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Abstract

A general theory is presented for the diffusion of a multi-ion species plasma across magnetic field due to electrostatic fluctuations. The theory is based on the weak turbulence theory for an isothermal plasma in a slab geometry. It is shown that the diffusion flux due to wave-particle interactions is not necessarily proportional to the density gradient of the particle species of interest. The result is applied to the impurity diffusion due to those drift waves which exist in a single ion--species plasma. The dominant nonlinear effect is the induces scattering of waves on ions and the resulting diffusion constant of the host ions is given by the Kadomtsev formula, $D{\sim}\gamma^2/k^2\omega$, where γ, ω and k are respectively the growth rate, frequency and wavenumber of the drift wave. The impurity diffusion constant is positive (or negative) if the impurity Larmour radius is smaller (or greater) than the host-ion Larmour radius, and becomes greater than the host diffusion constant if the spectrum is localized in the unstable region.

§1. Introduction

Anomalous diffusion of plasma particles across the magnetic field due to drift-wave fluctuations has been a subject of extensive investigations.[1-8] In this paper, we discuss the problem of mutual diffusion of different ion components. Coppi et al.[9-13] have argued the effect of impurity mode on the impurity diffusion. Here we first derive a general expression for the diffusion flux in a multi-ion species plasma due to the general electrostatic fluctuations, and then consider specifically the problem of impurity diffusion due to the ordinary drift waves, i.e. those drift waves which exist in a single ion component plasma.

Drift waves are accompanied by an electrostatic field, $\vec{E}=-\nabla\varphi$, and a density fluctuation, δn. The electric field produces an ExB drift motion of particles and this drift motion yields a diffusion flux across the magnetic field when there exist a phase difference between φ and δn. One can alternatively interpret this diffusion flux as arising from the drift motion of particles due to the force exerted by the wave via wave-particle interaction. One important feature of this diffusion flux is that it is not always proportional to the density gradient of the particle species under consideration. Indeed, the direction of the diffusion flux depends not only on the density gradient, but also on the wave momentum.

For quantitative estimates, one needs to know the saturation level of the drift waves. Restricting ourselves to the ordinary drift waves, we determine this saturation level by the weak turbulence theory. Quasilinear effects can be included phenomenologically as determining the average backgroud properties of the plasma, but are assumed to be insufficient for the ultimate stabilization of the drift waves. Since the linear dispersion curve for the drift wave in the wavenumber region $k_\perp \rho_i \sim 1$ is of non-decay type, the dominant nonlinear effect is the induced scattering of waves on the ions*. Taking this effect into account, we find the saturation level of the order of $(e\varphi/T_e)^2 \sim (\kappa/k_\perp)^2(\gamma/\omega)$, where T_e is the electron temperature, κ the reciprocal of the density scale length of the electrons, $k_\perp$ the typical wavenumber of the drift wave perpendicular to the magnetic field, γ the growth rate and ω the real frequency. The resulting electron diffusion constant agrees with the Kadomtsev formula[14], $D_e \sim \gamma^2/\omega k_\perp{}^2$. The impurity diffusion constant can substantially exceed this value, and its sign depends on the Larmour radius; it is positive (or negative) if the Lamour radius of the impurity ion is smaller (or greater) than that of the host ion.

In §2, we derive a general expression for the diffusion flux in terms of the polarizabilities, including the nonlinear mode--coupling effect whithin the framework of the weak turbulence theory. Then in §3, we discuss the contribution of the linear polarizabilities in some detail, showing explicit results for the

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* There may be a decay into the impurity mode which we shall neglect here by assuming that it is by some means unimportant, e.q. by a large collision frequency of the impurity ion.

case of Maxwellian distributions. Up to this Section, the results obtained are quite general, being valid for arbitrary electrostatic waves and arbitrary composition of the plasma. Sections 4 and 5 are devoted to the problem of impurity diffusion due to the ordinary drift waves. In §4, we estimate the saturation level and the diffusion flux of the host ions by considering a single-ion species plasma and by taking into account the induced scattering on ions. Then in §5, we discuss the resulting impurity diffusion flux in some detail. Finally in §6, a brief discussion is given on the overall feature of the diffusion flux as well as on the quasilinear effects which we shall practically ignore. Possible application of the present formalism to the mutual diffusion of a D-T mixture is also added.

Unless otherwise stated, we shall for simplicity consider an isothermal plasma and use a slab geometry of a constant magnetic field along the z-axis with a density gradient in the x-direction.

§ 2. Diffusion Flux Due to Electrostatic Fluctuations

The diffusion flux along the density gradient (i.e. in the x-direction) due to electrostatic fluctuations can be written as

$$\Gamma_\sigma = \frac{n_\sigma c}{B} \langle \tilde{n}_\sigma(\vec{r},t)\, E_y(\vec{r},t) \rangle \tag{1}$$

where B is the magnitude of the magnetic field, c is the speed of light, $n_\sigma(x)$ is the average density, $\tilde{n}_\sigma(\vec{r},t)$ is the normalized density perturbation, $\tilde{n}_\sigma=\delta n_\sigma/n_\sigma$, $E_y(\vec{r},t)$ is the y-component of the fluctuating electric field and the angular bracket denotes the ensemble average; the suffix σ denotes the species of the particle with σ=e for the electron, σ=i for the host ion and σ=z for the impurity ion. Equation (1) describes the average flux due to the E×B drift in the x-direction. Derivation of eq.(1), as well as a discussion concerning its validity, are given in the Appendix.

We introduce the linear and nonlinear polarizabilities, $\chi_1^\sigma(\vec{k}, \omega)$, $\chi_2^\sigma(\vec{k}', \omega'; \vec{k}'', \omega'')$, ---, which relate the Fourier component of $\tilde{n}_\sigma(\vec{r}, t)$ to that of the normalized potential $\phi(\vec{r}, t) \equiv e\varphi(\vec{r}, t)/T_e$ as

$$\tilde{n}_\sigma(k) = \chi_1^\sigma(k)\,\phi(k) + \sum_{k'+k''=k} \chi_2^\sigma(k', k'')\,\phi(k')\,\phi(k'')$$

$$+ \sum_{k'+k''+k'''=k} \chi_3^\sigma(k', k'', k''')\,\phi(k')\,\phi(k'')\,\phi(k''') + \cdots \tag{2}$$

where we introduced the simplified notation, $k \equiv (\vec{k}, \omega)$ and

$$\sum_{k'+k''=k} \equiv \sum_{\vec{k}'} \sum_{\vec{k}''} \int d\omega' \int d\omega''\, \delta_{\vec{k}'+\vec{k}'', \vec{k}}\ \delta(\omega'+\omega''-\omega) \ ,$$

etc., and used the local approximation by assuming $|k_x| \gg |\kappa|$. We also assume that the nonlinear polarizabilities are symmetrized with respect to the permutation of the arguments, i.e. $\chi_2^\sigma(k', k'') = \chi_2^\sigma(k'', k')$, etc. Substituting eq.(2) into eq.(1) and neglecting the terms higher than the fourth order, we obtain,

$$\Gamma_\sigma = n_\sigma \frac{cT_e}{eB} \, \mathrm{Im} \sum_{k,k'} k_y' \Big[\chi_1^\sigma(k) \langle \phi(k)\phi(k') \rangle$$

$$+ \sum_{k_1+k_2=k} \chi_2^\sigma(k_1, k_2) \langle \phi(k_1)\phi(k_2)\phi(k') \rangle$$

$$+ \sum_{k_1+k_2+k_3=k} \chi_3^\sigma(k_1, k_2, k_3) \langle \phi(k_1)\phi(k_2)\phi(k_3)\phi(k') \rangle \Big] .$$

$$(3)$$

We calculate the nonlinear terms (the last two terms on the right-hand side) by the standard method of the weak turbulence theory.[14, 15] Namely, we consider a uniform and stationary turbulence and introduce the spectral function $I_{\vec{k}}$ by the relation

$$\langle \phi(k)\phi(k') \rangle = (2\pi)^2 \, \delta_{\vec{k},-\vec{k}'} \, \delta(\omega+\omega') \, \delta(\omega-\omega_{\vec{k}}) \; I_{\vec{k}} \qquad (4)$$

where $\omega_{\vec{k}}$ is the frequency of the drift wave of wavenumber $\vec{k}$. Then, using the Poisson equation, we calculate $\phi(k)$ by iteration to express $\langle\phi\phi\phi\rangle$ by $\langle\phi\phi\phi\phi\rangle$ which is decomposed into a sum of products of $\langle\phi\phi\rangle$ using the random phase approximation. The final result can be written in the form,

$$\Gamma_\sigma = -n_\sigma \frac{cT_e}{eB} \sum_{\vec{k}} k_y \Big[\mathrm{Im}\, \chi_1^\sigma(\vec{k}, \omega_{\vec{k}}) \, I_{\vec{k}}$$

$$+ \sum_{\vec{k}'} \alpha^\sigma_{\vec{k}\,\vec{k}'} I_{\vec{k}'} I_{\vec{k}} + \sum_{\vec{k}'} \beta^\sigma_{\vec{k}\,\vec{k}'} I_{\vec{k}'} I_{\vec{k}-\vec{k}'} \Big] . \qquad (5)$$

Here, the coefficients $\alpha^{\sigma}_{\vec{k}\vec{k}'}$ and $\beta^{\sigma}_{\vec{k}\vec{k}'}$ are respectively given by

$$\alpha^{\sigma}_{\vec{k}\vec{k}'} = \mathrm{Im}\Big\{ 3\,\chi_3^{\sigma}(\vec{k}', \omega_{\vec{k}'}; -\vec{k}', -\omega_{\vec{k}'}; \vec{k}, \omega_{\vec{k}})$$

$$- 4\, \frac{\chi_2^{\sigma}(\vec{k}-\vec{k}', \omega_{\vec{k}}-\omega_{\vec{k}'}; \vec{k}', \omega_{\vec{k}'})\, \epsilon_2(-\vec{k}', -\omega_{\vec{k}'}; \vec{k}, \omega_{\vec{k}})}{\epsilon_1(\vec{k}-\vec{k}', \omega_{\vec{k}}-\omega_{\vec{k}'})} \Big\} , \tag{6}$$

$$\beta^{\sigma}_{\vec{k}\vec{k}'} = -\mathrm{Im}\left[\frac{2\chi_2^{\sigma}(\vec{k}', \omega_{\vec{k}'}; \vec{k}-\vec{k}', \omega_{\vec{k}-\vec{k}'})\, \epsilon_2(-\vec{k}', -\omega_{\vec{k}'}; -\vec{k}+\vec{k}', -\omega_{\vec{k}-\vec{k}'})}{\epsilon_1(-\vec{k}, -\omega_{\vec{k}'}-\omega_{\vec{k}-\vec{k}'})} \right] \tag{7}$$

where

$$\epsilon_1(k) = k^2/k_e^2 - \sum_{\sigma} (e_\sigma n_\sigma / e n_e)\, \chi_1^{\sigma}(k) , \tag{8}$$

$$\epsilon_2(k, k') = - \sum_{\sigma} (e_\sigma n_\sigma / e n_e)\, \chi_2^{\sigma}(k, k') \tag{9}$$

with e_σ being the charge of the σ-th species of particle (e_e=-e) and k_e the electron Debye wavenumber.

The coefficients $\alpha^{\sigma}_{\vec{k}\vec{k}'}$ and $\beta^{\sigma}_{\vec{k}\vec{k}'}$ are related to those which appear in the wave kinetic equation that can be derived in the same approximation as above:

$$\frac{1}{2}\left[\frac{\partial}{\partial\omega}\mathrm{Re}\,\epsilon_1(\vec{k},\omega)\right]_{\omega=\omega_{\vec{k}}} \frac{\partial}{\partial t} I_{\vec{k}} + \mathrm{Im}\,\epsilon_1(\vec{k},\omega_{\vec{k}})\, I_{\vec{k}}$$

$$= \sum_{\vec{k}'} \left[\alpha_{\vec{k}\vec{k}'} I_{\vec{k}'} I_{\vec{k}} + \beta_{\vec{k}\vec{k}'} I_{\vec{k}'} I_{\vec{k}-\vec{k}'} \right] . \tag{10}$$

The relations are

$$A_{\vec{k}\vec{k}'} = \sum_{\sigma} (e_\sigma n_\sigma / e n_e)\, \alpha^{\sigma}_{\vec{k}\vec{k}'} \,,$$
$$B_{\vec{k}\vec{k}'} = \sum_{\sigma} (e_\sigma n_\sigma / e n_e)\, \beta^{\sigma}_{\vec{k}\vec{k}'} \,. \tag{11}$$

Using these relations, one then obtains the following general formula:

$$\sum_{\sigma} e_\sigma \Gamma_\sigma = \frac{n_e}{2} \frac{cT_e}{B} \sum_{\vec{k}} k_y \left[\frac{\partial}{\partial \omega} \mathrm{Re}\, \epsilon_1(\vec{k}, \omega) \right]_{\omega = \omega_{\vec{k}}} \frac{\partial I_{\vec{k}}}{\partial t} \,. \tag{12}$$

This formula implies that in the stationary state, where $\partial I_{\vec{k}}/\partial t=0$, the diffusion is strictly ambipolar. This is consistent with the result derived in the Appendix.

§3. Linear Contribution

In this Section, we consider the contribution of the first term inside the square bracket on the right-hand side of eq.(3). Using eq.(4), we can write this term as

$$\Gamma_\sigma^{(1)} = - n_\sigma \frac{cT_e}{eB} \sum_{\vec{k}} k_y\, \mathrm{Im}\, \chi_1^{\sigma}(\vec{k}, \omega_{\vec{k}})\, I_{\vec{k}} \,. \tag{13}$$

The factor $\mathrm{Im}\, \chi_1^{\sigma}(\vec{k}, \omega_{\vec{k}})$ is proportional to the phase difference between the potential and the density perturbation and can in general be written in the form[15)]

$$\mathrm{Im}\, \chi_1^{\sigma}(\vec{k}, \omega_{\vec{k}}) \propto \left\{ \left[k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\Omega_\sigma} \frac{\partial}{\partial x} \right] f_\sigma(x, v) \right\}_{v_z = \omega/k_z} , \tag{14}$$

where $\Omega_\sigma = e_\sigma B/m_\sigma c$ and $f_\sigma(x, v)$ is the average distribution function. From this relation, one can immediately infer that there are two contributions to the diffusion flux $\Gamma_\sigma^{(1)}$; one is proportional to the density gradient dn_σ/dx (contribution of $\partial f_\sigma/\partial x$) and the other is independent of dn_σ/dx (contribution of $\partial f_\sigma/\partial v_z$). This result is generally true and can be given the following physical interpretation.

We first note that $\Gamma_\sigma^{(1)}$ as given by eq.(13) is simply the sum of the drift motions due to the momentum flux exerted by the waves. Indeed, the time rate of change of the wave momentum density $\vec{P}_{\vec{k}}$ of wavenumber $\vec{k}$ due to wave-particle interactions can be written as

$$\frac{d\vec{P}_{\vec{k}}}{dt} = \frac{T_e \vec{k}}{e} \sum_\sigma n_\sigma e_\sigma \, \mathrm{Im}\, \chi_1^\sigma(\vec{k}, \omega_{\vec{k}}) \; I_{\vec{k}} \quad ,$$

so that the force acted by this wave on one particle of species σ along the y-direction is given by

$$F_{\sigma y} = - T_e k_y \frac{e_\sigma}{e} \, \mathrm{Im}\, \chi_1^\sigma(\vec{k}, \omega_{\vec{k}}) \; I_{\vec{k}} \quad ,$$

which causes a drift of velocity $F_{\sigma y} c/e_\sigma B$ in the x-direction. Collecting this drift motion for all the wave modes, we get the diffusion flux given by eq.(13).

Now, the linear wave-particle interaction consists of absorption and emission of waves by the particle, and the net absorption or emission results in the damping or growth of the wave. Associated with this, there occurs a net momentum gain or loss of the particle. The direction of this net momentum transfer is independent of the density gradient of the particles of given species, so is their resulting drift motion. This accounts for the term which is independent of dn_σ/dx. The term proportional to dn_σ/dx arises from the usual effect; namely, even when there is no net absorption or emission of the wave, there exists a drift motion associated with the individual absorption or emission processes, and in the presence of the density gradient the flux due to this drift motion does not average to zero.

In order to obtain a more explicit result, we consider the case when the average distribution functions are given by local Maxwellian forms:

$$f_\sigma(x, v) = n_\sigma(x + v_y/\Omega_\sigma)\, \pi^{-\frac{3}{2}}\, v_\sigma^{-3} \exp(-v^2/v_\sigma^2) \quad , \qquad (15)$$

where $v_\sigma(=[2T_\sigma/m_\sigma]^{1/2})$ is the thermal speed. Stating from the Vlasov equation with the collision term expressed by the Krook model of collision frequency ν_σ, we then obtain the following expressions for the linear polarizabilities[14,16]:

$$\chi_1^\sigma(k) = -\frac{e_\sigma T_e}{e\,T_\sigma}\left[1 - \frac{\omega - \omega_\sigma^*}{|k_z| v_\sigma} F_\sigma(\vec{k}, \omega)\right] \qquad (16)$$

where ω_σ^* $(=-k_y \kappa_\sigma T_\sigma c/e_\sigma B)$ is the drift frequency associated with the density gradient, $\kappa_\sigma=-d\ln n_\sigma/dx$, of the σ-th species of the particle and the function $F_\sigma(\vec{k},\omega)$ is given by

$$F_\sigma(\vec{k}, \omega) = \frac{\Lambda(s_\sigma)\, Y[(\omega + i\nu_\sigma)/|k_z| v_\sigma]}{1 - i(\nu_\sigma/|k_z| v_\sigma)\Lambda(s_\sigma)\, Y[(\omega + i\nu_\sigma)/|k_z| v_\sigma]} . \qquad (17)$$

Here $s_\sigma = k_\perp^2 \rho_\sigma^2 = k_\perp^2 T_\sigma / m_\sigma \Omega_\sigma^2$, $\Lambda(s) = I_0(s)\exp(-s)$ with I_0 being the zero-th order modified Bessel function, and $Y(z)$ is the plasma dispersion function defined by

$$Y(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \frac{\exp(-x^2)}{z - x + i\delta} \qquad (\delta > 0) \quad . \qquad (18)$$

If one used the asymptotic formulas for $Y(z)$, one can derive more explicit expressions for $\mathrm{Im}\chi_1^\sigma(k)$. As an example, we give the expression for the case when $\omega \gg |k_z| v_\sigma$, ν_σ and $s_\sigma \gg (k_z v_\sigma/\omega)^2$:

$$\mathrm{Im}\,\chi_1^\sigma(k) = -\frac{e_\sigma T_e}{e T_\sigma} \Lambda(s_\sigma)(\omega - \omega_\sigma^*) \left\{ \frac{\sqrt{\pi}}{|k_z| v_\sigma} \exp\left(-\frac{\omega^2}{k_z^2 v_\sigma^2}\right) + (\nu_\sigma/\omega^2)\left[1 - \Lambda(s_\sigma)\right] \right\} . \qquad (19)$$

As seen from eq.(16), $\mathrm{Im}\chi_1^\sigma(\vec{k}, \omega_{\vec{k}})$ is proportional to $(\omega_{\vec{k}} - \omega_\sigma^*)$ of which the term $\omega_{\vec{k}}$ represents the contribution of $\partial f_\sigma/\partial v_z$ and the term ω_σ^* that of $\partial f_\sigma/\partial x$. For the case of the ordinary drift wave in a plasma with small impurity concentration, $\omega_{\vec{k}}$ is approximately given by ω_e^*, so that the ion diffusion flux is proportional to

$$\Gamma_\sigma^{(1)} \propto \frac{T_e}{e} \frac{d}{dx} \ln n_e + \frac{T_\sigma}{e_\sigma} \frac{d}{dx} \ln n_\sigma \quad .$$

Thus if the impurity ion has a density gradient opposite to the host ion and hence to the electron, the two contributions to the impurity flux tend to cancel each other. In other words, this type of drift wave tends to prevent the penetration of the impurity ions from the low plasma-density region. Unfortunately, the magnitude of $\mathrm{Im}\chi_1^z(\vec{k}, \omega_{\vec{k}})$ for the ordinary drift wave is extremely small in a thermonuclear plasma; i.e. the Landau damping term in eq.(19) is negligibly small because of the large impurity mass and the collisional contribution for $n_e=10^{14}/cm^3$ and $T_e=10KeV$ becomes of the order $\mathrm{Im}\chi_1^z \sim \nu_z/\omega_{\vec{k}} \sim (e_z/e)\times 10^{-5}$. Therefore, as far as the linear contribution is concerned, the impurity flux, Γ_1^z, is mostly determined by the low-frequency impurity mode.* In the following Sections, however, we shall show the possibility of a large impurity flux due to the ordinary drift waves when the nonlinear polarizabilities are taken into consideration.

§4. Saturation Level and Diffusion Constant for a Single Ion Species Plasma.

We now examine the relevant nonlinear effect which limits the linear growth of the drift waves and estimate their saturation level. We shall do this by restricting ourselves to the effect of the ordinary drift waves in a plasma which contains only a small fraction of impurity ions, i.e. $n_i e >> n_z e_z$. We entirely neglect the effect of impurity modes, which may be justified, say for a relatively low temperature plasma where the impurity collision frequency becomes sufficiently large to suppress the low-frequency

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* In a low temperature plasma, the situation is different because of the relatively large collision frequency ν_z.

impurity modes. In order to investigate the saturation level of the drift wave and the diffusion constant of the host ion in such a plasma, we can to the lowest order disregard the presence of the impurity ion and consider a plasma of single ion species.

For a single ion-component plasma, the dispersion relation, $\varepsilon_1(\vec{k}, \omega_{\vec{k}} + i\gamma_{\vec{k}})=0$, for the local Maxwellian distributions (15) yields the following result:

$$\omega_{\vec{k}} = \frac{\omega_e^* \Lambda}{2-\Lambda} \left[1 + \frac{(2-\Lambda)}{\Lambda^2} \frac{k_z^2 v_i^2}{\omega_e^{*2}} \right] , \quad (20)$$

$$\gamma_{\vec{k}} = \frac{\omega_e^* \Lambda}{(2-\Lambda)^2} \, I_m \left[\chi_1^i(\vec{k}, \omega_{\vec{k}}) - \chi_1^e(\vec{k}, \omega_{\vec{k}}) \right] , \quad (21)$$

where for simplicity we assumed that $T_e = T_i$ and $|\gamma_{\vec{k}}| << |\omega_{\vec{k}}|$. For $\mathrm{Im}\ \chi_1^\sigma$, we use eqs.(16) through (18). In eq.(20), the second term inside the square bracket is much smaller than unity, and the function $\omega_e^* \Lambda/(2-\Lambda)$ is plotted in Fig.1. One can see that in the region where $k_\perp \rho_i$ is of order unity, where the growth rate $\gamma_{\vec{k}}$ becomes maximum, the dispersion curve is of non-decay type. Moreover, for $T_e = T_i$, ion acoustic waves are highly damped, so that decay into ion acoustic waves does not occur. Therefore, the dominant nonlinear effect will be the induced scattering of drift waves on the ions.

The resonance condition for this process is

$$\omega_{\vec{k}} - \omega_{\vec{k}'} = (k_z - k_z')\, v_z \quad .$$

Since the right-hand side for thermal ions is much smaller than the left-hand side for the typical pair of unstable waves, one can approximate this resonance condition by $\omega_{\vec{k}}=\omega_{\vec{k}'}$. This means that the energy redistribution of waves takes place over the equi-frequency surface. The drift waves tend to give the y-component of the wave momentum to the ions, so that the wave energy is carried toward the direction of small k_y. In Fig.2, we show the equi-frequency surface. One can see from this figure that as k_y decreases below the value $k_y\rho_i=0.6$, k_x decreases and k_z increases, Now, a slight increase of k_z causes a strong enhancement of the ion Landau damping. Thus this nonlinear process transfers the energy from the linearly growing region to the damped region, as shown in Fig.3, and can thereby bring the system to an ultimate saturation.

For the induced scattering on ions, the coefficients $\beta^{\sigma}_{\vec{k}\vec{k}'}$ and $\alpha^{e}_{\vec{k}\vec{k}'}$ vanish, while the coefficient $\alpha^{i}_{\vec{k}\vec{k}'}$ is calculated in ref.(15). The calculation was carried out by neglecting the collision term, but by assuming the local Maxwellian distribution for the resonant ions which are in the high energy tail region since the typical value of $|\omega_{\vec{k}}-\omega_{\vec{k}'}|$ is much greater than $|k_z-k'_z|v_i$ for $k_\perp\rho_i\sim 1$. It is not quite clear whether this nonlinear process brings the system to a complete stationary state. The final state is likely to be a quasi-steady state where the fluctuation spectrum oscillates around a mean value $\langle I_{\vec{k}}\rangle$. We estimate this mean value by the time average of eq.(10) which yields

$$\mathrm{Im}\left[\chi_1^e(\vec{k},\omega_{\vec{k}})-\chi_1^i(\vec{k},\omega_{\vec{k}})\right]=\sum_{\vec{k}'}\alpha^i_{\vec{k}\vec{k}'}\langle I_{\vec{k}'}\rangle \quad . \qquad (22)$$

We shall be content with an order of magnitude estimate of the saturation level. From eq.(21), the left-hand side is estimated to be of the order of $\gamma_{\vec{k}}/\omega_{\vec{k}}$, while the coefficient $\alpha^{i}_{\vec{k}\,\vec{k}'}$, whose explicit expression is given in the next Section, eq.(25), is estimated to be $(k_\perp/\kappa_e)^2$.[15] Thus we find

$$\sum_{\vec{k}} \langle I_{\vec{k}} \rangle \sim \left(\frac{\kappa_e}{k_\perp}\right)^2 \frac{\gamma}{\omega} \tag{23}$$

where γ and ω stand for the typical growth rate and frequency of the drift wave.

One can give a simple physical interpretation for this result as follows. First we note that the electric field E associated with the background drift-wave turbulence causes an E×B drift motion of the plasma fluid. Because of this drift motion, the frequency of a test drift wave suffers a random modulation due to the Doppler effect. The modulation amplitude is of the order of $k_\perp v_E$, where $v_E=cE/B$, and is much smaller than the frequency itself if the turbulence level is sufficiently low. Such a random frequency modulation causes a damping of the test wave with the damping rate given by $\langle(k_\perp v_E)^2\rangle\tau$[17], where τ is the auto-correlation time of the modulation and is approximately given by ω^{-1}. The saturation level is then determined by balancing this damping with the linear growth, i.e. $\langle(k_\perp v_E)^2\rangle/\omega\sim\gamma$. Noting that $\omega\sim k_\perp v_D$, where v_D is the diamagnetic drift velocity, and the relation $E\sim k_\perp\varphi = k\,T_e\phi/e$, we then find $\langle\phi^2\rangle\sim(\kappa/k_\perp)^2\gamma/\omega$ which gives the relation (23). Condition for weak modulation is satisfied since $\gamma/\omega\ll1$.

Let us finally evaluate the diffusion flux. Under the condition (22), the diffusion is ambipolar, so that it is sufficient if we calculate Γ_e. Assuming that the fluctuation spectrum is localized in the linearly unstable region, we neglect $\mathrm{Im}\,\chi_1^i(\vec{k}, \omega_{\vec{k}})$ and set $\mathrm{Im}\,\chi_1^e(\vec{k}, \omega_{\vec{k}}) \sim \gamma/\omega$. Then from eq.(5) and (23), we find

$$\Gamma_e = \frac{e_i}{e}\Gamma_i \sim \frac{dn_e}{dx}\frac{\gamma}{k_\perp^2}\frac{\gamma}{\omega}, \qquad (24)$$

which is in agreement with the Kadomtsev formula[14].

§5. Impurity Diffusion Due to Induced Scattering on Ions.

We now investigate the impurity diffusion flux associated with the induced scattering of the ordinary drift waves on ions. To this end, we extend the calculation given in ref.(15) to a plasma which contains a small fraction of impurity ions. The calculation is straight forward and the result can be written as follows:

$$\alpha^{\sigma}_{\vec{k}\vec{k}'} = -\pi\, K_\sigma\left(\frac{cT_e}{eB}\right)^3 \frac{[(\vec{k}\times\vec{k}')\cdot\vec{b}\,]^2}{\omega_k^2}(k_y - k_y')\,\delta(\omega_{\vec{k}} - \omega_{\vec{k}'})\,\mathcal{D}^{\sigma}, \qquad (25)$$

where $\sigma \neq e$, $\vec{b}(=\vec{B}/B)$ is the unit vector along the magnetic field and

$$\begin{aligned}\mathcal{D}^{\sigma} = &\int_0^\infty dt\, e^{-t} J_0^2(\alpha_\sigma\sqrt{t})\, J_0^2(\alpha_\sigma'\sqrt{t}) \\ &- \int_0^\infty dt\, e^{-t} J_0(\alpha_i\sqrt{t})\, J_0(\alpha_i'\sqrt{t})\, J_0(|\vec{\alpha}_i - \vec{\alpha}_i'|\sqrt{t}) \\ &\times \int_0^\infty dt\, e^{-t} J_0(\alpha_\sigma\sqrt{t})\, J_0(\alpha_\sigma'\sqrt{t})\, J_0(|\vec{\alpha}_\sigma - \vec{\alpha}_\sigma'|\sqrt{t}) \Big/ \Lambda([\vec{\alpha}_i - \vec{\alpha}_i']^2/2),\end{aligned} \qquad (26)$$

with $\vec{\alpha}_\sigma=\vec{k}_\perp v_\sigma/\Omega_\sigma$, $\vec{\alpha}_\sigma'=\vec{k}'_\perp v_\sigma/\Omega_\sigma$, $\alpha_\sigma =|\vec{\alpha}_\sigma|$ etc. and J_0 is the zero-th order ordinary Bessel function. Substitution of eq.(25) into eq.(5) with $\alpha^e_{\vec{k}\vec{k}'}=\beta^\sigma_{\vec{k}\vec{k}'}=0$ yields the associated diffusion flux. In the derivation of eqs.(25) and (26), we assumed that $\omega_{\vec{k}}\sim\omega_{\vec{k}'}>>|\omega_{\vec{k}}-\omega_{\vec{k}'}|$ and $n_i e_i>>n_z e_z$, and that both the host and the impurity ions have Maxwellian distributions.

We first note that $\alpha^\sigma_{\vec{k}\vec{k}'}$, and hence the resulting diffusion flux, is proportional to the density gradient κ_σ, in contrast to the contribution of the linear polarizabilities. This can be explained as follows. In the induced scattering process under consideration, there is no absorption or emission of the wave evergy since $\omega_{\vec{k}}=\omega_{\vec{k}'}$, but there exists momentum exchange between the wave and the particle associated with the angular scattering of the wave. In each scattering event, however, the probability for increasing k_y and that for decreasing k_y are the same, so that for a uniform distribution of the particles there is no net momentum exchange between the wave and the particle. A finite diffusion flux arises only when the particles are nonuniformly distributed and as a result the drift motion due to the individual scattering event yields a net drift flux. Note that it is this same effect that causes a net decrease of k_y by the induced scattering on ions.

We next note that the relative magnitude of the diffusion flux of the host ion to that of the impurity ion due to a given scattering process, say from wavenumber $\vec{k}$ to wavenumber $\vec{k}'$, is given by $\kappa_i\mathfrak{D}^i/\kappa_z\mathfrak{D}^z$ (see eqs.(5) and (25)). The function $\mathfrak{D}^\sigma$ is, on the other hand, entirely determined by the Larmour radius, $\rho_\sigma=|v_\sigma/\sqrt{2}\Omega_\sigma|$, as seen from eq.(26). Therefore, for given density gradients, the difference between the contributions to the diffusion fluxes of the host and the impurity ions arises only from the difference between their Larmour radii.

To obtain an explicit information, we numerically calculated $\mathfrak{D}^\sigma$ for several different cases. Fig.4 shows the results for the case $k_\perp\rho_i=k_\perp'\rho_i=.8/\sqrt{2}$ and $\rho_z/\rho_i=.4$, 1 and 2; θ is the angle between $\vec{k}$ and $\vec{k}'$ and is restricted to the region $0\leq\theta\leq\pi/2$, since the drift-wave spectrum is presumably confined to the quadrant, $k_y>|k_x|>0$. From this Figure, one can observe the following two features: first, except for the case when $\rho_z=\rho_i$, the impurity diffusion is substantially greater than the host diffusion, and secondly, $\mathfrak{D}^z$ becomes negative when $\rho_z>\rho_i$. These two features can also be seen from the limiting expression of $\mathfrak{D}^\sigma$ at $k_\perp\rho_i$, $k_\perp\rho_z\ll 1$. In this case, we find by retaining the lowest order terms,

$$\mathfrak{D}^z = (\rho_i^2-\rho_z^2)\, k_\perp k_\perp' \cos\theta \quad ,$$
$$\mathfrak{D}^i = \rho_i^4\, k_\perp^2 k_\perp'^2 \cos^2\theta \quad , \qquad (27)$$

which show that $\mathfrak{D}^i$ is of higher order than $\mathfrak{D}^z$ with respect to $k_\perp\rho_\sigma$ and that $\mathfrak{D}^z$ changes sign depending on whether $\rho_i^2>\rho_z^2$ ($\mathfrak{D}^z>0$) or $\rho_i^2<\rho_z^2$ ($\mathfrak{D}^z<0$).

We now give a physical explanation of these results. As shown by Tsytovich[18], the physical mechanism of the induced scattering is the emission of the scattered wave due to the oscillating particle current produced by the incident wave which is absorbed. The resonant particle is accompanied by a screening charge, and the oscillating current consists of both the resonant particle and its screening charge. When the screening charge consists of the same species of particles as the resonant particle, the two contributions to the oscillating current almost cancel each other, so that the cross section for the induced scattering becomes very small. On the other hand, when the screening charge consists of the particle species different from the resonant particle, the oscillation amplitudes of the screening charge and the resonant particle are different, so that there is no cancellation of the associated current. In the present case, the screening charge mainly consists of the host ions; the electrons do not contribute to the screening, since they are magnetized in the scale length of the ion Larmour radius and are unable to follow the rapid gyrating motion of the resonant ion ($k_z v_e < |\Omega_i|$, $|\Omega_z|$). Therefore, the induced scattering on the host ion is strongly reduced as compared with that on the impurity ion. This explains the first of the above two features. To account for the second feature, we note that the diffusion flux of the impurity ion consists of two parts, one as a resonant particle and the other as a screening charge to the host ion. Obviously, for $\rho_z = \rho_i$ the host ion and the impurity ion cannot be distinguished from each other, so that the two contributions just cancel each other. When $\rho_z > \rho_i$, the impurity ion can more efficiently screen the resonant particle than the host ion, so that the contribution as a screening charge becomed dominant. Since the screening charge is a "hole" of negative charge, its flux along the density gradient corresponds to a particle flux in the opposite direction, whence we get a negative diffusion constant.

§6. Discussions

We have investigated in Sections 3 and 5 the diffusion fluxes due to the linear and nonlinear scattering processes separately. The actual diffusion flux consists of the sum of both contributions. The relative importance of the linear and nonlinear terms depend sensitively on the fluctuation spectrum.

Let us consider the impurity diffusion due to the ordinary drift waves. Its velocity is proportional to (see eqs.(5) and (25)),

$$\sum_{\vec{k}} \left[\mathrm{Im}\, \chi_1^{z}(\vec{k}, \omega_{\vec{k}}) + \sum_{\vec{k}'} \frac{\kappa_z \mathcal{D}^z}{\kappa_i \mathcal{D}^i} \hat{\alpha}_{\vec{k}\vec{k}'} \langle I_{\vec{k}'} \rangle \right] \langle I_{\vec{k}} \rangle . \qquad (28)$$

We have pointed out in §3 that the linear contribution of the ordinary drift wave to the impurity diffusion is relatively small. To evaluate the nonlinear contribution, we replace $\kappa_z \mathcal{D}^z / \kappa_i \mathcal{D}^i$ by its typical value and take it outside the summation. Then the saturation condition (22) yields the following expression for the nonlinear term:

$$\sum_{\vec{k}} \frac{\kappa_z \mathcal{D}^z}{\kappa_i \mathcal{D}^i} \mathrm{Im} \left[\chi_1^{e}(\vec{k}, \omega_{\vec{k}}) - \chi_1^{i}(\vec{k}, \omega_{\vec{k}}) \right] \langle I_{\vec{k}} \rangle . \qquad (29)$$

We compare this quantity with the corresponding expression for the host ion which is given by

$$\sum_{\vec{k}} \mathrm{Im}\, \chi_1^{e}(\vec{k}, \omega_{\vec{k}}) \langle I_{\vec{k}} \rangle . \qquad (30)$$

Now, the term $\mathrm{Im}\, \chi_1^{e}(\vec{k}, \omega_{\vec{k}})$ is large in the linearly unstable region whereas $\mathrm{Im}\chi_1^{i}(\vec{k}, \omega_{\vec{k}})$ is large in the linearly damped region. Therefore, if the spectrum $\langle I_{\vec{k}} \rangle$ is more localized to the linearly unstable region than to the damped region, the impurity diffusion will be greater than the host diffusion, since in most cases $|\kappa_z \mathcal{D}^z| > |\kappa_i \mathcal{D}^i|$, as we have shown in §5. The direction of the impurity diffusion depends on ρ_z/ρ_i as well as on κ_z. On the other hand, if the spectrum spreads almost equally to both the linearly unstable and damped region, then there is a large cancellation in eq.(29), so that the impurity diffusion becomes even slower than the host diffusion.

The basic feature of the result obtained in the present paper will also be applicable to the mutual diffusion of a D-T mixture, although the explicit expressions for the nonlinear coupling coefficients, i.e. eqs.(25) and (26), are to be modified. In this case, since the tritium has a greater Larmour radius than the deuterium, the nonlinear scattering process tends to carry the deuterium outward and the tritium inward.

Let us finally discuss the quasilinear effects which we assumed to be insufficient for the ultimate saturation of the drift-wave instability. There are three quasilinear effects: the modification of the electron velocity distribution, that of the ion velocity distribution and that of the plasma density profile. First, the quasilinear modification of the electron velocity distribution can readily be suppressed by weak electron--electron collisions[19)] since the resonance width is extremely small for the present low-level saturation of the instability. Secondly, the modification of the ion velocity distribution becomes important when a high-energy tail, of order $v_{\parallel} \sim \omega/k_{\parallel}$, is produced. The time needed for formation of such a tail may be estimated as $\tau \sim (\omega/k_{\parallel})^2/D$, where $D(\sim[e_i k_{\parallel} \varphi/m_i]^2/\omega)$ is the velocity-space diffusion constant. Using the saturation level (23), we find $\tau \sim (\kappa_e k_{\perp}/k_{\parallel}^2)^2 \gamma^{-1}$ which is much greater than the growth time γ^{-1}. Therefore, this quasilinear effect is also unimportant for the present case. Finally, the quasilinear modification of the density profile becomes important when the drift waves are strongly localized due to a magnetic shear. In a Tokamak plasma, each drift wave is localized near one of the rational surfaces which are distributed in space with the spacing Δx of the order of $\Delta q(dq/dx)^{-1} \sim k_{\perp}^{-1} \rho_i L_s \kappa_e^2 (\varepsilon/q)$, where L_s is the magnetic shear length, ε the aspect ratio, q the safety factor. The drift wave can be stabilized by a local flattening of the density profile only when Δx is greater than the wave localization length. The latter is of the order of $\rho_e \kappa_e L_s$ [20], so that the quasilinear effect is unimportant unless $\kappa_e/k_{\perp} > (q/\varepsilon)(\rho_e/\rho_i)$. If this inequality is satisfied, on the other hand, the drift wave is stabilized at an extremely low level and no anomalous diffusion can be expected.

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Appendix

We start from the microscopic momentum transpart equation,

$$n_\sigma \left[\frac{\partial \vec{V}_\sigma}{\partial t} + \vec{V}_\sigma \cdot \frac{\partial}{\partial \vec{r}} \vec{V}_\sigma \right] + \frac{1}{m_\sigma} \frac{\partial}{\partial \vec{r}} P_\sigma + \frac{1}{m_\sigma} \frac{\partial}{\partial \vec{r}} \cdot \overleftrightarrow{\Pi}_\sigma$$

$$= \frac{e_\sigma}{m_\sigma} \left[n_\sigma \frac{\vec{V}_\sigma \times \vec{B}}{c} + \langle \hat{n}_\sigma(\vec{r},t)\, \vec{E}(\vec{r},t) \rangle \right] \qquad \text{(A-1)}$$

where

$$P_\sigma = \frac{n_\sigma m_\sigma}{3} \sum_{j\in\sigma} \langle |\vec{v}_j(t) - \vec{V}_\sigma|^2 \, \delta[\vec{r} - \vec{r}_j(t)] \rangle , \qquad \text{(A-2)}$$

$$\overleftrightarrow{\Pi}_\sigma = n_\sigma m_\sigma \sum_{j\in\sigma} \langle \left[(\vec{v}_j(t) - \vec{V}_\sigma)^2 - \frac{|\vec{v}_j(t) - \vec{V}_\sigma|^2}{3} \right] \delta[\vec{r} - \vec{r}_j(t)] \rangle ,$$

$$\text{(A-3)}$$

$$\hat{n}_\sigma(\vec{r},t) = \sum_{j\in\sigma} \delta[\vec{r} - \vec{r}_j(t)] , \qquad \text{(A-4)}$$

$$\vec{E}(\vec{r},t) = -\sum_{\sigma'} e_{\sigma'} \sum_{j\in\sigma'} \delta[\vec{r} - \vec{r}_j(t)] \frac{\partial}{\partial \vec{r}} \frac{1}{|\vec{r} - \vec{r}_j(t)|} ,$$

$$\text{(A-5)}$$

the summation symbol, $\Sigma_{j\epsilon\sigma}$, denoting the sum over the particles of species σ, and the rest of the notation is the same as in the text. We consider a stationary state and take the y-component of eq.(A-1), obtaining

$$n_\sigma V_{\sigma x} \frac{\partial}{\partial x} V_{\sigma y} + \frac{1}{m_\sigma} \left(\frac{\partial}{\partial \vec{r}} \cdot \overleftrightarrow{\Pi}_\sigma \right)_y + n_\sigma \omega_\sigma V_{\sigma x}$$

$$= \frac{e_\sigma}{m_\sigma} \langle \hat{n}_\sigma(\vec{r},t)\, E_y(\vec{r},t) \rangle . \qquad \text{(A-6)}$$

In the absence of the first two terms on the left-hand side, eq.(A-6) yields the flux density given by eq.(1). It is therefore sufficient to show that the first two terms on the left-hand side of eq.(A-6) are small as compared with the third term.

The first term of eq.(A-6) can be estimated by approximating $v_{\sigma y}$ by the diamagnetic drift velocity, $v_{\sigma y}=\kappa_\sigma T_\sigma / m_\sigma \Omega_\sigma$. The ratio of the first term to the third term in eq.(A-6) then becomes of order $\kappa_\sigma^2 \rho_\sigma^2$ which is assumed to be much less than unity.

Consider next the second term of eq.(A-6). In the quiescent case, this term can be written as[21)]

$$\frac{1}{m_\sigma}\left(\frac{\partial}{\partial \vec{r}}\cdot \overleftrightarrow{\Pi}_\sigma\right)_y = -\frac{1}{m_\sigma}\frac{\partial}{\partial x}\frac{n_\sigma T_\sigma}{\Omega_\sigma}\left(\frac{\nu_\sigma}{\Omega_\sigma}\frac{\partial V_{\sigma y}}{\partial x}+\frac{1}{2}\frac{\partial V_{\sigma x}}{\partial x}\right)$$

$$\sim n_\sigma \Omega_\sigma \rho_\sigma^2 \kappa_\sigma^2 \left(v_{cl}+v_{\sigma x}\right) \qquad \text{(A-7)}$$

where $v_{c\ell}(=\rho_\sigma^2 \nu_\sigma \kappa_\sigma)$ is the classical diffusion velocity. In a turbulent plasma, $v_{c\ell}$ is enhanced to $v_{\sigma x}$. Equation (A-7) then shows that this term is again of order $\kappa_\sigma^2 \rho_\sigma^2$ as compared with the third term of eq.(A-6). We therefore conclude that the first two terms on the left-hand side of eq.(A-6) are smaller by a factor $\kappa_\sigma^2 \rho_\sigma^2$ than the terms retained in eq.(1). We note that in this approximation the local current density across the magnetic field vanishes.

Figure Captions

Fig.1. The dispersion curve of the drift wave in a single ion component plasma with $k_z=0$.

Fig.2. The equifrequency surface of the ordinary drift wave in the wavenumber space.

Fig.3. The equifrequency line of the ordinary drift wave in the k_y-k_z plane and the stable and the unstable regions.

Fig.4. The coefficients $\mathfrak{D}^i$ and $\mathfrak{D}^z$ as functions of the scattering angle θ for $\rho_z/\rho_i=.4$, 1 and 2 with $k_\perp \rho_i = k'_\perp \rho_i = .8/\sqrt{2}$.

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Fig.1.

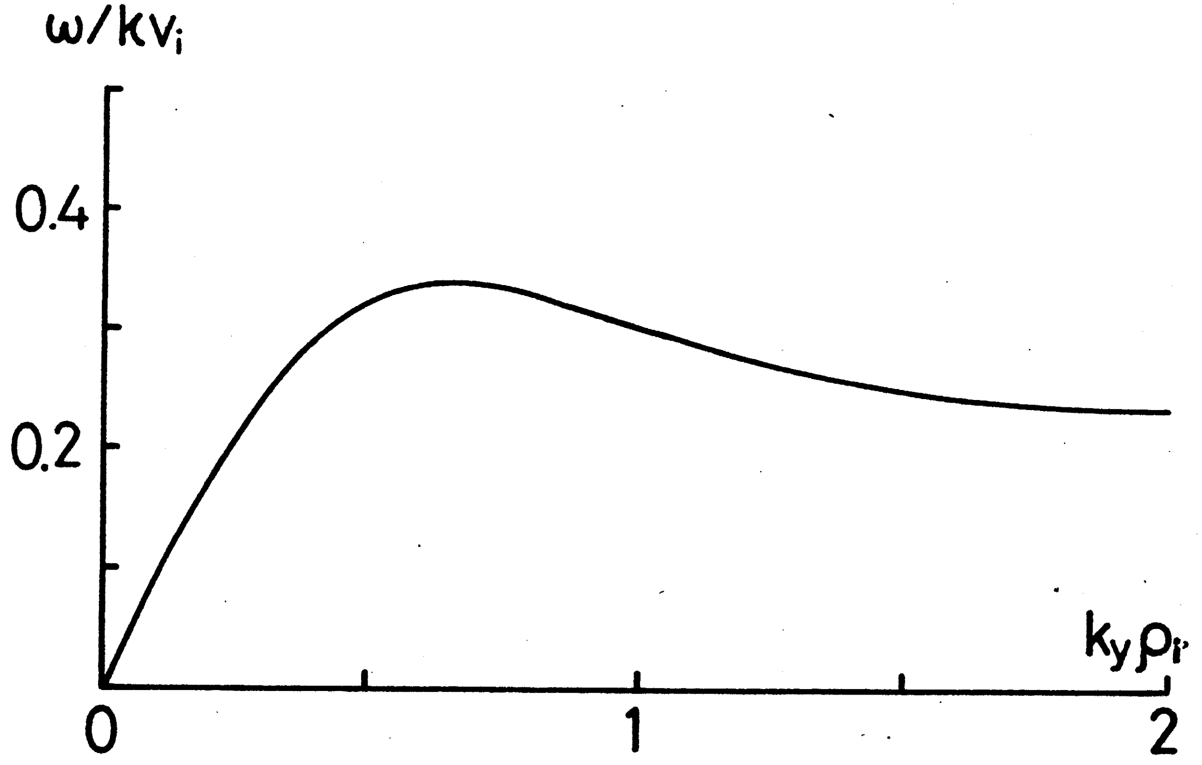
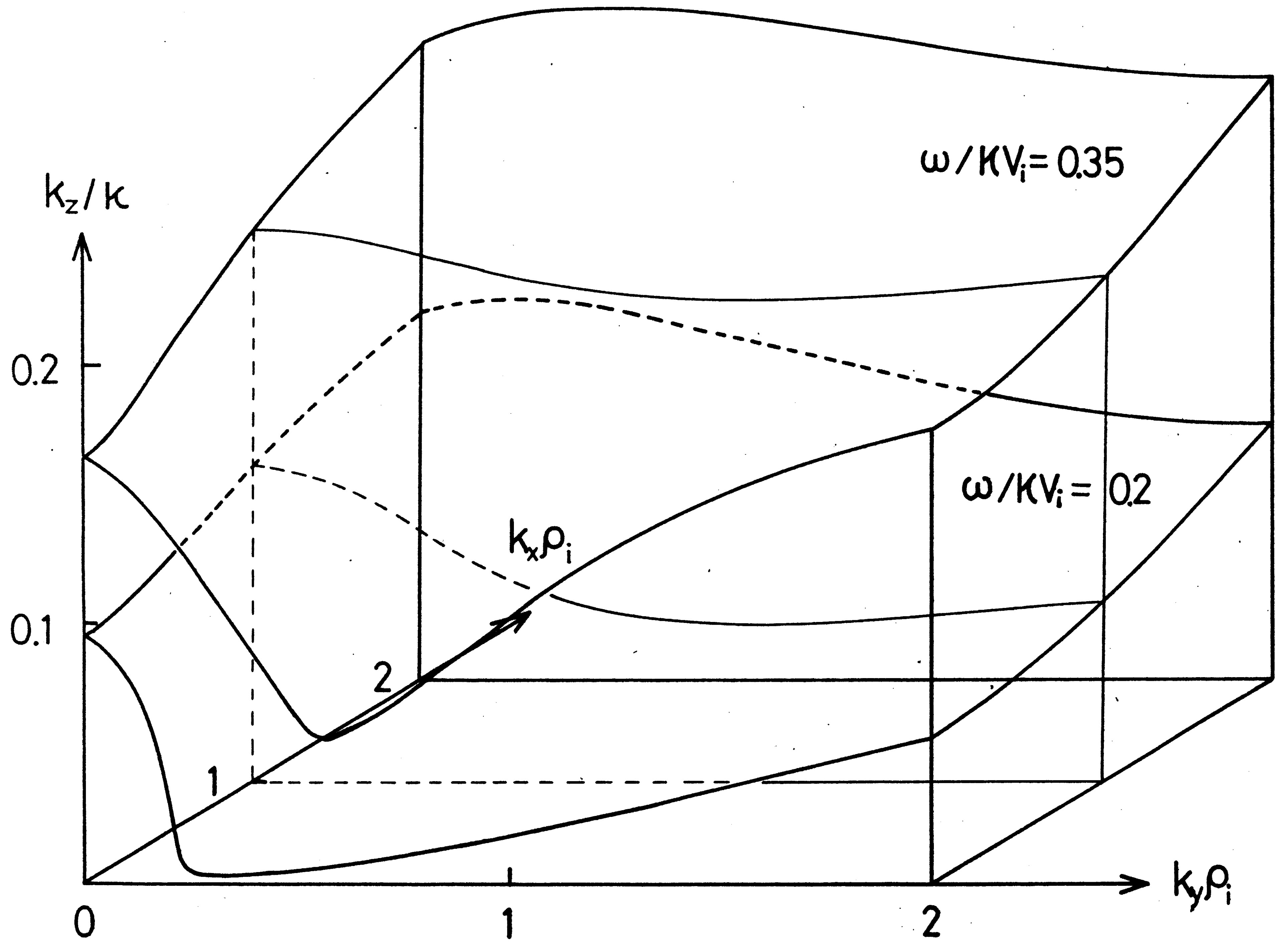
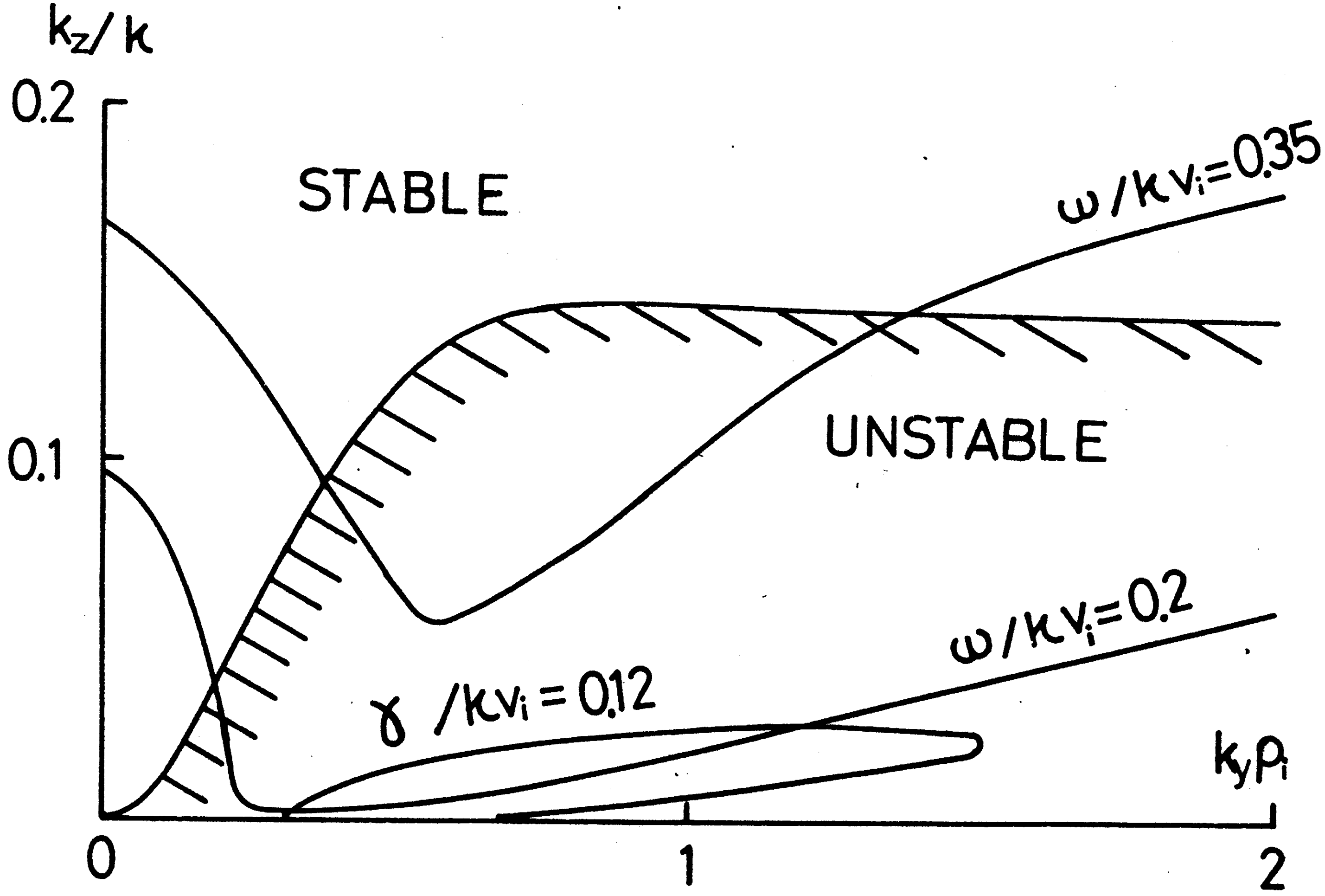


Fig.2.



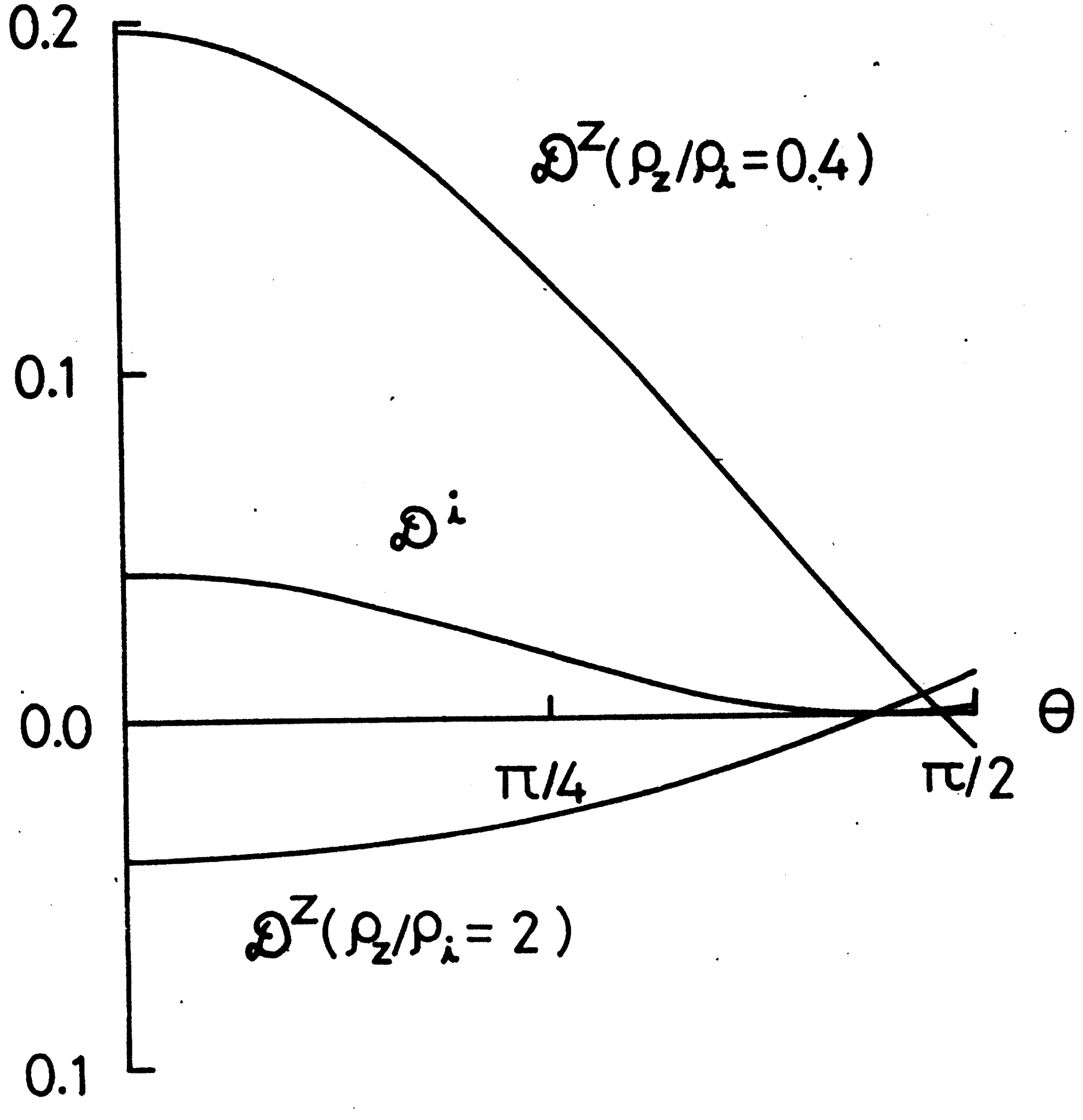
k_z/κ
0.2
0.1
0
1
2
$k_y\rho_i$
$k_x\rho_i$
1
2
$\omega/\kappa V_i = 0.35$
$\omega/\kappa V_i = 0.2$

Fig.3.



k_z/k
0.2
0.1
0
1
2
$k_y\rho_i$
STABLE
UNSTABLE
$\omega/kv_i=0.35$
$\omega/kv_i=0.2$
$\gamma/kv_i=0.12$

Fig.4.



0.2
0.1
0.0
0.1
$\mathcal{D}^Z(\rho_z/\rho_i=0.4)$
$\mathcal{D}^i$
$\mathcal{D}^Z(\rho_z/\rho_i=2)$
$\pi/4$
$\pi/2$
θ