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Helically Symmetric Equilibrium of Current-Carrying Finite Beta Plasma

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Abstract

Characteristics of helically symmetric equilibrium of longitudinal current-carrying finite beta plasma are studied analytically.

Analytical expression of the rotational transform angle indicates that the contribution due to the helical component of the plasma current shoud be taken into account in addition to the rotational transform angle due to the external helical current and the longitudinal plasma current.

1. Introduction

When a finite beta plasma is produced by ohmic heating in a stellarator field, the magnetic surfaces are deformed from those of the vacuum stellarator field.

On the other hand, if helical perturbations occur in tokamak plasma, helical structures appear in the magnetic surface.

In this paper characteristics of helically symmetric equilibrium of finite beta plasma with current are discussed on the base of the helical equilibrium solutions.

2. Solution of Helical Equilibrium

The magnetohydrodynamic equilibrium equation of helically symmetric system is well known[1)~3)] and the flux function must satisfy the following equation with use of the cylindrical coordinates (r,θ,z) and helical angle $\phi = \theta - \alpha z$,

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{r}{1+(\alpha r)^2}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{F^*(B^*)'}{1+(\alpha r)^2} - \frac{2\alpha B^*}{\{1+(\alpha r)^2\}^2}$$

$$= -\mu_o p' \quad , \tag{2-1}$$

where $p(\Psi)$ is the plasma pressure and $B^*(\Psi)$ is the function of Ψ only, μ_o being the permeability of vacuum. The prime of $(B^*)'$ and p' means the differentiation with respect to Ψ.

The magnetic field and the current density are given by

$$B_r = \frac{1}{r}\frac{\partial\Psi}{\partial\phi} \quad , \tag{2-2}$$

$$B_\theta = \frac{-\frac{\partial \Psi}{\partial r} + (\alpha r)B^*}{1 + (\alpha r)^2} \quad , \tag{2-3}$$

$$B_z = \frac{\alpha r\frac{\partial \Psi}{\partial r} + B^*}{1 + (\alpha r)^2} \quad , \tag{2-4}$$

$$\mu_o\vec{j} = (B^{*'})\vec{B} + \mu_o p'(\alpha r\vec{e}_\theta + \vec{e}_z) \quad , \tag{2-5}$$

respectively where $\vec{e}_\theta$ and $\vec{e}_z$ are unit vectors in θ and z directions. There is a relation

$$B^* = \alpha r B_\theta + B_z \quad . \tag{2-6}$$

Let us consider the following case

$$B^* = B_o + \gamma(\Psi - \Psi_o) \quad , \tag{2-7}$$

$$p = p_o(1 - \frac{\Psi}{\Psi_o}) \quad , \tag{2-8}$$

inside the plasma boundary $\Psi = \Psi_o$ and

$$B^* = B_o \quad , \tag{2-9}$$

$$p = 0 \quad , \tag{2-10}$$

outside the plasma boundary. Here we take $\Psi = 0$ at plasma center

$r = 0$. This model corresponds to the case of nearly uniform current distribution and parabolic pressure distribution.

The solution in the force free case was discussed in ref.1). If the effect of the plasma pressure is taken into account, the flux function Ψ_p in the plasma region is written as[2)∿3)]

$$\Psi_p = \Psi_p^{(0)}(r) + \sum_{n=1}^{\infty}\Psi_p^{(n)}(r)\cos n\phi \quad , \tag{2-11}$$

$$\Psi_p^{(0)}(r) = -\frac{B_o - \gamma\Psi_o}{\gamma} + \frac{\mu_o p_o}{\Psi_o \gamma^2}\{1 + (\alpha r)^2 + \frac{2\alpha}{\gamma}\}$$

$$+ A_o\{\varepsilon r J_o'(\gamma r) - \frac{\gamma}{\alpha} J_o(\varepsilon r)\} \quad , \tag{2-12}$$

$$\Psi_p^{(n)}(r) = A_n\{\varepsilon_n r I_n'(\varepsilon_n r) - \frac{\gamma}{\alpha} I_n(\varepsilon_n r)\} \quad , \tag{2-13}$$

where A_n is constant , $\varepsilon_n = (n^2\alpha^2 - \gamma^2)^{1/2}$, J_o is Bessel function and I_n is modified Bessel function. Since $\Psi_p = 0$ is assumed at $r = 0$, the constant A_o is

$$A_o = -\frac{\alpha(B_o - \gamma\Psi_o)}{\gamma^2} + \frac{\mu_o p_o \alpha}{\Psi_o \gamma^3}(1 + \frac{2\alpha}{\gamma}) \quad . \tag{2-14}$$

The terms of Bessel functions are force free terms.[1)]

The flux function Ψ_v in the region outside the plasma ($\Psi > \Psi_o$) and inside the helical winding ($r < b$) is given by

$$\Psi_v = \Psi_v^{(0)}(r) + \sum_{n=1}^{\infty}\Psi_v^{(n)}(r)\cos\phi \ , \qquad (2\text{-}15)$$

$$\Psi_v^{(0)}(r) = \frac{B_o\alpha r^2}{2} + E_o + F_o\{\ell n\frac{r}{r_o} + \frac{(\alpha r)^2}{2}\} \ , \qquad (2\text{-}16)$$

$$\Psi_v^{(n)}(r) = C_n n\alpha r I_n'(n\alpha r) + D_n n\alpha r K_n'(n\alpha r) . \qquad (2\text{-}17)$$

The flux function Ψ_c outside the helical winding $(r > b)$ is

$$\Psi_c = \frac{B_o\alpha r^2}{2} + E_o + F_o\{\ell n\frac{r}{r_o} + \frac{(\alpha r)^2}{2}\}$$

$$+ \{\sum_{n=1}^{\infty}\frac{I_n'(n\alpha r)}{K_n'(n\alpha r)}\bigg|_{r=b} C_n + D_n\}\ n\alpha r\ K_n'(n\alpha r)\cos n\phi \ . \qquad (2\text{-}18)$$

The current distribution on the helical sheet coil placed at $r = b$ is given by the following equation

$$i = \frac{\ell I_h^\ell}{2b}\cos\ell\phi \ , \qquad (2\text{-}19)$$

where I_h^ℓ is the helical coil current per one coil. The coefficient C_ℓ is fixed by the boundary condition at the $r = b$ and

$$C_\ell = -\frac{\mu_o I_h^\ell}{2}\alpha b\{1 + (\alpha b)^2\}^{-1/2}\ K_\ell'(\ell\alpha b) . \qquad (2\text{-}20)$$

The boundary conditions at the plasma surface Σ are

$$\Psi_p(r,\phi) = \Psi_v(r,\phi) = \Psi_o \quad , \tag{2-21}$$

$$\frac{\partial \Psi_p}{\partial r} = \frac{\partial \Psi_v}{\partial r} \quad , \tag{2-22}$$

$$\frac{\partial \Psi_p}{\partial \phi} = \frac{\partial \Psi_v}{\partial \phi} \quad , \tag{2-23}$$

and these conditions fix the coefficients of E_o, F_o, A_n, D_n.

The boundary surface itself shoud be determined by eqs.(2-21) -(2-23). Generally this boundary value problem is difficult to be solved. We assume that the boundary surface is not much different from a circle, which is given in the form of $r = r_o + \Delta(r_o)\cos\ell\phi$ $(\Delta \ll r_o)$, and only the zeroth and ℓ-th harmonic terms of the flux function are dominant, that is,

$$\Psi(r,\phi) = \Psi_o(r) + \Psi_\ell(r)\cos\ell\phi \quad , \tag{2-24}$$

where $|\Psi_\ell| < |\Psi_o|$.

Then the boundary conditions are reduced to

$$\Psi_p^{(0)}(r_o) = \Psi_v^{(0)}(r_o) \quad , \tag{2-25}$$

$$\Psi_p^{(\ell)}(r_o) = \Psi_v^{(\ell)}(r_o) \quad , \tag{2-26}$$

$$\frac{\partial}{\partial r}\Psi_p^{(0)}(r) = \frac{\partial}{\partial r}\Psi_v^{(0)}(r) \quad \text{at } r = r_o \quad , \tag{2-27}$$

$$\frac{\partial}{\partial r}\Psi_p^{(\ell)}(r)+\Delta\frac{\partial^2}{\partial r^2}\Psi_p^{(0)}(r) = \frac{\partial}{\partial r}\Psi_v^{(\ell)}(r)+\Delta\frac{\partial^2}{\partial r^2}\Psi_v^{(0)}(r)$$

$$\text{at } r = r_o \quad , \qquad (2\text{-}28)$$

where

$$\Delta = -\frac{\Psi_p^{(\ell)}(r)}{\frac{\partial}{\partial r}\Psi_p^{(0)}(r)} \qquad (2\text{-}29)$$

When the plasma radius r_o is much smaller than the helical pitch α^{-1}, that is $|\alpha r| << 1$, the modified Bessel function can be expanded and the solution are

$$\Psi_p(r,\phi) = \frac{\alpha B_o r^2}{2}\left(1+\frac{\mu_o p_o}{2\alpha B_o\Psi_o}-\frac{\gamma}{2\alpha}\right)$$

$$+\; C_\ell\left\{1+\frac{\frac{\mu_o p_o}{2\alpha B_o\Psi_o}-\frac{\gamma}{2\alpha}}{\ell+(\ell-1)\left(\frac{\mu_o p_o}{2\alpha B_o\Psi_o}-\frac{\gamma}{2\alpha}\right)}\right\}\frac{\left(\frac{\ell\alpha r}{2}\right)^\ell}{(\ell-1)!}\cos\ell\phi, \qquad (2\text{-}30)$$

$$\Psi_v(r,\phi) = \frac{\alpha B_o r^2}{2}\left\{1+\left(\frac{\mu_o p_o}{2\alpha B_o\Psi_o}-\frac{\gamma}{2\alpha}\right)\left(\frac{r_o}{r}\right)^2\left(2\ell n\frac{r}{r_o}+1\right)\right\}$$

$$+\; C_\ell\left\{1+\frac{\frac{\mu_o p_o}{2\alpha B_o\Psi_o}-\frac{\gamma}{2\alpha}}{\ell+(\ell-1)\left(\frac{\mu_o p_o}{2\alpha B_o\Psi_o}-\frac{\gamma}{2\alpha}\right)}\left(\frac{r_o}{r}\right)^{2\ell}\right\}\frac{\left(\frac{\ell\alpha r}{2}\right)^\ell}{(\ell-1)!}\cos\ell\phi, \qquad (2\text{-}31)$$

$$\Psi_c(r,\phi) = \frac{\alpha B_o r^2}{2}\{1+\left(\frac{\mu_o p_o}{2\alpha B_o \Psi_o} - \frac{\gamma}{2\alpha}\right)\left(\frac{r_o}{r}\right)^2 (2\ln\frac{r}{r_o} + 1)\}$$

$$+ C_\ell\{ \left.\frac{I_\ell'(\ell\alpha r)}{K_\ell'(\ell\alpha r)}\right|_{r=b} \left(-\frac{(\ell-1)\ \ell}{2}\right)\left(\frac{2}{\ell\alpha r}\right)^{2\ell}$$

$$+ \frac{\frac{\mu_o p_o}{2\alpha B_o \Psi_o} - \frac{\gamma}{2\alpha}}{\ell+(\ell-1)\left(\frac{\mu_o p_o}{2\alpha B_o \Psi_o} - \frac{\gamma}{2\alpha}\right)} \left(\frac{r_o}{r}\right)^{2\ell} \} \frac{\left(\frac{\ell\alpha r}{2}\right)^\ell}{(\ell-1)!} \cos\ell\phi \ , \tag{2-32}$$

where

$$\Psi_o = \frac{B_o \alpha r_o^2}{2}\left(1 + \frac{\mu_o p_o}{2\alpha B_o \Psi_o} - \frac{\gamma}{2\alpha}\right) \ . \tag{2-33}$$

3. Characteristics of Equilibrium

When the following quantities are introduced,

$$\frac{R}{m} = \frac{1}{\ell\alpha} \ , \tag{3-1}$$

$$q = \frac{2}{R\gamma} \ , \tag{3-2}$$

$$\beta = (\alpha r_o)^2 \frac{\mu_o p_o}{\alpha B_o \Psi_o} \ , \tag{3-3}$$

the linear helical field with the pitch parameter α corresponds to a toroidal helical field with the pole number ℓ, the field period m and the major radius R. The variable ϕ is expressed by

$\ell\phi = \ell\theta - \frac{m}{R} z$. The quantities q are approximately equal to the safety factor and β is approximately equal to the beta ratio at the axis.

With use of these quantities, the magnetic flux functions are

$$\Psi_p(r,\phi) = \frac{\alpha B_o r^2}{2} (1 + \varepsilon_\beta - \varepsilon_\gamma)$$

$$+ C_\ell \{1 + \frac{1}{\ell} \cdot \frac{\varepsilon_\beta - \varepsilon_\gamma}{1+ \frac{\ell-1}{\ell}(\varepsilon_\beta-\varepsilon_\gamma)}\} \frac{(\frac{mr}{2R})^\ell}{(\ell-1)!} \cos\ell\phi \quad , \qquad (3\text{-}4)$$

$$\Psi_v(r,\phi) = \frac{\alpha B_o r^2}{2} \{1+(\varepsilon_\beta - \varepsilon_\gamma)(\frac{r_o}{r})^{2\ell} (2\ell n \frac{r}{r_o} + 1)\}$$

$$+ C_\ell \{1+ \frac{1}{\ell} \cdot \frac{\varepsilon_\beta - \varepsilon_\gamma}{1+\frac{\ell-1}{\ell} (\varepsilon_\beta-\varepsilon_\gamma)} (\frac{r_o}{r})^{2\ell}\} \frac{(\frac{mr}{2R})^\ell}{(\ell - 1)!} \cos\ell\phi \quad ,$$

$$(3\text{-}5)$$

$$\Psi_c(r,\phi) = \frac{\alpha B_o r^2}{2} \{1+(\varepsilon_\beta-\varepsilon_\gamma)(\frac{r_o}{r})^2 (2\ell_n \frac{r}{r_o} + 1)\}$$

$$+ C_\ell \{ \left. \frac{I_\ell'(\frac{mr}{R})}{K_\ell'(\frac{mr}{R})} \right|_{r=b} (- \frac{\ell(\ell - 1)}{2}) (\frac{2R}{mr})^{2\ell}$$

$$+ \frac{1}{\ell} \cdot \frac{\varepsilon_\beta - \varepsilon_\gamma}{1+ \frac{\ell-1}{\ell} (\varepsilon_\beta - \varepsilon_\gamma)} (\frac{r_o}{r})^{2\ell}\} \frac{(\frac{mr}{2R})^\ell}{(\ell - 1)!} \cos\ell\phi \quad ,$$

$$(3\text{-}6)$$

where

$$\varepsilon_\beta = \frac{\beta}{2} \left(\frac{\ell R}{m r_o} \right)^2 \quad , \tag{3-7}$$

and

$$\varepsilon_\gamma = \frac{\ell}{mq} \quad . \tag{3-8}$$

The zeroth harmonic term of the magnetic field B^o in the plasma becomes

$$B_r^{\,o} = 0 \quad , \tag{3-9}$$

$$B_\theta^{\,o} = \left(\frac{1}{Rq} - \frac{\beta \ell R}{m r_o^2} \right) B_o r \quad , \tag{3-10}$$

$$B_z^{\,o} = B_{zo} \left\{ 1 + \left(\frac{r_o}{Rq} \right)^2 \left(1 - \frac{r^2}{r_o^2} \right) - \frac{\beta}{2} \left(1 - \frac{r^2}{r_o^2} \right) \right\} \quad , \tag{3-11}$$

where

$$B_{zo} = B_o \left\{ 1 - \frac{m}{\ell q} \left(\frac{r_o}{R} \right)^2 + \frac{\beta}{2} \right\} \quad . \tag{3-12}$$

The second term of eq.(3-11) is the paramagnetic term of force free current and the third term of eq.(3-11) is that of the diamagnetic current.

Rotational transform angle can be obtained by the average method[4] as follows:

$$\frac{d\bar{\theta}}{dz} = \frac{\bar{b}_\theta}{\bar{r}B_o} + \frac{1}{\bar{r}B_o}\frac{\partial}{\partial \bar{r}}\left(\frac{\overline{\langle b_r \rangle b_\theta}}{B_o}\right) , \tag{3-13}$$

$$\frac{d\bar{r}}{dz} = 0 , \tag{3-14}$$

when the magnetic field is expressed by $\vec{B} = B_z\vec{e}_z + b_r\vec{e}_r + b_\theta\vec{e}_\theta + b_z\vec{e}_z$. The notations $\bar{a}$ and $\langle a \rangle$ mean

$$\bar{a} = L^{-1}\int_o^L a dz , \tag{3-15}$$

$$\langle a \rangle = \int_o^z (a-\bar{a}) dz - \overline{\int_o^z (a-\bar{a}) dz} . \tag{3-16}$$

The rotational transform angle ι is expressed in the form

$$\iota \equiv 2\pi R \frac{d\bar{\theta}}{dz}$$

$$= \frac{2\pi}{q} - \beta\frac{\pi\ell}{m}\left(\frac{R}{r_0}\right)^2 + \iota_h\left[1 + \frac{\beta - \frac{2}{q}\left(\frac{r_o}{R}\right)^2\frac{m}{\ell}}{2\ell\left(\frac{m}{\ell}\right)^2\left(\frac{r_o}{R}\right)^2 + (\ell-1)\{\beta - \frac{2}{q}\left(\frac{r_o}{R}\right)^2\frac{m}{\ell}\}}\right]^2$$

$$= \frac{2\pi}{q} + \iota_h - \frac{2}{mq}\iota_h - \pi\beta\frac{\ell}{m}\left(\frac{R}{r_o}\right)^2 , \tag{3-17}$$

where

$$\iota_h = 2\pi R^2\frac{\ell}{m}\left(\frac{\ell^2 c_\ell}{B_o}\right)^2\left(\frac{1}{\ell}\right)^2\left(\frac{m}{2R}\right)^{2\ell}\left(1 - \frac{1}{\ell}\right)r^{2\ell-4} . \tag{3-18}$$

The first term is the term due to the plasma current of the zeroth harmonic component and the second one is the term due to the external helical current. The third one is the term due to the plasma current of the ℓ-th harmonic component. The ratio of the magnitude of the third term to that of the second one is $\frac{2}{mq}$. This ratio is not very small. The fourth term represents the finite beta effect.

When the rotational transform angle due to the plasma current has the same direction as that of the external helical current, the radius of the separatrix becomes small as the plasma current increases. When q becomes to a critical value

$$q_c = \frac{\frac{\ell}{m}}{1 + \frac{\beta}{2} \left(\frac{mr}{R} \right)^2} , \qquad (3\text{-}19)$$

the magnetic surface shrinks to a point. When $\ell = 2$, $m = 1$ and β is negligible, $q = 2$ becomes already critical value to shrink the magnetic surface in this configuration.

§4. Conclusion

The magnetohydrodynamic equilibrium of a current carrying plasma in helically symmetric system is studied analytically.

It is shown that the total rotational transform angle consists of four components; that is, the component due to the plasma current of zeroth harmonic, the component due to external helical current, the component due to the plasma current of helical component and the component due to the finite beta effect.

The effect of the plasma current to the magnetic flux function is clarified in the case of nearly uniform current distribution and parabolic pressure distribution.

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