 is a solution of p=q, is always negative.

To conclude this section, we may give a rather pessimistic prediction that, although there is a frequency range over which the O-mode envelop is stable against modulational isntabilities, the envelop will quickly collapse due to the predominant steepening effect ($p \ll q$). In this respect it is an interesting problem to investigate quantitatively a relationship between the ratio p/q (or its reciprocal) and a "life time" of the envelop.

§5. Concluding discussions

The NLS equation describing the nonlinear modulation of the O-mode has been explicitly established within the framework of the well-known reductive perturbation technique and the limiting expressions of the dispersion and the nonlinear coupling coefficients for a cold electron plasma are derived. It is noteworthy to mention that a structure of the nonlinear couping coefficient of the NLS equation for the O-mode is in some way different from that for the whistler. While in the case of whistlers the second-order second-harmonic components are not induced, the components $\vec{A}^{(2)}_{\perp,2}$ transverse to $\vec{B}_0$, are excited in the present case and give an additional contribution to the nonlinear coupling coefficient. With regards the "slow" mode $\vec{A}^{(2)}_0$, emphasis should be placed on the structural difference of governing equations which determine $f^{(2)}_{\alpha,0}$. In the whistler case (Kako et al 1973) those equations are written as

$$\frac{\partial}{\partial \psi} f^{(2)}_{\alpha,0} = 0 \qquad (51)$$

and $- \omega_{c\alpha}\frac{\partial}{\partial\psi} f_{\alpha,0}^{(3)} + (\frac{\partial}{\partial\sigma} + v_z \frac{\partial}{\partial\eta}) f_{\alpha,0}^{(2)} = \psi$-independent terms

$$+ \text{ periodic terms in } \psi \qquad (52)$$

from which they had deduced

$$(\frac{\partial}{\partial\sigma} + v_z \frac{\partial}{\partial\eta}) \; f_{\alpha,0}^{(2)} = \psi \text{ - independent terms} \qquad (53)$$

In the case of the O-mode we have obtained a couple of equations

$$\frac{\partial}{\partial\psi} f_{\alpha,0}^{(2)} = \text{ aperiodic terms in } \psi \qquad (54)$$

and

$$- \omega_{c\alpha} \frac{\partial}{\partial\psi} f_{\alpha,0}^{(3)} + (\frac{\partial}{\partial\sigma} + v_x \frac{\partial}{\partial\eta}) f_{\alpha,0}^{(2)} = (\text{periodic + aperiodic}) \text{ terms in } \psi \; , \qquad (55)$$

from which it follows

$$(\frac{\partial}{\partial\sigma} + v_x \frac{\partial}{\partial\eta}) f_{\alpha,0}^{(2)} = \text{ aperiodic terms in } \psi \qquad (56)$$

since $f_{\alpha,0}^{(3)}$ is required to be periodic in ψ. A non-zero integration constant, resulting from the integration of (54), has been uniquely determined by the procedure of taking an angular average of (56).

Also have we to notice that, in the previous paper written by one of us (Y.F.), there was an error of having completely neglected the contribution of $\vec{A}_{\perp,2}^{(2)}$ to q^{NL}. In the forthcoming paper we shall deal with the NLS equation for the extraordinary mode.

The hospitality extended to one of us (Y.F.), during his stay in summer 1976, by the Laboratoire de physique des plasmas, Université Paris-Sud (Orsay), where final part of this work has been acieved, is gratefully acknowledged.

Appendix A

Determination on the Integration Constant K.

When we integrate (16), we have to take care of an integration constant, which we denote by K. Note that K can not be set a priori equal to zero. The reason in that, on the one hand, it gives a non-zero contribution to the nonlinear coupling coefficient q [see (43b) and (43c)] and, on the other hand, it indeed ensures a posteriori the identities $\vec{J}_0^{(2)}=\rho_0^{(2)}=0$. In order to determine K, we first write

$$f_{\alpha,0}^{(2)} = H(\psi) + K \tag{A-1}$$

where $H(\psi)$, dependent of ψ, is given by (17a) of the text and then invoke an equation for $f_{\alpha,0}^{(3)}$ which reads

$$-\omega_{c\alpha}\frac{\partial}{\partial\psi} f_{\alpha,0}^{(3)} + \left(\frac{\partial}{\partial\sigma} + v_x\frac{\partial}{\partial\eta}\right) f_{\alpha,0}^{(2)} = R \tag{A-2}$$

where

$$R = \frac{e_\alpha}{m_\alpha}\left(\frac{\partial}{\partial\eta}\phi_0^{(2)}\frac{\partial}{\partial v_x} + \frac{1}{c}\frac{\partial}{\partial\sigma}\vec{A}_0^{(2)}\cdot\frac{\partial}{\partial\vec{v}}\right)F_\alpha + \frac{e_\alpha}{m_\alpha c}\Big[\Big\{\left(\frac{\partial}{\partial\sigma} + v_x\frac{\partial}{\partial\eta}\right)$$

$$A_{z,1}^{(1)*}\frac{\partial}{\partial v_z} - v_z\frac{\partial}{\partial\eta}A_{z,1}^{(1)*}\frac{\partial}{\partial v_x}\Big\} f_{\alpha,1}^{(1)} - i\omega A_{z,1}^{(2)} L_z f_{\alpha,1}^{(1)*}$$

$$+ i\omega A_{z,1}^{(1)*} L_z f_{\alpha,1}^{(2)} + c.c\Big] \tag{A-3}$$

Now we take the angular average of (A2) through the operation

$$\langle T(\psi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\psi T(\psi)$$

Recalling that $f_{\alpha,0}^{(3)}$ is a periodic function of ψ and that $\langle v_x K\rangle = K\langle v_x\rangle = 0$, we obtain immediately

Appendix B

Polarization Vector Associated with the second Harmonic

The amplitude of the second harmonic of the second order, given by (28b), can be explicitly evaluated in terms of the polarization vector $\vec{\Gamma}$ defined through (30). Since $\vec{\Gamma}$ is colinear with $\vec{C}_\perp$, it is sufficient to calculate the latter by direct substitution into (29) of the explicit expressions of L_z and $P_{\alpha,z}$ defined, respectively, by (7) and (9), i.e., we have

$$\vec{C}_\perp = \frac{1}{4k} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha \omega^2 \omega_{c\alpha}} \int d\vec{v}\, \vec{v}_\perp \int^\psi d\psi' G_{2k}(\psi,\psi') \{(\omega - kv'_x)\frac{\partial}{\partial v_z}$$

$$+ kv_z \frac{\partial}{\partial v'_x}\} \times \int^{\psi'} d\psi'' G_k(\psi',\psi'') \{(\omega - kv_x'')\frac{\partial}{\partial v_z} + kv_z \frac{\partial}{\partial v_x''}\} F_\alpha \tag{B1}$$

The term associated with the operator $(\omega - kv'_x)\partial/\partial v_z$ vanishes upon integration by parts over v_z. We then obtain

$$\vec{C}_\perp = \frac{1}{4} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha \omega\, \omega_{c\alpha}} \int d\vec{v}\, v_z\, \vec{v}_\perp \int^\psi d\psi'\, G_{2k}(\psi,\psi') \frac{\partial}{\partial v'_x}$$

$$\times \int^{\psi'} d\psi'' G_k(\psi',\psi'') \{(\omega - kv''_x)\frac{\partial}{\partial v_z} + kv_z \frac{\partial}{\partial v_x''}\} F_\alpha$$

$$= \frac{1}{4} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha \omega\omega_{c\alpha}} \int d\vec{v}\, v_z\, \vec{v}_\perp \{-i\omega_{c\alpha} \frac{\partial}{\partial v_z} \int^\psi d\psi' G_{2k}(\psi,\psi') \frac{\partial}{\partial v'_x} F_\alpha$$

$$+ k\, v_z \int^\psi d\psi' G_{2k}(\psi,\psi') \frac{\partial}{\partial v'_x} \int^{\psi'} d\psi'' G_k(\psi',\psi'') \cos\psi'' \frac{\partial}{\partial v_\perp} F_\alpha\}$$

$$= \frac{1}{4} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha \omega\omega_{c\alpha}} [i\omega_{c\alpha} \int d\vec{v}\, \vec{v}_\perp (1 - \frac{v_z^2}{a_\alpha^2}) \int^\psi d\psi' G_{2k}(\psi,\psi') \frac{\partial}{\partial v'_x} F_\alpha$$

$$- \frac{\omega}{a_\alpha^2} \int d\vec{v}\, \vec{v}_\perp\, v_z^2 \int^\psi d\psi' G_{2k}(\psi,\psi') \frac{\partial}{\partial v'_x} \int^{\psi'} d\psi'' G_k(\psi',\psi'') F_\alpha]$$

$$= -\frac{1}{4} \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha \omega\omega_{c\alpha}} \int d\vec{v}_\perp\, \vec{v}_\perp \int^\psi d\psi' G_{2k}(\psi,\psi') \frac{\partial}{\partial v'_x} \int^{\psi'} d\psi'' G_k(\psi',\psi'') \hat{F}_\alpha \tag{B2}$$

where use has been made of the identity:

$(\omega-kv_x')G_k(\psi,\psi') = -i\omega_{c\alpha}\frac{\partial}{\partial\psi'}G_k(\psi,\psi')$ and we have assumed F_α to be Maxwellian. Also $\bar{F}_\alpha(\vec{v}_\perp) = \int_{-\infty}^{+\infty} d\,v_z F_\alpha(\vec{v}_\perp, v_z)$.

Finally, successive integration over ψ'', ψ' and $\vec{v}_\perp$ yields

$$\vec{C}_\perp = \frac{1}{4}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\omega_{c\alpha}}\frac{1}{\lambda_\alpha}\int_0^\infty d\mu\,\mu\exp(-\frac{\mu^2}{2\lambda_\alpha})\sum_{\ell,m=-\infty}^{\infty}\frac{1}{(\nu_\alpha-m)(2\nu_\alpha-\ell-m)}$$
$$\times\begin{pmatrix}\frac{\ell+m}{2\mu}J_{\ell+m}(2\mu)\\ -iJ'_{\ell+m}(2\mu)\end{pmatrix}[\ell\, J_\ell(\mu)\{J'_m(\mu)-\frac{\mu}{\lambda_\alpha}J_m(\mu)\}-mJ_m(\mu)J'_\ell(\mu)], \tag{B3}$$

where $\lambda_\alpha = (k\, a_\alpha/\omega_{c\alpha})^2$ and $\nu_\alpha=\omega/\omega_{c\alpha}$ and $a_\alpha{}^2=T_\alpha/m_\alpha$ is the squared thermal velocity. The dash denotes the derivative with respect to the argument. Thus we obtain for $\vec{\Gamma}$

$$\vec{\Gamma} = \overleftrightarrow{D}_X(2k,\ 2\omega)^{-1}\cdot\vec{C}_\perp \tag{B4}$$

where $\overleftrightarrow{D}_X{}^{-1}$ is the inverse matrix of $\overleftrightarrow{D}_X$.

In the small λ_α limit, we can expand the Bessel functions in ascending series of μ and integrate. The resulting expresions, correct to first order in λ_α, are

$$C_x = -\frac{1}{2}\sum_\alpha\frac{R_\alpha}{\nu_\alpha(4\nu_\alpha{}^2-1)}\{1+\lambda_\alpha\frac{3(2\nu_\alpha{}^2-1)}{(\nu_\alpha{}^2-1)^2}\} \tag{B5}$$

and

$$C_y = \frac{i}{4}\sum_\alpha\frac{R_\alpha}{\nu_\alpha{}^2(4\nu_\alpha{}^2-1)}\{1+\lambda_\alpha\frac{2(7\nu_\alpha{}^2-4)}{(\nu_\alpha{}^2-1)^2}\} \tag{B6}$$

$$R_\alpha = \omega_{p\alpha}^2/\omega_{c\alpha}^2$$

of which the limiting formulae for a cold plasma are quoted in the text. Finally we evaluate Γ_x and Γ_y up to first order in λ_α. We first need the explicit expressions for the components of the propagation tensor $\overleftrightarrow{D}_X(2) \equiv \overleftrightarrow{D}_X(2k,\ 2\omega)$

$$D_{XX} \equiv \varepsilon_{xx}(2) = 1 - \sum_\alpha \frac{R_\alpha}{4\nu_\alpha{}^2-1}\left(1 + \lambda_\alpha \frac{3}{\nu_\alpha^2-1}\right) \tag{B7}$$

$$D_{xy} \equiv \chi_{xy}(2) = -i \sum_\alpha \frac{R_\alpha}{2\nu_\alpha(4\nu_\alpha{}^2-1)} \left(1+\lambda_\alpha \frac{6}{\nu_\alpha{}^2-1}\right) = D^*_{yx}(2) \tag{B8}$$

and

$$D_{yy} \equiv \varepsilon_{yy}(2) - N^2 = \varepsilon_{yy}(2) - \varepsilon_{zz}(2)$$

$$= \sum_\alpha \frac{R_\alpha}{\nu_\alpha^2(4\nu_\alpha{}^2-1)}\left(3\nu_\alpha{}^2-1+\lambda_\alpha \frac{3(5\nu_\alpha{}^2-2)}{\nu_\alpha{}^2-1}\right) \tag{B9}$$

Now, since the O-mode is propagative over the frequency range $\omega \gtrsim \omega_{pe}$, the ion dynamics can be neglected. The sum over the particle species is delected and thus we only give the electron contributions to Γ_x and Γ_y which read

$$\Gamma_x = \frac{1}{D}\left(D_{yy}\ C_x - D_{xy}\ C_y\right)$$

$$= -\frac{1}{D}\ \frac{3R^2}{8\nu^3(4\nu^2-1)}\left\{1 + \frac{2}{3}\ \frac{\lambda}{(4\nu^2-1)(\nu^2-1)^2}(66\nu^4-62\nu^2+11)\right\} \tag{B10}$$

$$\Gamma_y = \frac{1}{D}\left(-D_{yx}\ C_x + D_{xx}\ C_y\right)$$

$$= \frac{i}{D}\ \frac{R}{4\nu^2(4\nu^2-1)}\left\{1+ \frac{\lambda}{(4\nu^2-1)(\nu^2-1)^2}[-(5\nu^2-2)R+2(7\nu^2-4)(4\nu^2-1)]\right\} \tag{B11}$$

where

$$D \equiv \det \overleftrightarrow{D}_X(2) = D_{XX}\ D_{yy} - D_{xy}\ D_{yz}$$

$$= -\frac{R}{4\nu^2(4\nu^2-1)}\left[3R - 4(3\nu^2-1)+ \frac{12\lambda}{\nu^2-1}\{2R-(5\nu^2-2)\}\right] \tag{B12}$$

Appendix C

Evaluation of q^{NL} for a Maxwellian Plasma

When the distribution function $F_\alpha(\vec{v})$ is Maxwellian, messy expressions (44a) and (44b) can be considerably simplified. First we can show that the last term of q^I identically vanishes. To prove this, suffice it to carry out the ψ'-integration. The result is

$$\int^{\psi} d\psi' \; G_k(\psi,\psi') \; \frac{\partial}{\partial\psi'} P^*_{\alpha,x}(1) = \frac{1}{2}\{P_{\alpha,x}(1) + P^*_{\alpha,x}(1)\}$$

We then integrate over $\vec{v}$ to have

$$\int d\vec{v} \; v_z^2 P_{\alpha,x}(1) = -i \; \frac{\omega_{c\alpha}}{k}\{1 - \nu_\alpha \; \exp(-\lambda_\alpha) \sum_n \frac{I_n(\lambda_\alpha)}{\nu_\alpha - n}\}$$

where $I_n(\lambda_\alpha)$ is the modified Bessel function and parameters ν_α and λ_α are defined in Appendix B. Thus, $\int d\underline{v} \; v_z^2\{P_{\alpha,x}(1) + \text{c.c.}\} = 0$. Next, terms involving v_z^4 in q^{II} can be shown to be at least of the order of λ_α^2. In the small λ_α limit where we restrict ourselves to the first-order correction in λ_α, we can delete them completely. Then, in terms of the following seven integrals

$$a_1 = -i \; \frac{\omega_{c\alpha}}{k} \int d\vec{v} \; P_{\alpha,x}(2) \qquad b_1 = -i \; \frac{\omega_{c\alpha}}{k} \int d\vec{v} \; P_{\alpha,y}(2)$$

$$a_2 = \int d\vec{v} \; v_z^2 \int^{\psi} d\psi' G_k(\psi,\psi') \frac{\partial}{\partial v_x'} P_{\alpha,x}(2), \quad b_2 = \int d\vec{v} \; v_z^2 \int^{\psi} d\psi' G_k(\psi,\psi') \frac{\partial}{\partial v_x'} P_{\alpha,y}(2)$$

$$a_3 = \int d\vec{v} \; v_z^2 \int^{\psi} d\psi' G_k(\psi, \psi') \frac{\partial}{\partial v_x'} P^*_{\alpha,x}(1), \quad b_3 = \int d\vec{v} \; v_z^2 \int^{\psi} d\psi' G_k(\psi,\psi') \frac{\partial}{\partial v_y'} P^*_{\alpha,x}(1)$$

$$a_4 = \int d\vec{v} \; v_z^2 \int^{\psi} d\psi' G_{2k}(\psi,\psi') \frac{\partial}{\partial v_x'} \; P_{\alpha,x}(1)$$

the nonlinear coupling coefficient q^I and q^{II} are given as

$$q^{I} = 2\,\frac{k^2}{B_O^{\,2}}\;\sum_\alpha \frac{\omega_{p\alpha}^{\;2}}{\omega\,\omega_{c\alpha}}\,\{(a_1+a_2+a_3)\Gamma_x + (b_1+b_2+b_3)\Gamma_y\}$$

$$q^{II} = -\,\frac{k^2}{B_O^{\,2}}\;\sum_\alpha \frac{\omega_{p\alpha}^{\;2}}{\omega^2}\,(a_1+a_2+a_3-a_4)$$

After tedious calculations we obtain to first order in λ_α, the following expressions

$$a_1=\frac{2}{4\nu_\alpha^2-1}\left(1+\lambda_\alpha\frac{3}{\nu_\alpha^2-1}\right) \qquad b_1=\frac{i}{\nu_\alpha(4\nu_\alpha^{\,2}-1)}\left(1+\lambda_\alpha\frac{6}{\nu_\alpha^2-1}\right)$$

$$a_2=\lambda_\alpha\frac{3(3-2\nu_\alpha^{\,2})}{(\nu_\alpha^2-1)(4\nu_\alpha^2-1)(\nu_\alpha^2-4)}\,, \qquad b_2=-i\lambda_\alpha\frac{3(3\nu_\alpha^{\,2}-2)}{\nu_\alpha(\nu_\alpha^2-1)(4\nu_\alpha^2-1)(\nu_\alpha^2-4)}$$

$$a_3=\lambda_\alpha\frac{3(\nu_\alpha^{\,2}-3)}{(\nu_\alpha^2-1)^2(\nu_\alpha^2-4)}\,, \qquad b_3=i\lambda_\alpha\frac{2(2\nu_\alpha^{\,2}-5)}{\nu_\alpha(\nu_\alpha^2-1)^2(\nu_\alpha^2-4)}$$

$$a_4=\lambda_\alpha\frac{6\nu_\alpha^{\,2}+1}{(\nu_\alpha^2-1)^2(4\nu_\alpha^2-1)}$$

Collecting them, we find

$$q^{I}=2\,\frac{k^2}{B_O^{\,2}}\;\sum_\alpha \frac{R_\alpha}{\nu_\alpha(4\nu_\alpha^{\,2}-1)}\left[2\Gamma_x\left\{1+\lambda_\alpha\frac{3(2\nu_\alpha^2-1)}{(\nu_\alpha^{\,2}-1)^2}\right\}+i\frac{\Gamma_y}{\nu_\alpha}\left\{1+\frac{13\nu_\alpha^{\,2}-7}{(\nu_\alpha^{\,2}-1)^2}\right\}\right]$$

and

$$q^{II}= -2\,\frac{k^2}{B_O^{\,2}}\;\sum_\alpha \frac{R_\alpha}{\nu_\alpha^{\,2}(4\nu_\alpha^{\,2}-1)}\left\{1+\frac{\lambda_\alpha}{2}\,\frac{6\nu_\alpha^{\,2}-5}{(\nu_\alpha^{\,2}-1)^2}\right\}$$

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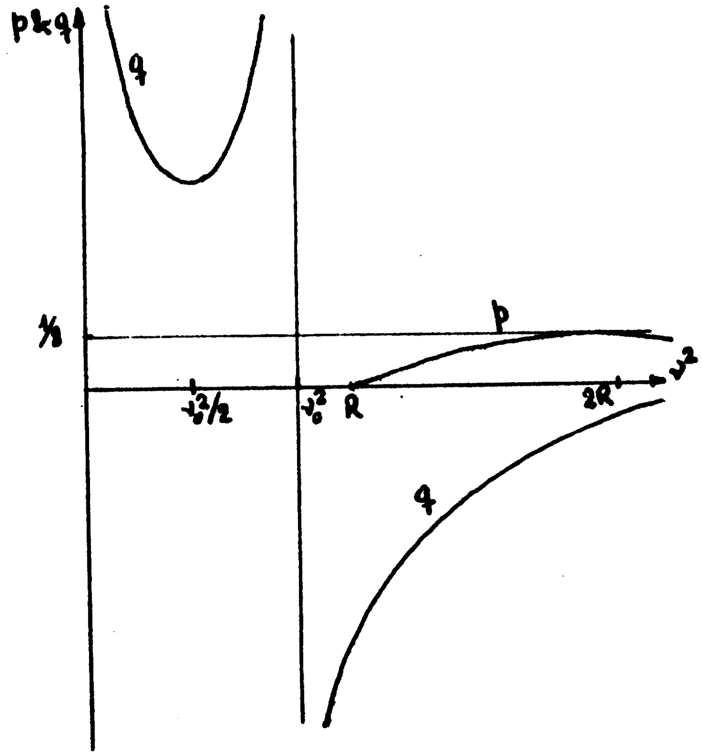
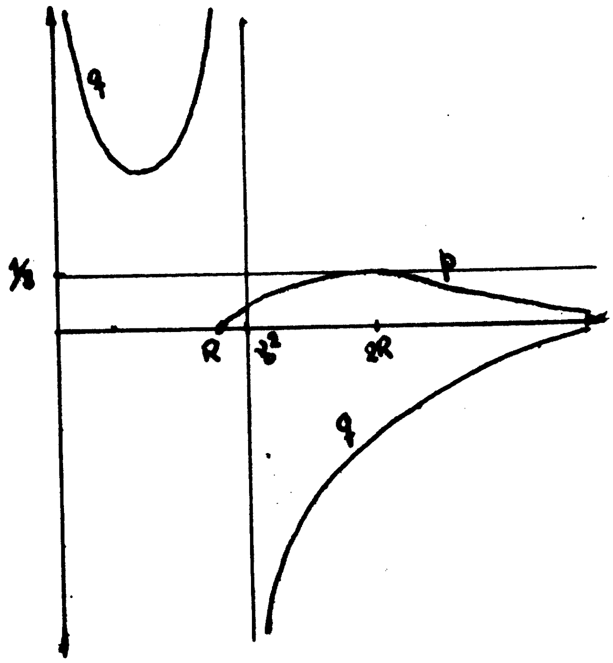
Figure Captions

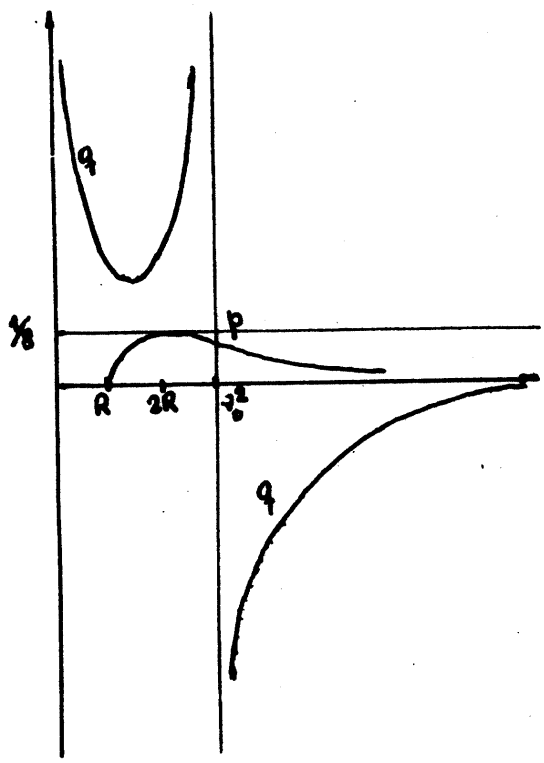
Variation of the dispersion coefficient p and of the nonlinear coupling coefficient q as a function of ν^2. Four different cases (a) $\nu_0{}^2 < R$, (b) $R < \nu_0{}^2 < 2R$, (c) $2R < \nu_0{}^2$ and (d) $2R < \nu_0{}^2/2$, according to a magnitude of R, are shown qualitatively.



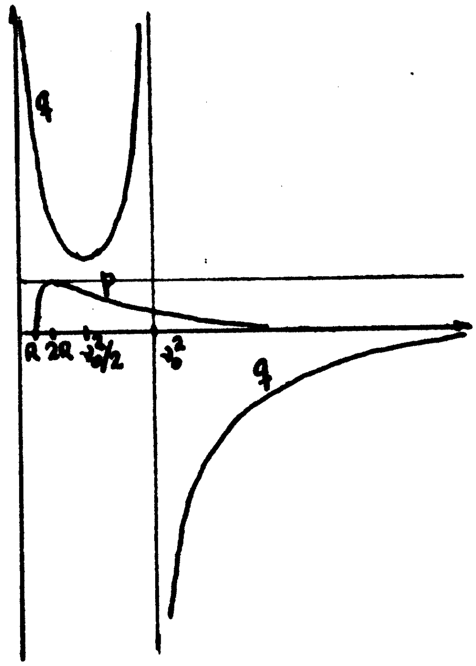
(a) $v_0^2 < R$



(b) $R < v_0^2 < 2R$



(c) $2R < v_0^2$



(d) $2R < v_0^2/2$